Statistical Signal Processing of Complex-Valued Data

The Theory of Improper and Noncircular Signals

Peter J. Schreier and Louis L. Scharf

CAMBRIDGE

Statistical Signal Processing of Complex-Valued Data

The Theory of Improper and Noncircular Signals

PETER J. SCHREIER
University of Newcastle, New South Wales, Australia

LOUIS L. SCHARF
Colorado State University, Colorado, USA

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521897723

First published 2010

Printed in the United Kingdom at the University Press, Cambridge

A catalog record for this publication is available from the British Library

Library of Congress Cataloging in Publication data

ISBN 978-0-521-89772-3 Hardback

Statistical Signal Processing of Complex-Valued Data

Complex-valued random signals are embedded into the very fabric of science and engineering, yet the usual assumptions made about their statistical behavior are often a poor representation of the underlying physics. This book deals with improper and noncircular complex signals, which do not conform to classical assumptions, and it demonstrates how correct treatment of these signals can have significant payoffs.

The book begins with detailed coverage of the fundamental theory and presents a variety of tools and algorithms for dealing with improper and noncircular signals. It provides a comprehensive account of the main applications, covering detection, estimation, and signal analysis of stationary, nonstationary, and cyclostationary processes.

Providing a systematic development from the origin of complex signals to their probabilistic description makes the theory accessible to newcomers. This book is ideal for graduate students and researchers working with complex data in a range of research areas from communications to oceanography.

PETER J. SCHREIER is an Associate Professor in the School of Electrical Engineering and Computer Science, The University of Newcastle, Australia. He received his Ph.D. in electrical engineering from the University of Colorado at Boulder in 2003. He currently serves on the Editorial Board of the *IEEE Transactions on Signal Processing*, and on the IEEE Technical Committee *Machine Learning for Signal Processing*.

LOUIS L. SCHARF is Professor of Electrical and Computer Engineering and Statistics at Colorado State University. He received his Ph.D. from the University of Washington at Seattle. He has since received numerous awards for his research contributions to statistical signal processing, including an IEEE Distinguished Lectureship, an IEEE Third Millennium Medal, and the Technical Achievement and Society Awards from the IEEE Signal Processing Society. He is a Life Fellow of the IEEE.

Preface

Complex-valued random signals are embedded into the very fabric of science and engineering, being essential to communications, radar, sonar, geophysics, oceanography, optics, electromagnetics, acoustics, and other applied sciences. A great many problems in detection, estimation, and signal analysis may be phrased in terms of two channels' worth of real signals. It is common practice in science and engineering to place these signals into the real and imaginary parts of a complex signal. Complex representations bring economies and insights that are difficult to achieve with real representations.

In the past, it has often been assumed – usually implicitly – that complex random signals are *proper* and *circular*. A *proper* complex random variable is uncorrelated with its complex conjugate, and a *circular* complex random variable has a probability distribution that is invariant under rotation in the complex plane. These assumptions are convenient because they simplify computations and, in many aspects, make complex random signals look and behave like real random signals. Yet, while these assumptions can often be justified, there are also many cases in which proper and circular random signals are very poor models of the underlying physics. This fact has been known and appreciated by oceanographers since the early 1970s, but it has only recently been accepted across disciplines by acousticians, optical scientists, and communication theorists.

This book develops the tools and algorithms that are necessary to deal with *improper* complex random variables, which are correlated with their complex conjugate, and with *noncircular* complex random variables, whose probability distribution varies under rotation in the complex plane. Accounting for the improper and noncircular nature of complex signals can have big payoffs. In digital communications, it can lead to a significantly improved tradeoff between spectral efficiency and power consumption. In array processing, it can enable us to estimate with increased accuracy the direction of arrival of one or more signals impinging on a sensor array. In independent component analysis, it may be possible to blindly separate Gaussian sources – something that is impossible if these sources are *proper*.

In the electrical engineering literature, the story of improper and noncircular complex signals began with Brown and Crane, Gardner, van den Bos, Picinbono, and their co-workers. They have laid the foundations for the theory we aim to review and extend in this research monograph, and to them we dedicate this book. The story is continuing, with work by a number of our colleagues who are publishing new findings as we write this preface. We have tried to stay up to date with their work by referencing it as carefully as we have been able. We ask their forbearance for results not included.

Outline of this book

The book can be divided into three parts. Part I (Chapters 1 and 2) gives an overview and introduction to complex random vectors and processes. In Chapter 1, we describe the origins and uses of complex signals. The chapter answers the following question: why do engineers and applied scientists represent real measurable effects by complex signals? Chapter 2 lays the foundation for the remainder of the book by introducing important concepts and definitions for complex random vectors and processes, such as widely linear transformations, complementary correlations, the multivariate improper Gaussian distribution, and complementary power spectra of wide-sense stationary processes. Chapter 2 should be read before proceeding to any of the later chapters.

Part II (Chapters 3–7) deals with complex random vectors and their application to correlation analysis, estimation, performance bounding, and detection. In Chapter 3, we discuss in detail the second-order description of a complex random vector. In particular, we are interested in those second-order properties that are invariant under either widely unitary or widely linear transformation. This leads us to a test for impropriety and applications in independent component analysis (ICA). Chapter 4 treats the assessment of multivariate association between two complex random vectors. We provide a unifying treatment of three popular correlation-analysis techniques: canonical correlation analysis, multivariate linear regression, and partial least squares. We also present several generalized likelihood-ratio tests for the correlation structure of complex Gaussian data, such as sphericity, independence within one data set, and independence between two data sets.

Chapter 5 is on estimation. Here we are interested in linear and widely linear least-squares problems, wherein parameter estimators are constrained to be linear or widely linear in the measurement and the performance criterion is mean-squared error or squared error under a constraint. Chapter 6 deals with performance bounds for parameter estimation. We consider quadratic performance bounds of the Weiss–Weinstein class, the most notable representatives of which are the Cramér–Rao and Fisher–Bayes bound. Chapter 7 addresses detection, where the problem is to determine which of two or more competing models best describes experimental measurements. In order to demonstrate the role of widely linear and widely quadratic forms in the theory of hypothesis testing, we concentrate on hypothesis testing within Gaussian measurement models.

Part III (Chapters 8–10) deals with complex random processes, both continuous- and discrete-time. Throughout this part, we focus on second-order spectral properties, and optimum linear (or widely linear) minimum mean-squared error filtering. Chapter 8 discusses wide-sense stationary (WSS) processes, with a focus on the role of the complementary power spectral density in rotary-component and polarization analysis. WSS processes admit a spectral representation in terms of the Fourier basis, which allows a frequency interpretation. The transform-domain description of a WSS signal is a spectral process with *orthogonal* increments. For nonstationary signals, we have to sacrifice either the Fourier basis and thus its frequency interpretation, or the orthogonality of the transform-domain representation. In Chapter 9, we will discuss both possibilities, which leads either to the Karhunen–Loève expansion or the Cramér–Loève spectral representation. The latter is the basis for bilinear time–frequency representations. Then, in Chapter 10 we treat cyclostationary processes. They are an important class of nonstationary processes that have periodically varying correlation properties. They can model periodic phenomena occurring in science and technology, including communications, meteorology, oceanography, climatology, astronomy, and economics.

Three appendices provide background material. Appendix 1 presents rudiments of matrix analysis. Appendix 2 introduces Wirtinger calculus, which enables us to compute generalized derivatives of a *real* function with respect to *complex* parameters. Finally, Appendix 3 discusses majorization, which is used at several places in this book. Majorization introduces a preordering of vectors, and it will allow us to optimize certain scalar real-valued functions with respect to real vector-valued parameters.

This book is mainly targeted at researchers and graduate students who rely on the theory of signals and systems to conduct their work in signal processing, communications, radar, sonar, optics, electromagnetics, acoustics, oceanography, geophysics, and geography. Although it is not primarily intended as a textbook, chapters of the book may be used to support a special-topics course at a second-year graduate level. We would expect readers to be familiar with basic probability theory, linear systems, and linear algebra, at a level covered in a typical first-year graduate course.

Acknowledgments

We would like to thank Dr. Patrik Wahlberg for giving us detailed feedback on many chapters of this book. We further thank Dr. Phil Meyler of Cambridge University Press for his support throughout the writing of this book. Peter Schreier acknowledges financial support from the Australian Research Council (ARC) under its Discovery Project scheme, and thanks Colorado State University, Ft. Collins, USA, for its hospitality during a five-month study leave in the winter and spring of 2008 in the northern hemisphere. Louis Scharf acknowledges years of research support by the Office of Naval Research and the National Science Foundation (NSF), and thanks the University of Newcastle, Australia, for its hospitality during a one-month study leave in the autumn of 2009 in the southern hemisphere.

Peter J. Schreier
Newcastle, New South Wales, Australia

Louis L. Scharf
Ft. Collins, Colorado, USA

Notation

Conventions

$\langle x, y \rangle$	inner product
$\|x\|$	norm (usually Euclidean)
$\hat{x}$	estimate of x
$\widetilde{x}$	complementary quantity to x
$x \perp y$	x is orthogonal to y

Vectors and matrices

$\mathbf{x}$	column-vector with components x_i
$\mathbf{x} \prec \mathbf{y}$	$\mathbf{x}$ is majorized by $\mathbf{y}$
$\mathbf{x} \prec_{\mathrm{w}} \mathbf{y}$	$\mathbf{x}$ is weakly majorized by $\mathbf{y}$
$\mathbf{X}$	matrix with components $(\mathbf{X})_{ij} = X_{ij}$
$\mathbf{X} > \mathbf{Y}$	$\mathbf{X} - \mathbf{Y}$ is positive definite
$\mathbf{X} \geq \mathbf{Y}$	$\mathbf{X} - \mathbf{Y}$ is positive semidefinite (nonnegative definite)
$\mathbf{X}^*$	complex conjugate
$\mathbf{X}^{\mathrm{T}}$	transpose
$\mathbf{X}^{\mathrm{H}} = (\mathbf{X}^{\mathrm{T}})^*$	Hermitian (conjugate) transpose
$\mathbf{X}^{\dagger}$	Moore–Penrose pseudo-inverse
$\langle \mathbf{X} \rangle$	subspace spanned by columns of $\mathbf{X}$
$\underline{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x}^* \end{bmatrix}$	augmented vector
$\underline{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 \\ \mathbf{X}_2^* & \mathbf{X}_1^* \end{bmatrix}$	augmented matrix

Functions

$x(t)$	continuous-time signal
$x[k]$	discrete-time signal
$\hat{x}(t)$	Hilbert transform of $x(t)$; estimate of $x(t)$
$X(f)$	scalar-valued Fourier transform of $x(t)$
$\mathbf{X}(f)$	vector-valued Fourier transform of $\mathbf{x}(t)$
$\mathbb{X}(f)$	matrix-valued Fourier transform of $\mathbf{X}(t)$

$X(z)$	scalar-valued z-transform of $x[k]$
$\mathbf{X}(z)$	vector-valued z-transform of $\mathbf{x}[k]$
$\mathbb{X}(z)$	matrix-valued z-transform of $\mathbf{X}[k]$
$x(t) * y(t) = (x * y)(t)$	convolution of $x(t)$ and $y(t)$

Commonly used symbols and operators

$\arg(x) = \measuredangle x$	argument (phase) of complex x
$\mathbb{C}$	field of complex numbers
$\mathbb{C}_*^{2n}$	set of augmented vectors $\underline{\mathbf{x}} = [\mathbf{x}^T, \mathbf{x}^H]^T$, $\mathbf{x} \in \mathbb{C}^n$
$\delta(x)$	Dirac δ-function (distribution)
$\det(\mathbf{X})$	matrix determinant
$\mathbf{diag}(\mathbf{X})$	vector of diagonal values $X_{11}, X_{22}, \ldots, X_{nn}$ of $\mathbf{X}$
$\mathbf{Diag}(x_1, \ldots, x_n)$	diagonal or block-diagonal matrix with diagonal elements $x_1, \ldots, x_n$
$\mathbf{e}$	error vector
$E(x)$	expectation of x
$\mathbf{ev}(\mathbf{X})$	vector of eigenvalues of $\mathbf{X}$, ordered decreasingly
$\mathbf{I}$	identity matrix
$\operatorname{Im} x$	imaginary part of x
$\mathbf{K}$	matrix of canonical/half-canonical correlations k_i
$\mathbf{\Lambda}$	matrix of eigenvalues λ_i
$\mathbf{m}_x$	sample mean vector of $\mathbf{x}$
$\boldsymbol{\mu}_x$	mean vector of $\mathbf{x}$
$p_x(x)$	probability density function (pdf) of x (often used without subscript)
$\mathbf{P}_{\mathbf{U}}$	orthogonal projection onto subspace $\langle \mathbf{U} \rangle$
$P_{xx}(f)$	power spectral density (PSD) of $x(t)$
$\widetilde{P}_{xx}(f)$	complementary power spectral density (C-PSD) of $x(t)$
$\underline{\mathbb{P}}_{xx}(f)$	augmented PSD matrix of $x(t)$
$\mathbf{Q}$	error covariance matrix
ρ	correlation coefficient; degree of impropriety; coherence
$\mathbb{R}$	field of real numbers
$\mathbf{R}_{xy}$	cross-covariance matrix of $\mathbf{x}$ and $\mathbf{y}$
$\widetilde{\mathbf{R}}_{xy}$	complementary cross-covariance matrix of $\mathbf{x}$ and $\mathbf{y}$
$\underline{\mathbf{R}}_{xy}$	augmented cross-covariance matrix of $\mathbf{x}$ and $\mathbf{y}$
$\mathbb{R}_{xy}$	covariance matrix of composite vector $[\mathbf{x}^T, \mathbf{y}^T]^T$
$r_{xy}(t, \tau)$	cross-covariance function of $x(t)$ and $y(t)$
$\tilde{r}_{xy}(t, \tau)$	complementary cross-covariance function of $x(t)$ and $y(t)$
$\operatorname{Re} x$	real part of x
$\operatorname{sgn}(x)$	sign of x
$\mathbf{sv}(\mathbf{X})$	vector of singular values of $\mathbf{X}$, ordered decreasingly

$\mathbf{S}_{xx}$	sample covariance matrix of $\mathbf{x}$
$\widetilde{\mathbf{S}}_{xx}$	sample complementary covariance matrix of $\mathbf{x}$
$\underline{\mathbf{S}}_{xx}$	augmented sample covariance matrix of $\mathbf{x}$
$S_{xx}(\nu, f)$	(Loève) spectral correlation of $x(t)$
$\widetilde{S}_{xx}(\nu, f)$	(Loève) complementary spectral correlation of $x(t)$
$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathrm{j}\mathbf{I} \\ \mathbf{I} & -\mathrm{j}\mathbf{I} \end{bmatrix}$	real-to-complex transformation
$\mathrm{tr}(\mathbf{X})$	matrix trace
$\mathbf{W}$	Wiener (linear or widely linear minimum mean-squared error) filter matrix
$\mathcal{W}^{m \times n}$	set of $2m \times 2n$ augmented matrices
$\mathbf{x} = \mathbf{u} + \mathrm{j}\mathbf{v}$	complex message/source
$\boldsymbol{\xi}$	internal (latent) description of $\mathbf{x}$
$x(t) = u(t) + \mathrm{j}v(t)$	complex continuous-time message/source signal
$\xi(f)$	spectral process corresponding to $x(t)$
$\mathbf{y} = \mathbf{a} + \mathrm{j}\mathbf{b}$	complex measurement/observation
$y(t)$	complex continuous-time measurement/observation signal
$\upsilon(f)$	spectral process corresponding to $y(t)$
$\boldsymbol{\omega}$	internal (latent) description of $\mathbf{y}$
Ω	sample space
$\mathbf{0}$	zero vector or matrix

Contents