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G. E. SHILOV

CALCULUS OF RATIONAL FUNCTIONS

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

Г. Е. Шилов

МАТЕМАТИЧЕСКИЙ АНАЛИЗ В ОБЛАСТИ РАЦИОНАЛЬНЫХ ФУНКЦИЙ

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

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CALCULUS OF RATIONAL FUNCTIONS

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Contents

Foreword

The concepts of derivative and integral are basic for the calculus. They are not elementary; in any systematic textbook on calculus the presentation of these concepts is preceded by the theory of real numbers, the theory of limits, and the theory of continuous functions. This preliminary procedure is necessary to formulate the concepts of derivative and integral in sufficiently universal form, to be applied to the widest possible class of functions. If, however, we restrict ourselves to a comparatively narrow class of rational functions and utilize the illustrative language of graphs, we can present the concepts of derivative and integral in a few pages, sufficiently accurately and at the same time pithily. And this is the purpose of the pamphlet intended for a wide circle of readers; the knowledge of secondary school students is sufficient to insure understanding of everything that will be discussed.

1. Graphs

Though we assume that the reader is conversant with graphs, we shall anyway remind the basic points.

Let us draw two mutually perpendicular straight lines, one horizontal and one vertical, and denote by O their intersection point. The horizontal line will be referred to as the *axis of abscissas* and the vertical line—as the *axis of ordinates*. The point O divides each line into two semi-axes, a positive and a negative one; the right-hand semi-axis of abscissas and the upper semi-axis of ordinates are called positive, while the left-hand semi-axis of abscissas and the lower semi-axis of ordinates are called negative. We mark the positive semi-axes by arrows. Now the position of each point M on the plane can be defined by a pair of numbers. To do this we drop perpendiculars from the point M onto each of the axes; the perpendiculars will cut on the axes the segments OA and OB (Fig. 1). The length of the segment OA, taken with the sign "+" if A is located on the positive semi-axis and with the sign "−" if it lies on the negative semi-axis, will be called the *abscissa* of the point M and will be denoted by x. Similarly, the length of the segment OB (with the same rule of signs) will be called the *ordinate* of the point M and denoted by y. The numbers, x and y, are called the *coordinates* of the point M. Each point on the plane is determined by coordinates. Points of the abscissa axis have the ordinate equal to zero, while the points of the ordinate axis have zero abscissa. The origin of coordinates O (the point of intersection of axes) has both coordinates equal to zero. Conversely, if two arbitrary numbers x and y of any signs are given, we can always plot, and this is very impor-

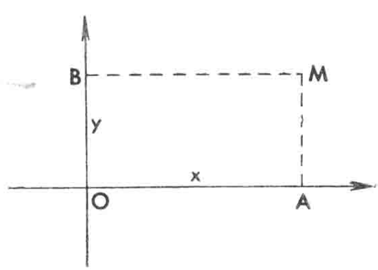

Fig. 1

tant, exactly a unique point M with the abscissa x and the ordinate y; to achieve this we have to lay off the segment $OA = x$ on the abscissa axis and to erect a perpendicular $AM = y$ (signs being taken into account); the point M will be the one sought for.

Let the rule be given which indicates the operations that should be performed over the independent variable (denoted by x) to obtain the value of the quantity of interest (denoted by y).

Each such rule *defines*, in the language used by mathematicians, *the quantity y as function of the independent variable x.* It can be said that a *given function* is just that specific *rule* by which the values of y are obtained from the values of x.

For instance, the formula

$$y = \frac{1}{1+x^2}$$

indicates that to obtain the values of y we have to square the independent variable x, add it to unity and then divide unity by the obtained result. If x takes on some numerical value x_0 then by virtue of our formula y takes on a certain value y_0. The values x_0 and y_0 define a point M_0 in the plane of the drawing. We can then replace x_0 by another number x_1 and calculate by the formula the new value y_1; the pair of numbers x_1, y_1 defines a new point M_1 on the plane. The geometric locus of all points of the plane, whose ordinates are related to abscissas by the given formula, is called the *graph* of the corresponding function.

Generally speaking, the set of graph points is infinite so that we cannot hope to plot all of them without exception by using the foregoing rule. But we shall not have to do that. In most cases a certain moderate number of points is sufficient for us to be able to realize the general shape of the graph.

The method of plotting a graph "point-by-point" consists just in plotting a certain number of graph points and in joining these points by as smooth a curve as possible.

As an example we shall consider the graph of the function

$$y = \frac{1}{1+x^2} \tag{1}$$

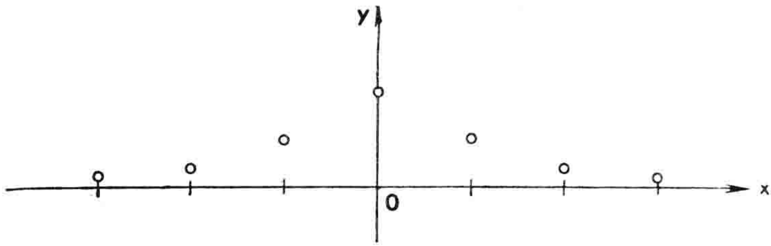

Fig. 2

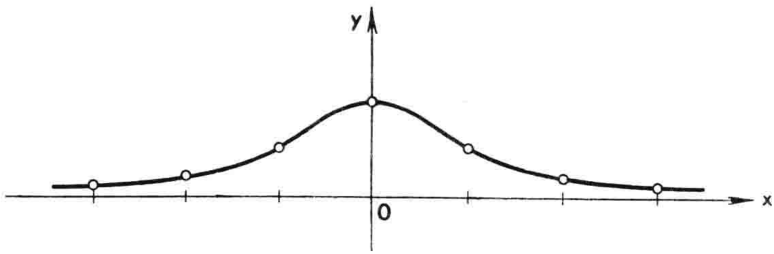

Fig. 3

Let us compile the following table

x	0	1	2	3	—1	—2	—3
y	1	1/2	1/5	1/10	1/2	1/5	1/10

The first line lists the values of $x = 0, 1, 2, 3, -1, -2, -3$. As a rule, integral values of x are more useful for calculations. The second line lists the corresponding values of y calculated by formula (1). Plotting the corresponding points on the plane (Fig. 2) and connecting them by a smooth curve, we obtain the graph (Fig. 3).

The rule of plotting a graph "point-by-point" is, as we have seen, extremely simple and requires no "science". Nevertheless, it may be for this very reason that blind adhering to this "point-by-point" rule may be fraught with serious errors.

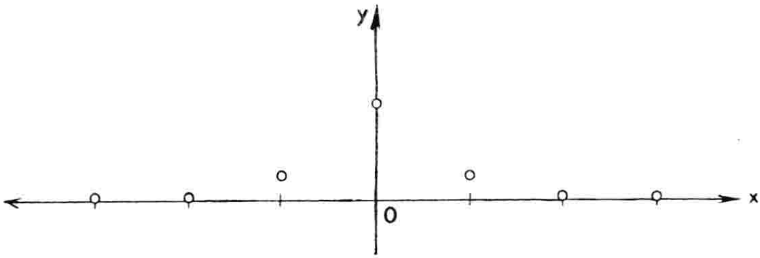

Fig. 4

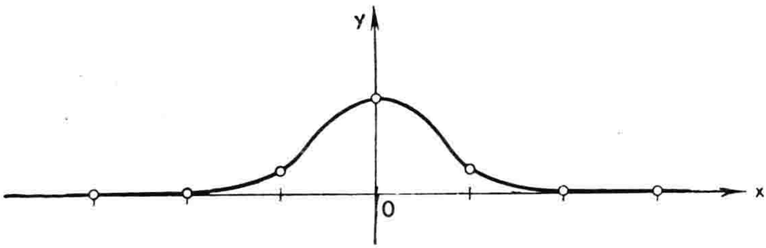

Fig. 5

Let us plot "point-by-point" the curve specified by the equation

$$y = \frac{1}{(3x^2 - 1)^2} \tag{2}$$

The table of x and y values corresponding to this equation is as follows

x	0	1	2	3	−1	−2	−3
y	1	1/4	1/121	1/676	1/4	1/121	1/676

The corresponding points on the plane are plotted in Fig. 4 which is very similar to Fig. 2. Connecting the plotted points with a smooth curve we obtain the graph (Fig. 5). It may seem that we could put the pen away and feel satisfied: the art of plotting graphs has been grasped! But for the sake of a test let us calculate y for some intermediate value of x, for example, for $x = 0.5$. After performing the calculations, we obtain an unexpected result: $y = 16$. This is in

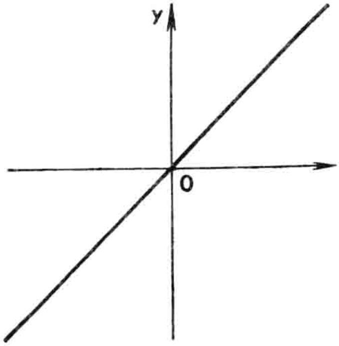

Fig. 6

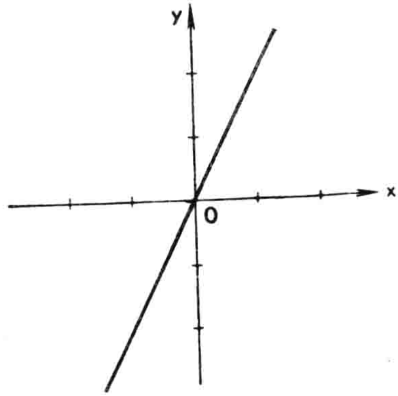

Fig. 7

striking disagreement with our graph. And we cannot guarantee that calculation of y for other intermediate values of x—and their number is infinite—will not produce even greater discrepancies. Unfortunately, the method of tracing the graph "point-by-point" proves rather unreliable.

We shall discuss below another method of graph plotting which is more reliable in the sense of safeguarding us from surprises similar to one we have encountered above. Using this method we shall be able to plot the correct graph of Eq. (2). In this method—let us term it, for instance, "by successive operations"—we have to perform directly in the graph all the operations which are written down in a given formula, viz. addition, subtraction, multiplication, division, etc.

Let us consider a few simplest examples. We shall plot a graph corresponding to the equation

$$y = x \tag{3}$$

This equation expresses that all points of the curve of interest have equal abscissas and ordinates. The locus of the points for which ordinates are equal to abscissas is the bisectrix of the angle between positive semi-axes and of the angle between negative semi-axes (Fig. 6).

The graph corresponding to the equation $y = kx$ with a coefficient k is obtained from the foregoing graph by multiplying each ordinate by the same number k. Let us set, for example, $k = 2$; each ordinate of the foregoing graph

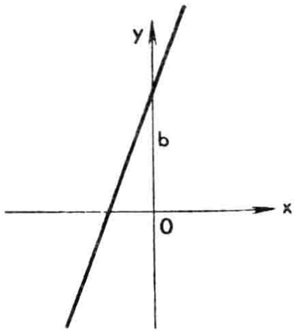

Fig. 8

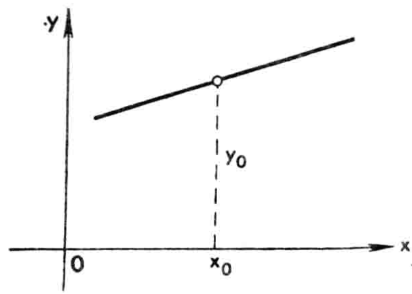

Fig. 9

must be doubled, so that as a result we obtain a straight line rising more steeply (Fig. 7). With each rightward step along the x-axis the line rises two steps up along the y-axis. By the way, this enables us to perform readily the plotting on squared paper. In the general case of the equation $y = kx$ with an arbitrary k a straight line is obtained. If $k > 0$, then with each rightward step the line will rise k steps up along the y-axis. If $k < 0$, the line will descend.

Consider the formula

$$y = kx + b \tag{4}$$

To plot the corresponding graph we have to add to each ordinate of the already known line the same number b. This will shift the straight line $y = kx$ as a whole upward in the plane by b units (for $b > 0$; if $b < 0$ the original curve will naturally be shifted downward). As a result we shall obtain a straight line parallel to the original one;it does not pass any more through the origin of coordinates but cuts on the ordinate axis the segment b (Fig. 8).

The number k is called the *slope* of a straight line $y = kx + b$; we already mentioned that this number k shows by what number of steps the straight line moves upward per each rightward step. In other words, k is the tangent of the angle between the direction of the x-axis and the straight line $y = kx + b$.

The equation

$$y = k(x - x_0) + y_0 \tag{4'}$$

corresponds to the straight line with the slope k; it passes through the point (x_0, y_0) (Fig. 9), since setting $x = x_0$ gives $y = y_0$.

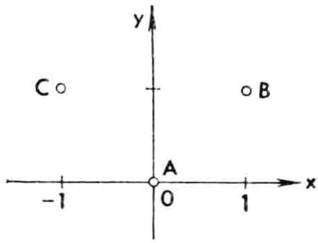

Fig. 10

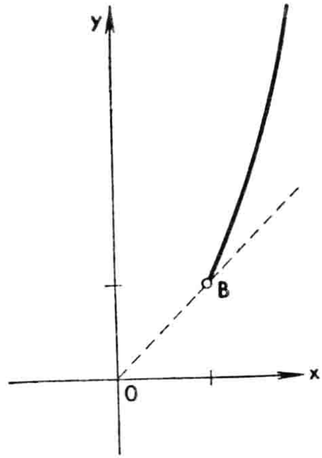

Fig. 11

Thus, the graph of any first-degree polynomial in x is a straight line which is plotted according to the aforesaid rules. Let us pass over to the second-degree polynomials.

Consider the formula

$$y = x^2 \tag{5}$$

It can be presented in the form

$$y = y_1^2, \text{ where } y_1 = x$$

In other words, the required graph will be obtained if each ordinate of the already known line $y = x$ is squared. Let us find out what this should produce.

Since $0^2 = 0$, $1^2 = 1$, $(-1)^2 = 1$, we obtain three reference points A, B, C (Fig. 10). If $x > 1$, then $x^2 > x$; therefore to the right of the point B the curve will be above the bisectrix of the quadrant angle (Fig. 11). If $0 < x < 1$, then $0 < x^2 < x$; therefore the curve between the points A and B will be under the bisectrix. Moreover, we state that, as it approaches the point A, the curve will enter an angle bounded above by the line $y = kx$ (however small k) and below by the x-axis; indeed, the inequality $x^2 < kx$ is satisfied for all $x < k$. This fact means that the sought-for curve is *tangent* to the abscissa axis at the point O (Fig. 12). Let us move now leftward along the x-axis from the point O. We know that the numbers $-a$ and $+a$ when squared give

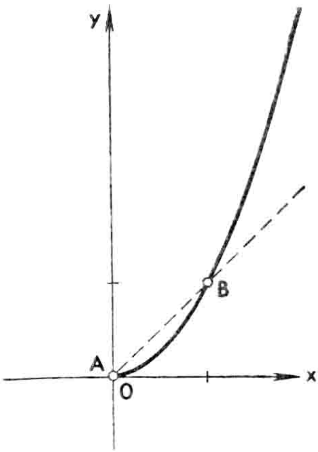

Fig. 12

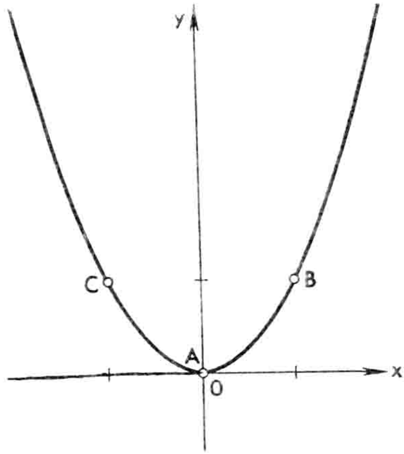

Fig. 13

the same result a^2. The ordinate of our curve will therefore be the same both for $x = +a$ and for $x = -a$. In geometrical terms this means that the graph of the curve in the left-hand semi-plane can be obtained by reflection relative to the ordinate axis of the curve already plotted in the right-hand semi-plane. We obtain the curve which is called the *parabola* (Fig. 13)

Now, following the same procedure, we sketch a more complicated curve

$$y = ax^2 \qquad (6)$$

and a still more complicated one

$$y = ax^2 + b \qquad (7)$$

The first of these curves is obtained by multiplying all ordinates of parabola (5)—we shall refer to it as a *reference parabola*—by a number a.

If $a > 1$ the curve will be similar to (5) but will rise more steeply (Fig. 14).

If $0 < a < 1$ the curve will be less steep (Fig. 15), and when $a < 0$ its branches will turn downward (Fig. 16). Curve (7) will be obtained from curve (6) by shifting it upward by a segment b if $b > 0$ (Fig. 17). If $b < 0$, we have to shift the curve downward (Fig. 18). All these curves are also called parabolas.

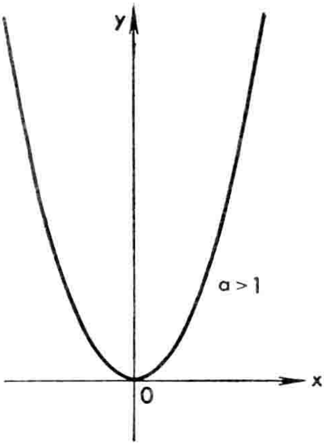

Fig. 14

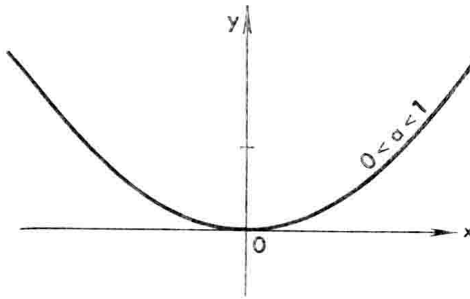

Fig. 15

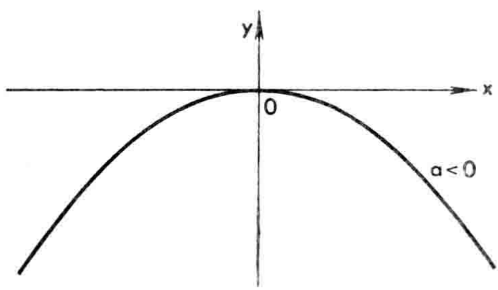

Fig. 16

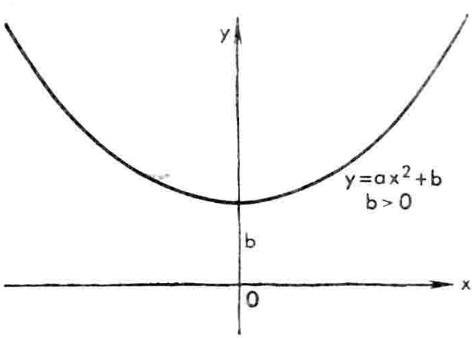

Fig. 17

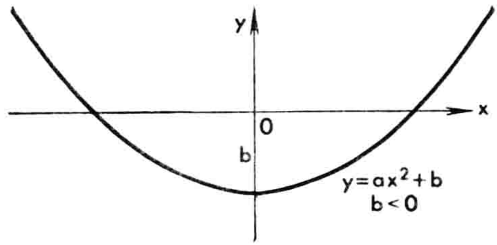

Fig. 18

Consider now a more complicated example of plotting graphs by means of multiplication. Let the problem be to plot the graph of the equation

$$y = x(x-1)(x-2)(x-3) \tag{8}$$

Here we have the product of four multipliers. Let us plot each of them separately: all of them are straight lines parallel to the bisectrix of the quadrant angle and cutting the segments on the ordinate axis (see Fig. 19)

$$0, -1, -2, -3$$

At the points 0, 1, 2, 3 of the x-axis the sought-for curve will have the ordinate 0 since the product is equal to zero if at least one of the factors is equal to zero. At other points the product will differ from zero and its sign can easily be found by considering the signs of the co-factors. Thus, all factors are positive to the right of the point 3; hence, the

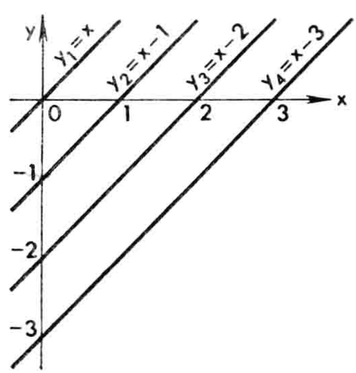

Fig. 19