Rational Approximation and its Applications in Mathematics and Physics 1985

Edited by J. Gilewicz, M. Pindor, W. Siemaszko

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FOREWORD

One of the main purposes of the Łancut Conference was the direct exchange of experiences and results between specialists in rational approximation who have not had any occasion to meet until now. At the first European meetings in Marseille -Toulon (1975), Lille (1978), Antwerp (1979), Amsterdam (1980), Bad Honnef (1983), Bar le Duc (1984), Segovia (1985), Marseille(1985) there were only a few participants from Poland. The first French -Polish meeting on rational approximation at Warsaw, took place in June 1981. The proof of the need for such meetings as that of Łancut, for exemple, is the constatation made by Ukrainian mathematicians that some of their results obtained more than ten years before, though published in a journal little known in the West, were rediscovered in 1985.
We would like to explain the reasons for the choice of the topics of the Conference. As is well known, continued fractions and rational approximations constitute the same domain expressed in different languages. Considering these problems in a wider aspect of approximation theory is necessary today for their further development. It should not be forgotten that interest in these problems comes from their spectacular applications to numerical and physical problems. Thus, all these subjects have found their place in the program of the Łancut Conference.
A very serious disease did not allow our friend, Prof. Dr. Helmut Werner, to participate in the Conference. He did his best in sending us his article . . . a few days before he died. A little In Memoriam is devoted to him. We would like to thank his wife, Mrs. Ingrid Werner, for writing a few words about her husband's life. We also thank his devoted secretary, Mrs. Elisabeth Becker, and his colleague, Dr. Paul Janssen, for their cooperation. All Helmut's friends' thoughts are expressed by Dr. Annie Cuyt.
The Conference was sponsored by the "Komitet Nauk Fizycznych PAN" of Warsaw, the "Towarzystwo Naukowe" of Rzeszów, the "Instytut Fizyki Teoretycznej" of Warsaw University and by the "Politechnika Rzeszowska im. I. Łukasiewicza" of Rzeszów, to all of whom our grateful thanks.
The Organizing Committee of the Łancut Conference expresses its gratitude to Springer-Verlag for kindly publishing the Proceedings in the series Lecture Notes in Mathematics.

IN MEMORIAM

Prof. Dr. Helmut Werner
1931-1985

November 22, 1985

Dear friend and colleague,

"Nur wer den Gipfel des Berges erstiegen, vermag in die weiteste Ferne zu sehn". However true this proverb may be, today it announced the sad event of your decease. At the conference in Łańcut all of us were still hoping that you would get better again. Although your farewell was not completely unexpected, it came far too soon.

Many have known you and worked with you and I'm sure that as many have loved and appreciated you. You were always such a busy man. Even when your health was not what it used to be anymore, you only felt really happy when you could be very active and were trying to do several things simultaneously. You were often short of time but never short of appointments! You were also a very precise man. When mathematical formulas had to be checked, when a paper had to be written down, you took all the time to make sure that everything was correct. If necessary you went over the same thing several times until you found it satisfactory.

We have all learnt from you and so have many students at the Universities of Münster and Bonn. What's more, you also cared about people. The large number of reports on the use of mathematics and computer science in medicine, especially to improve the situation of the blind, can testify to this

I am sure to speak for all the participants when I say: "May you rest in peace !"

Annie Cuyt

My husband was born on March 22, 1931 in Zwenkau near Leipzig. His father was a teacher at the Gymnasium. He went to school in his home town and in Leipzig, and after his Abitur in 1949 he was allowed to start studying mathematics and physics at the University of Leipzig which was a great privilege at that time.

In 1951 he moved with his parents to the German Federal Republic and continued his studies at the University of Göttingen. In between the terms of the academic year he worked in the oil fields near Lingen/Ems to earn his living. Later on he earned some money being a teaching assistant. His teachers were the Professors Beckert, Hölder and Kähler in Leipzig and the Professors Deuring, Heinz, Kaluza, Rellich and Siegel in Göttingen. He specialized in partial differential equations and prepared his thesis with Prof. Rellich who died of cancer before the thesis was finished. Prof. Siegel and Prof. Heinz, at that time assistant of Prof. Rellich, accepted his work and he got his doctor's degree in 1956.

A few months later we got married.

While preparing for his doctor's degree he could work at the Max-Planck Institute of Physics using (mainly at night) one of the first computers available in Germany. He was fascinated by that kind of work and therefore preferred a position in industry, at the AEG Research Center, rather than a position as assistant in pure mathematics at the university.

But soon he accepted an invitation to teach as assistant professor at the University of Southern California in Los Angeles. We stayed in California for two years and came back to Germany because my husband had met Prof. Collatz in Los Angeles who offered him the possibility to get his Habilitation at the University of Hamburg. Since some research had already been done in Los Angeles he obtained the Habilitation early 1962. Then he started teaching at the University of Hamburg. This period was interrupted by half a year of teaching and researching at Stanford University in California. There he was offered a full professorship to erect the Institute for Numerical and Instrumental Mathematics and the Computation Center of the University of Münster. He accepted the offer and we moved to Münster in 1964.

At the beginning my husband spent a lot of time running over the building plans for the Computation Center, deciding what kind of computer had to be bought, trying to get money for it and training students to work with it. Ten years later, when the IBM 360-50 became too small for the university he had to go through this procedure again.

He always tried to help a lot of people from other faculties of the university who wanted to use the computer for their own research projects. Over the years he developed very intense contacts with many colleagues not only from Science but also from Medicine, Theology, the Humanities and others. He became a member of the Sonderforschungsbereich Mittelalterforschung, helping historians to handle huge amounts of data and laborious publication procedures. This stimulated his interest in providing a text editing system.

As long as I have known him, in every private or professional situation, he had an open mind for other people's problems trying to help them with what he knew about mathematics and computer science. When we were newly wedded and visiting my girl friend who had married a blind teacher, we learnt a lot about braille. This inspired him to develop an automatic braille program for computers. During the next 25 years he developed this project to the extent that it is now used in Germany, Switzerland and Austria for the production of braille printing of all kind. Recently he was awarded the Louis Braille price (in 1984) and the Carl-Strehl medal (in

1985) for this work. When he met a professor in ophtalmology who tried to help patients having problems with their three-dimensional sight due to the removal of a lense in one of the eyes, he developed formulas which not only made use of spectacles, as was common practice, but also of contact lenses. He arranged those formulas in such a way that any ophtalmologist all over the country could easily use them.

In the late seventies more than 50 people were working at the Computation Center in Münster, including an academically trained staff of 23 and another 6 researchers at the Institute of Numerical Mathematics. He was Fachgutachter for mathematics for the Deutsche Forschungsgemeinschaft between 1972 and 1980. Though this responsibility took a lot of time and energy he found it very stimulating. Besides this he was a member of the senate of the Sonderforschungsbereiche of the Deutsche Forschungsgemeinschaft from 1974 till 1982. In this way he was closely connected to the most recent research projects.

He never wanted to return to pure research again (like at AEG) or accept a position as manager. He enjoyed giving lectures and advising students and doctorands. In total 26 students got their doctor's degree inspired by him. In 1980 he became director of the Institute of Applied Mathematics and the Department of Functional Analysis and Numerical Mathematics at the University of Bonn. He continued his research and teaching there, also being a member of the Sonderforschungsbereich 72 (applied mathematics).

He had to stop lecturing in the middle of a term, one week before Whit Sunday in 1985. He entered the hospital the next day because he was very much in pain but he hoped to be able to continue his lecturing after Whit Sunday. During all the following months, up to the last two weeks, he had some of his older students come to the hospital for discussions or examinations. He published a great number of technical notes and scientific papers. He wrote 11 books, of which many were reprinted, and was editor of another 10.

In 1978 he became a member of the Akademie der Naturforscher-Leopoldina in Halle and was very happy about it because it enabled him to make friends with many colleagues from his home region.

He loved the professional and social contact with colleagues all over the world. On his last main lecturing tour in September 1984, already ill with cancer, we visited several universities in China and he lectured almost every day. He was very sorry not to be able to come to Łańcut in Poland anymore as we had planned and hoped till the last moment. However, his last scientific work will appear in these proceedings together with the work of those mathematicians he felt so close to.

When travelling he took every opportunity to enrich himself culturally, using night hours to attend concerts or visit musea – he really got excited about modern paintings. What's more, he always tried to plan these things so that I could share his opportunities.

Working at home, he always listened to music, mainly Bach, Mozart, Brahms, Mahler and Prokoviev. In his spare time he enjoyed reading books on modern history and art or do some handicraft, especially with wood. He collected music on tapes, books and maps, lending the latter out to who ever needed them. On Sundays the family used to make excursions by bicycle or car, most of the times to a point from where the whole landscape could be overlooked. Several times he biked from Münster to Texel, together with our twins, while our youngest daughter and I went by car with the luggage for our holidays. During our last family holiday in 1983 we toured the western part of the USA and Helmut showed us all the places he had got to know at various earlier occasions.

So when he was seriously ill he had a profound reservoir of mathematical problems to be solved, of favourite art to enjoy, of fine experiences to remember and a lot of friends to care for and who cared for him !

He did hope to get his strength back again, supporting his doctors in every physical and mental way, but on the other hand he was prepared to accept his fate, if necessary, having "set his house in order".

He died on November 22, 1985.

Ingrid Werner

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CONTENTS

Problems related to physics

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A survey of bounds for the zeros of analytic functions obtained by continued fraction methods.

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1. Introduction

We mainly consider sequences of polynomials $q_n(z)$, which satisfy a three term recurrence relation

(1) $q_n(z)=b_n(z)q_{n-1}(z)-a_n(z)q_{n-2}(z), \quad n \in \mathbb{N}$

where $q_{-1}=0$, $q_0=1$, and where a_n, b_n are complex polynomials $\not\equiv 0$. All results below are concerned with the construction of various subsets of $\mathbb{C}$ from the coefficients of a_n, b_n, such that these subsets contain all zeros of $q_n(z)$ for $n \geq 2$ or $2 \leq n \leq N$. Most of these results easily can be extended to power series.

2. The first continued fraction method

The method which is underlying all relevant proofs consists in associating the sequence q_n from (1) to a sequence of Moebius transforms and utilizing their various mapping properties. For example, (1) yields

$$q_{n-1}/q_n = \frac{1}{b_n-a_n(q_{n-2}/q_{n-1})}, \quad n \geq 1,$$

or, using $\mu_n:=q_{n-1}/q_n$, $n \geq 2$, $\mu_1:=1/b_1$, $\mu_0:=0$,

$$\mu_n = \frac{1}{b_n-a_n\mu_{n-1}}, \quad n \geq 1.$$

Hence, we obtain for each $N \geq 2$ the finite continued fraction representation

(2) $$\mu_N = \frac{1}{b_N} - \frac{a_N}{b_{N-1}} - \cdots - \frac{a_2}{b_1} .$$

(For definitions and notations see [19], [25], [34].)
Therefore, μ_N can be written as a composition of Moebius transforms

$$\mu_N = T_N \circ T_{N-1} \circ \cdots \circ T_2(\mu_1)$$
$$= T_N \circ T_{N-1} \circ \cdots \circ T_2 \circ T_1(0), \text{ where}$$

$$T_n(u) := \frac{1}{b_n-a_n u}, \quad n \geq 2, \quad T_1(u) := \frac{1}{b_1-u} .$$

Applying equivalence transformations to the continued fraction (2) yields other representations of μ_N as composition of suitable Moebius transforms.

Observing that $\mu_n = T_n \circ T_{n-1} \circ \ldots \circ T_1(0)$ holds for $n \geq 1$, conditions on T_n of the following type are formulated. Choose $V_n \subset \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, such that the boundary of each V_n is a circle or straight line, $n \geq 0$. Assume, furthermore, that $0 \in V_0$ and $T_n(V_{n-1}) \subset V_n$ holds for $n \geq 1$. Since $\mu_0 = 0$ and $\mu_n = T_n(\mu_{n-1}) \in V_n$ for $n \geq 1$, the condition $\infty \notin V_N$ yields $\mu_N \neq \infty$. In most of the applications the V_n are closed half-planes and $T_n(V_{n-1})$ are closed disks. Therefore, $\mu_2, \ldots, \mu_N \neq \infty$ holds in this case. In many special cases this then implies $q_2, \ldots, q_N \neq 0$.

The above formulated conditions on T_n are conditions on the polynomials $a_n(z)$, $b_n(z)$ leading to the required subsets of $\mathbb{C}$ which contain the zeros of $q_2(z), \ldots, q_N(z)$. In this connection see also chapter 10 of [34].

3. The Parabola theorem of E.B. Saff and R.S. Varga

As an application of the first continued fraction method we now consider polynomials $q_n(z)$ satisfying

(3) $q_n(z) = (z+\beta_n) q_{n-1}(z) - \alpha_n z\, q_{n-2}(z)$, $n \in \mathbb{N}$

where $q_{-1} = 0$, $q_0 = 1$, α_{n+1}, $\beta_n \in \mathbb{C}$, $\alpha_{n+1} \neq 0$, $n \geq 1$.

Let, for example, $f(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$ be a formal power series and put $c_{-\nu} := 0$ for $\nu \geq 1$, $A_m^{(0)} := 1$ and

$$A_m^{(n)} := \begin{vmatrix} c_m & c_{m-1} & \cdots & c_{m-n+1} \\ c_{m+1} & c_m & \cdots & c_{m-n+2} \\ & & & \\ c_{m+n-1} & \cdots\cdots & & c_m \end{vmatrix} , \quad m \geq 0,\ n \geq 1.$$

If $A_m^{(n)} \neq 0$ for all m, $n \geq 0$, and if $U_{m,n}(z)$ and $V_{m,n}(z)$ denote the Padé-numerator and Padé-denominator ([10], [25], [34]) of the (m,n)-Padé approximant to $f(z)$, then one obtains (see [9], [27], [32])

<u>Proposition 1.</u> For <u>fixed</u> $n \geq 0$ $q_m(z) := U_{m,n}(z) A_m^{(n)} / A_m^{(n+1)}$

satisfies (3) with $\alpha_{m+1} = A_{m+1}^{(n)} A_{m-1}^{(n+1)} / A_m^{(n)} A_m^{(n+1)}$,

$\beta_m = A_{m-1}^{(n+1)} A_m^{(n)} / A_{m-1}^{(n)} A_m^{(n+1)}$, and

$\beta_m - \alpha_m = A_m^{(n)} \; A_{m-1}^{(n+2)} \;/ A_m^{(n+1)} \; A_{m-1}^{(n+1)}, \; m \geq 1 \; (\alpha_1 = 0)$

Especially, $U_{m,0}(z)$ is the m-th partial sum of f(z) and in this case

(4) $\beta_m = \alpha_{m+1} = c_{m-1}/c_m, \; m \geq 1$.

Proposition 2. For fixed $m \geq 0$ $q_n(z) := V_{m,n}(-z) A_m^{(n)} / A_{m+1}^{(n)}$

satisfies (3) with $\alpha_{n+1} = A_m^{(n+1)} A_{m+1}^{(n-1)} / A_m^{(n)} A_{m+1}^{(n)}$,

$\beta_n = A_m^{(n)} \; A_{m+1}^{(n-1)} / A_m^{(n-1)} \; A_{m+1}^{(n)}$, and

$\beta_n - \alpha_n = A_m^{(n)} \; A_{m+2}^{(n-1)} / A_{m+1}^{(n)} \; A_{m+1}^{(n-1)}, \; n \geq 1 \; (\alpha_1 = 0)$.

If $f(z) = e^z$, i.e. $c_\nu = 1/\nu!$, $\nu \geq 0$, then

(5) $A_m^{(n)} = \prod_{j=1}^{n} \frac{(j-1)!}{(m+j-1)!}$, m, n $\geq$ 1 (see [32])

Theorem 1 (Parabola theorem of E.B. Saff and R.S. Varga).

Assume that the polynomials $q_n(z)$, $n \geq 1$, satisfy (3) such that $\beta_n > 0$, $1 \leq n \leq N$, $\alpha_n > 0$, $2 \leq n \leq N$ and $D_N := \min_{1 \leq n \leq N} (\beta_n - \alpha_n) > 0$ (with $\alpha_1 = 0$). Then $q_n(z) \neq 0$ for $2 \leq n \leq N$ and all $z \in P$, where

$$P := \{ \zeta \in \mathbb{C} : |\zeta| \leq \operatorname{Re}\zeta + 2 D_N \} .$$

For the proof of Theorem 1 and all examples below, see [32]. Using (4), this result immediately can be applied to partial sums of power series $f(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$, if $c_\nu > 0$, $\nu \geq 0$ and $\frac{c_{\nu-1}}{c_\nu} - \frac{c_{\nu-2}}{c_{\nu-1}} > 0$, $\nu \geq 2$.

In particular, $\sum_{\nu=0}^{m} \frac{1}{\nu!} z^\nu$ is $\neq 0$ for $m \geq 2$ and all $z \in \mathbb{C}$ satisfying $|z| \leq \operatorname{Re} z + 2$, since $D_N = 1$ for each $N \geq 2$ in this case. See also [23], [24], and [33].

Besides many other examples also generalized Bessel polynomials are treated successfully in [32] by means of Theorem 1. The n-th generalized Besselpolynomial $Y_n^{(\delta)}$ is defined by (see [16], [32])

(6) $Y_n^{(\delta)} := 1 + \sum_{j=1}^{n} \binom{n}{j} (n+\delta+1)(n+\delta+2)\dots(n+\delta+j)(-z/2)^j$

for $n \in \mathbb{N}$, $\delta \in \mathbb{C}$.

For each <u>fixed</u> $m \geqq 2$ $q_n^{(m+\delta)}(z) := z^n Y_n^{(m+\delta-n)}(-2/z)$ satisfies (3) with $\beta_n = n+m+\delta$, $\alpha_n = n-1$, $n \geqq 1$.

Observe that the substitution $z \to -2/z$ maps the exterior of P (in Theorem 1) onto the interior of a cardioid region. Then Theorem 1 yields for n=m [32].

<u>Theorem 2</u>. If $\delta \in \mathbb{R}$ and $m+\delta+1=c>0$, then all zeros of $Y_m^{(\delta)}(z)$, $m \geqq 2$, are contained in the open cardioid region

$$C := \{\zeta = re^{i\theta} \in \mathbb{C} : 0 < r < (1+\cos\theta)/c, \ |\theta| < \pi\}.$$

For further results concerning the zeros of generalized Bessel polynomials see [20], [22], [26] and section 8 of this paper. In [6] K. Dočev showed that all zeros of $Y_m^{(\delta)}(z)$, $m+\mathrm{Re}\,\delta+1=c>0$ are contained in $\{z \in \mathbb{C} : |z| \leq 2/c\}$. Especially, M.G. de Bruin, E.B. Saff and R.S. Varga have proved in [1], [31], [33], that the result of Theorem 2 is sharp in the sense that each boundary point of $\{z = re^{i\theta} \in \mathbb{C} : 0 < r < (1+\cos\theta), |\theta| < \pi\}$ is an accumulation point of zeros of the set of normalized Bessel polynomials $Y_m^{(\delta)}(z/(m+\delta+1))$, $m \in \mathbb{N}$, $m+\delta+1>0$.

4. Results of M.G. de Bruin

By applying a method of proof which is similar to the first continued fraction method (introduced by E.B. Saff and R.S. Varga in [32]) M.G. de Bruin obtained the following results.

<u>Theorem 3</u>. Let the sequence of polynomials $\{P_n(z)\}_{n=0}^{\infty}$ be generated by

$$P_n(z) = (1+a_n z)P_{n-1}(z) + b_n z\, P_{n-2}(z) + c_n z^2\, P_{n-3}(z), \quad n \geqq 3, \quad \text{where}$$

$$P_0(z)=1,\ P_1(z)=1+a_1 z,\ P_2(z)=(1+a_1 z)(1+a_2 z)+b_2 z, \text{ with } a_n \in \mathbb{R}\setminus\{0\},$$

$n \geqq 1$, $b_n \in \mathbb{R}$, $n \geqq 2$, $c_n \in \mathbb{R}\setminus\{0\}$, $n \geqq 3$.

Let $\{A_n^{(k)}\}_{n=1}^{\infty}$ $(k=1,2,\ldots,5)$ be sequences of positive real numbers and define the sets V_k $(k=1,2,\ldots 5)$ of complex numbers z as follows