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Stochastic Differential Systems

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PREFACE

This volume contains the major part of contributions to the 4th Bad Honnef Conference on Stochastic Differential Systems held at Bad Honnef, West Germany, June 20–24, 1988.Following the tradition of the preceding Bad Honnef Conferences, the meeting was intended to highlight recent advances in the areas of stochastic control and filter theory as well as stochastic analysis.

As sort of thematic "domains of attraction", special emphasis has been given to two rather active fields of current research: the use of adaptive methods in stochastic systems analysis and the theory of random fields. In view of the overwhelming flood of information accumulated in these two areas in recent years, the most that could be hoped for at this conference was to offer a glimpse of the status of research and to inspire interest and discussion in these fields. Several survey lectures were intended to provide some introduction to the more mature parts of the theory for those less acquainted with the subject, complemented by contributions that should give some taste of the diversifying issues of current research both in theory and practice. We leave it to the reader to judge how well this goal has been achieved.

It is a privilege of the organizing committee to express its gratitude to the Deutsche Forschungsgemeinschaft, whose generous support made this conference possible. We are also indebted to the members of the International Advisory Committee; assisting us with many valuable suggestions concerning program and speakers they have a substantial share in the success of the conference. Last, but not least, our thanks go to the staff of the Elly–Hölterhoff–Stift for their kind hospitality as well as to G. Nöldeke and A. Schütt for their skilfull job in data processing for the conference and in preparing this volume.

Bonn, February 1989

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CONTENTS

Some Results on Newton Equation with an Additional Stochastic Force

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Abstract

In this report, based on joint work with E. Zehnder, we discuss stochastic perturbations of classical Hamiltonian systems by a white noise force. We give existence and uniqueness results for the solutions of the equation of motion, allowing for forces growing stronger than linearly at infinity. We prove that Lebesgue measure in phase space is a σ-finite invariant measure. Moreover we give a Girsanov formula relating the solutions for a nonlinear force to those for a linear force.

1. Introduction

The study of stochastic perturbations of classical dynamical systems has been developed in recent years, however, the case of stochastic perturbations of classical Hamiltonian systems has been much less investigated on a mathematical basis, despite its great interest in applications (celestial mechanics, vibrations in mechanical systems, wave propagation in solid state physics ...). One reason for this is the fact that the structure of the classical flows themselves is much more complicated. Orbits of very different long time behaviour, in general cannot be separated in finite time intervals, stable and unstable behaviour being mixed, see e.g. [Ar], [Mo], and [Mo-Ze]. The nature of the orbits depends on the dimension of the system. For degrees of freedom exceeding three the behaviour can vary between periodic motion, and Arnold diffusion. Often the difficult mathematically rigorous investigation of the longtime behaviour, has been replaced by heuristic numerical approaches, see e.g. [Li-Lie].

In case the perturbation is stochastic, hence typically non smooth, and under the restriction of one degree of freedom, Potter [Po] (see also McKean [McK]) analysed nonlinear oscillators perturbed by a white noise force, described by the equations

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= K(x) + \dot{w} \end{aligned} \tag{1.1}$$

where K is the deterministic force, $\cdot$ means time derivative, $\dot{w}$ is white noise (the derivative of Brownian motion $w(t)$). Under assumptions on the force $K(x) = -V'(x)$, $V \in C^1(\mathbb{R})$ being attracting towards the origin, i.e. $x \cdot K(x) \leq 0$, Potter proved the existence of global solutions and results about recurrence and the invariance of Lebesgue measure $dxdv$ under the flow given in (1.1). Some of these results have been recently extended by Markus and Weerasinghe [MaW] who also studied winding numbers associated with the solution process (x, v) around the origin, see also [AGQ].

Existence and uniqueness for solutions of higher (but finite) dimensional second order Ito equations, as the systems of the type given in (1.1) have been called by Borchers [Bo], have furthermore been deduced by Goldstein [Go] for systems with globally Lipschitz continuous force K, and Narita [Na] in case there exists a function, decreasing along the paths analogously to a Ljapunov function in the deterministic theory.

The present paper is based on joint work with E. Zehnder [AHZ], to which we refer for more details and further discussions. We shall study equations of the form (1.1) in the case where x, and v run in $\mathbb{R}^d$. In Section 2 we establish existence and uniqueness results for strong solutions of the equations, under assumptions on K which are of the type $K(x) = -\nabla V(x)$ for some $V \in C^1(\mathbb{R}^d)$, with either a condition of the form $V(x)$ quadratic or such that $x \cdot K(x) \leq 0$ for $|x|$ sufficiently large. Then the solution process possesses the Markov property and continuous sample paths, furthermore it depends continuously on the initial conditions.

In Section 3 we compare the solutions of the nonlinear system (1.1) with the ones of a corresponding linear system given by

$$\begin{aligned} dx &= vdt \\ dv &= -\gamma x dt + dw \end{aligned} \tag{1.2}$$

(w as above, and γ a constant $d \times d$-matrix with positive eigenvalues). This is done by establishing a Cameron - Martin - Maruyama - Girsanov type formula for the Radon Nikodym derivative of the probability measures. This result can be applied to prove some properties which hold with probability one for the nonlinear system by exploiting their validity for the associated linear system (see [AHZ],[H]).

In Section 4 we exhibit some features of the behaviour of the solution process of the nonlinear system for large times. In particular, we give estimates for the energy functional for the process. We introduce the generator of the diffusion, solving (1.1), and ensure its hypoellipticity (in the sense that the occurring vector fields span the tangent space to phase space). By a Hörmander's type theorem we demonstrate absolute continuity of the transition probability w.r. to Lebesgue measure without further restrictions but continuity of the coefficient functions. Moreover we show that Lebesgue measure is a σ-finite invariant measure.

2. Existence and Uniqueness of the Solution

We consider a Hamiltonian System with corresponding Newton equation

$$\frac{d}{dt}x(t) = v(t), \quad \frac{d}{dt}v(t) = K(x) \tag{2.1}$$

where $t \in \mathbb{R}_+$ is time, $x(t)$ is position in $\mathbb{R}^d$ at time t, $v(t)$ is velocity at time t, and $K(x)$ is the deterministic force. The initial conditions $x(0) = x_0, \quad v(0) = v_0, \quad (x_0, v_0) \in \mathbb{R}^{2d}$ are given. Adding a white noise force $\dot{w}$, we arrive at a system of stochastic differential equations in the phase space random variable $Y \equiv (y(t) \in \mathbb{R}^{2d} | y(t) = (x(t), v(t)),\ t \geq 0)$ of the form

$$dy(t) = \beta(y(t))dt + \sigma d\tilde{w}_t, \qquad y(0) = y_0 = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \tag{2.2}$$

with

$$\beta(y(t)) \equiv \begin{pmatrix} v(t) \\ K(x(t)) \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{w} = \begin{pmatrix} b_t \\ w_t \end{pmatrix},$$

where $((b_t, w_t), \mathcal{A}_t \otimes \mathcal{F}_t,\ t \geq 0)$ is a Brownian motion in $\mathbb{R}^{2d}$ issued from 0 at time 0, with independent families of σ-algebras $(\mathcal{A}_t)$, and $(\mathcal{F}_t)$.

Theorem 2.1

Each of the following conditions is sufficient for the existence of pathwise unique solutions of (2.2) for all $t \in I\!R_+$:
Let $\alpha, \beta \in I\!R^d$, and $R \geq 0$ then there exist constants $C_1, C_2 \geq 0$, where C_1 depends on R, such that

a) 1) $|K(\alpha) - K(\beta)| \leq C_1 |\alpha - \beta| \qquad \forall |\alpha|, |\beta| < R$
   2) $|K(\alpha)| \leq C_2(1 + |\alpha|) \qquad \forall \alpha \in I\!R^d$.

b) 1) $\alpha \mapsto K(\alpha)$ is a locally Lipschitz continuous function. Moreover,
   2) For $d \geq 1$: $K(\alpha) = -\nabla V(\alpha)$ for some $V \in C^1(I\!R^d)$
   3) $(\alpha - x_0)K(\alpha) \leq 0 \qquad \forall \alpha \in I\!R^d, \quad$ for some $x_0 \in I\!R^d$.

Proof:

a) Statement $a)$ is proven by a stochastic version of the Picard-Lindelöf method of iteration, see e.g. [Mc K] (Cor 6.3.4).

b) For $d = 1$ the statement (b) is a special case of a result of Potter[Po] see e.g. [Mc K], [Na 1].

We give a proof valid for $d \geq 1$ which uses a process which adopts the part played by the Ljapunov function in the deterministic theory. Let us introduce the energy functional

$$W(y) = \frac{1}{2} |v|^2 + V(x) - V(x_0). \tag{2.3}$$

From condition $b)2)$, i.e. $(\alpha - x_0) \cdot K(\alpha) = -|\alpha - x_0| \frac{\partial V(\alpha)}{\partial |\alpha|}$, where $\frac{\partial V}{\partial |\alpha|}$ is the derivative of V along the direction $(\alpha - x_0)$, we conclude

$$V(\alpha) = V(x_0) - \int_0^{|\alpha - x_0|} \frac{1}{|\beta|} (\beta \cdot K(\beta + x_0)) \, d|\beta| \geq V(x_0) \tag{2.4}$$

where we also used assumption $b)3)$.
For the energy functional in (2.3) this implies

$$W(y) \geq \frac{1}{2} |v|^2, \tag{2.5}$$

and

$$|\sigma \cdot \nabla_y W(y)|^2 = |v|^2 \leq 2W(y).$$

Since β is locally Lipschitz continuous, according to [Na 2], and [Ik-Wa], there exist local solutions Y.
Let us introduce stopping times

$$\sigma_n = \inf\{ t \geq 0 | \, y(t) \geq n\} \wedge n$$

of the process $Y \equiv (y(t), t \geq 0)$, $t < \sigma_n$, and define the explosion time $e(y_0)$ of Y for given initial condition $y(0) = y_0$, by

$$e(y_0) = \sup_{n \in I\!N} \sigma_n \qquad I\!P \ a.e. \tag{2.6}$$

For $n \to \infty$ the local solutions converge a.e. to the solution of (2.2), cf [Na 3]. The existence of a global solution Y with initial condition $y(t_0) = y_0$ is equivalent to an infinite explosion time $e(y_0)$, i.e.

$$I\!P(N_{y_0}) = I\!P(\{e(y_0) < \infty\}) = 1 \tag{2.7}$$

with

$$N_{y_0} = \{e(y_0) < \infty, \text{ and } \lim_{t \nearrow e(y_0)} |y(t)| = +\infty\}). \tag{2.8}$$

At times before any explosion can possibly occur we can reexpress the energy functional W (2.3), applying Ito's formula to its differential, by

$$W(y(t)) = W(y_0) + \int_0^t v(s) \cdot dw(s) + d\frac{t}{2}. \tag{2.9}$$

Let us set

$$\tau(t) \equiv \int_0^t |v(s)|^2 \, ds \tag{2.10}$$

and

$$a(t) \equiv \int_0^{\tau^{-1}} v(s) \cdot dw(s). \tag{2.11}$$

Then $a(t)$, with filtration $\mathcal{F}_{\tau(t)}$ and with a clock running according to the time τ is a new Brownian motion. Under the assumption of Theorem 2.1b) a global solution of (2.2) is established due to (2.7) by the following

<u>Lemma 2.2</u>

Under the hypothesis of Theorem 2.1b) we have

$$\mathbb{P}(e(y_0) = +\infty) = 1.$$

<u>Proof:</u>

The proof of the higher dimensional statement can be reduced to the one for the one dimensional case in [Po] with y being replaced by $|y|$. The proof is by construction, distinguishing the cases $\tau(e(y_0)) < \infty$ and $\tau(e(y_0)) = \infty$, and using the sample path properties of the Brownian Motion $a(\cdot)$. Thus the a.s. finiteness of $|y(t)|$, where $t_0 \leq t \leq e(y_0)$, in deduced, which yields the contradiction. ■

We are left with the proof of uniqueness

a) This case is covered e.g. by [Fr].

b) This case follows from [Ik-Wa], (Theorem 3.1), since the coefficients of the equation (2.2) are in particular locally Lipschitz continuous. This yields uniqueness for $t \in [0, e(y_0)(w)]$. Since $e(y_0)(w) = \infty$ a.s. , by the first part of this theorem, pathwise uniqueness holds for all $t \geq 0$. ■

<u>Remark 2.3</u>

The statement of the theorem holds for $t \geq t_0$ with initial condition $y(t_0)$ given. Furthermore, the condition $b3)$ may be generalized to

b3′) There exists a constant $r > 0$ s.t. $(\alpha - x_0) \cdot K(\alpha) \leq 0$ for $|\alpha - x_0| \geq r$

In fact the result on pathwise uniqueness is left untouched, in case we are able to establish the existence of a global solution. Proceeding as in the proof by contradiction of Lemma 2.2, assume that with probability different from zero $e(y_0) < \infty$. Then for w s.t. $e(y_0)(w) < \infty$, and $\tau(e(y_0))(\omega) \leq \infty$, the part showing boundedness of the configuration variable x(t) for $t \in [0, e(y_0)]$ does not involve the force K, and therefore remains unchanged. Splitting the integral in the expression for $V(x(t))$ in (2.4), and inserting the solution of (2.2) we receive a nonnegative contribution,

$$\int_r^{|x(t)-x_0|} |\beta|^{-1} (\beta \cdot K(\beta + x_0)) d|\beta| \tag{2.12}$$

plus a term

$$C_K \equiv \int_0^{\tilde{\gamma}} |\beta|^{-1} (\beta \cdot K(\beta + x_0)) d|\beta| \tag{2.13}$$

with $\tilde{\gamma} \equiv |x(t) - x_0| \wedge r$ which is bounded for $t \in [0, c(y_0)]$, since K is a continuous function. We can estimate the norm of the momentum v, using the energy process, by

$$|v(t)|^2 \leq 2(W(y_0) + d\frac{t}{2} + C_K + a(\tau(t))) \tag{2.14}$$

This yields the boundedness of the process Y in case the random time τ stays finite. In case $\tau(c(o, y_0)) = \infty$ the force is not involved in proving boundedness of the phase space process Y. So the same argument as in the proof of Lemma 2.2 applies. ∎

Remark 2.4

Let us look at some analytical properties of the solution of (2.2). We observe that by setting $z \equiv v(t) - w(t)$ we achieve a C^1 - (in time) solution (x, z) of the system of stochastic differential equations

$$\begin{aligned} \dot{x} &= z(t) + w(t) \\ \dot{z} &= K(x(t)) \end{aligned} \tag{2.15}$$

which is equivalent to the system given in (2.2). Since the Brownian motion $(w(t))$ possesses Hölder continuous paths of index $< \frac{1}{2}$ we have local in time solutions to which the theory of systems of ordinary differential equations can be applied. Finally we easily find continuous dependence of the initial data for the solution of (2.2).

Theorem 2.5

Let $K(\alpha, \mu)$ be a continuous function of α, μ satisfying a local-Lipschitz condition in α, uniformly in μ, for μ a parameter in some open connected domain in $I\!R^k$ of the form $\{\mu \in I\!R^k | \, |\mu - \mu_0| < c\}$ for some $c > 0$, $\mu_0 \in I\!R^k$. Consider the unique solution (2.2) for $t \geq t_0$ in a bounded domain D of $I\!R^{2d}$ with initial condition $y_0 = y(t_0)$, and coefficient $K(\alpha, \mu)$. Let us call $y(t, \mu, t_0, y_0)$ the solution, then the mapping $\mu \longmapsto y(t, \mu, t_0, y_0)$ is a.s. continuous. We also have

$$\lim_{\mu \to \mu_0} y(t, \mu, t_0, y_0) = y(t, \mu_0, t_0, y_0)$$

uniformly over every compact (t, y)-set.
The mappings $y_0 \mapsto y(t, \mu, t_0, y_0)$, and $t_0 \mapsto y(t, \mu, t_0, y_0)$ are locally Lipschitz continuous.

Proof:

Since there exists a pathwise (unique) solution of (2.2) this follows from results on ordinary differential equations, see e.g. [Am] (Theorem II.8).

3. A Girsanov Formula

In this section we shall answer the question whether almost sure properties about the solution to non linear stochastic differential equations can be reduced to ones of some Gaussian process. More precisely, we shall investigate whether the probability measure associated with the solution $Y = (y(t) = (x(t), v(t)),\ t \geq 0)$, $y(t_0) = y_0$ of the stochastic differential equation (2.2), is absolutely continuous with respect to the probability measure associated with the process given by the stochastic differential equation

$$d\eta(t) = a(\eta)\, dt + \sigma d\tilde{w}_t \tag{3.0}$$

with $\eta = \binom{z}{u}$, $a(\eta) = \binom{u}{-\gamma z}$, where γ is a positive constant $2d \times 2d$ matrix, and vice versa. This amounts to deriving a (Cameron-Martin-Maruyama-) Girsanov formula relating the probability measures.

Lemma 3.1

Let K, V be as in Theorem 2.1 and let Y be the corresponding solution of (2.2). Then for W defined as in (2.3) we have

$$\text{i) } \mathbb{E}(W(y(t))) = \mathbb{E}(W(y_0)) + d\frac{t}{2}$$

$$\text{ii) } \mathbb{E}\left(W(y(t))^2\right) = \mathbb{E}\left(W(y_0)^2\right) + \frac{d}{2}\int_0^t \mathbb{E}(W(y(s)))\,ds + \frac{1}{2}\int_0^t \mathbb{E}\left(|v(s)|^2\right) ds$$

Proof:

Statement $i)$ follows from 2.9 by taking expectation and using that $\int_0^t v(s)\cdot dw(s)$ is a martingale with expectation zero.

ii) For any $F \in C^2(\mathbb{R})$, we have, using Ito formula successively, for all $t \geq 0$

$$F(W(y(t))) = F(W(y_0)) + \int_0^t F'(W(y(s)))\,v(s)\cdot dw(s) + \frac{1}{2}\int_0^t \left(dF'(W(y(s))) + |v(s)|^2 F''(W(y(s)))\right) ds$$

Inserting $F(\lambda) = \lambda^2, \in \mathbb{R}$, we get the equation $ii)$ of the lemma.

■

Lemma 3.2

Let K be as in Theorem 2.1 and let $x(t)$ be the space component of the solution $y(t) = (x(t), v(t))$ of (2.2). In the case of hypothesis 2.1b assume furthermore that for all $\alpha \in \mathbb{R}^n$ and some constant $C > 0$:

$$|K(\alpha)|^2 \leq C\left(|V(\alpha) - V(x_0)|^2 + 1\right).$$

Then the stochastic integral

$$\int_0^t K(x(s))\cdot dw(s)$$

defines a square integrable martingale w.r. to the filtration $(\mathcal{G}_t)_{t\geq 0} \equiv (\mathcal{A}\otimes\mathcal{F}_t)_{t\geq 0}$, which has zero expectation.

Proof:

On Wiener space Ω over $\mathbb{R}^{2d}$ we are given the stochastic integral

$$\int_0^t \beta(y(s))\cdot\sigma d\tilde{w}(s) = \int_0^t K(x(s))\cdot dw(s). \tag{3.1}$$

a) For the assumption of Theorem 2.1a), i.e. for a linear growth condition, the statement of the Lemma follows by Gronwall's theorem, see [Am] (see also e.g. [Ik-Wa],[Lip-S],[McK]).

b) Now we turn to the case of assumption 2.1b in Theorem 2.1. Using $V(x(t)) - V(x_0) \geq 0$, and the definition of the energy function W (2.3), we get according to Lemma 3.1

$$\mathbb{E}\left(|V(x(t)) - V(x_0)|^2\right) \leq \mathbb{E}\left(\frac{|(v_0)|^2}{2}\right) + \frac{d+1}{2}\int_0^t \mathbb{E}(W(y(s)))\,ds \tag{3.3}$$

From the growth condition on K and (3.3) we deduce

$$\mathbb{E}\left(|K(x(t))|^2\right) \le C\left[C_1 + \frac{d+1}{2}\int_0^t \mathbb{E}\left(W(y(s))\right)ds\right] \tag{3.4}$$

with $C_1 \equiv (\mathbb{E}\left(\frac{|v_0|^2}{2}\right)$. Inserting in (3.4) the expectation $\mathbb{E}(W(y(s)))$ expressed in Lemma 3.1 we find

$$\mathbb{E}\left(\int_0^t |K(x(s))|^2 ds\right) \le p_1 t + p_2 t^2 + p_3 t^3 \equiv P_3(t) \tag{3.5}$$

where

$$p_1 = C\mathbb{E}(W(y_0)), \quad p_2 = C\frac{d+1}{4}\mathbb{E}(W(y_0)), \quad p_3 = C\frac{d(d+1)}{24} \quad .$$

As in part i) the statement of the lemma follows since by (3.1) combined with the estimate in (3.5), $\int_0^t K(x(s)) \cdot dw_s$ represents the stochastic integral of a nonanticipative function (in $L^2(\Omega, \mathbb{P})$),see c.g [Ik-Wa] (p. 48) or [Lip] (p. 97). In particular we have a martingale, with zero expectation (as seen from its form).

■

In order to prove the existence of a density of the probability on path space for the nonlinear solution w.r.t. the one corresponding to the linear system, we need the following special case of [Do].

Lemma 3.3

Under the assumptions of Lemma 3.2 the following estimate holds:

$$\mathbb{E}\left(\int_0^t |\gamma x(t) + K(x(t))|^2\right) dt < \infty,$$

where γ is a $d \times d$ matrix.

Proof:

a) We treat separately the cases a), b) corresponding to the assumptions 2.1a, 2.1b in Theorem 2.1. By Schwarz's inequality and the growth assumption on K

$$|\gamma x(t) + K(x(t))|^2 \le 2\left(2C_2^2 + (2C_2^2 + C_\gamma)|x(t)|^2\right) \tag{3.7}$$

with C_γ the norm of γ.

Replacing $x(t) = x_0 + \int_0^t v(s)ds$, it follows again by Schwarz's inequality

$$|x(t)|^2 \le 2\left(|x_o|^2 + t\int_0^t |v(s)|^2 ds\right). \tag{3.8}$$

Using $v(s) = v_0 + \int_0^s K(x(u))\,du + \int_0^s dw_u$ and the independence of v_0 of the rest, we get applying Ito's formula, taking into account a2) and the fact that the martingale in Lemma 3.2 has zero expectation:

$$\mathbb{E}\left(|v(s)|^2\right) \le \mathbb{E}\left(|\int_0^s K(x(u))\,du|^2\right) + s^2 + \mathbb{E}\left(|v_0|^2\right). \tag{3.9}$$

Combining (3.8) with (3.9) and applying Schwarz's inequality we arrive at the following estimate for the r.h.s. of (3.7)

$$\begin{aligned}\mathbb{E}\left(|x(t)|^2\right) &\le 2\left(\mathbb{E}\left(|x_0|^2\right) + 2\mathbb{E}\left(|v_0|^2\right)t^2 + C_2^2 t^3\,\mathbb{E}\left(\int_0^t 2(1+|x(u)|^2)du\right) + \frac{1}{3}t^4\right)\\ &\le C_1(t) + C_2(t)\mathbb{E}\left(\int_0^t |x(u)|^2 du\right)\end{aligned}$$

with $C_1(t) \equiv 2\mathbb{E}\left(|x_0|^2\right) + 2\mathbb{E}\left(|v_0|^2\right) t^2 + \frac{1}{3}t^4 + 2C_2^2 t^4$, and $C_2(t) \equiv 2C_2^2 t$

Using Gronwall-Lemma (see e.g. [Am]) in the variable $\mathbb{E}(|x(t)|^2)$ we arrive at

$$\mathbb{E}\left(|x(t)|^2\right) \leq C_1(t) + C_3(t) \tag{3.10}$$

with $C_3(t)$ a smooth function of t, bounded for $t \geq 0$.

From (3.9) and (3.10) we deduce the estimate required in statement a) of the Lemma, namely for $0 \leq t < \infty$:

$$\mathbb{E}\left(\int_0^t |\gamma x(s) + K(x(s))|^2 ds\right) \leq \tilde{C}(t) < \infty \tag{3.11}$$

with $\tilde{C}(t) \equiv 4C_2^2 t + 2(2C_2^2 + C_\gamma)\int_0^t [C_1(s) + C_3(s)]\, ds$.

b) Let K fulfill the conditions of Theorem 2.1b). Analogously to the first step in (3.7), we split the squared norm by Schwarz's inequality. Integrating, and taking expectation in a second and third step yields

$$\mathbb{E}\left(\int_0^t |\gamma x(s) + K(x(s))|^2 ds\right) \leq 2\int_0^t \left(\mathbb{E}\left(|\gamma x(s)|^2 + \mathbb{E}\left(|K(x(s))|^2\right)\right)\right) ds. \tag{3.12}$$

We have to estimate the linear and nonlinear term separately. Since the explosion time is a.s. infinite, combining (2.8) and (3.8) we get

$$|\mathbb{E}\left(x_0 \int_0^t v(s) ds\right)| \leq \int_0^t \left(\mathbb{E}|x_0 v(s)|\right) ds \leq \left(\mathbb{E}|x_0|^2\right)^{\frac{1}{2}} \int_0^t \left(\mathbb{E}\left(W(s)\right)\right)^{\frac{1}{2}} ds = P(t) \tag{3.13}$$

where $P(t)$ is a polynomial of degree 3 in $t^{\frac{1}{2}}$.

Then for all t

$$\mathbb{E}\int_0^t |x(s)|^2 ds \leq C_\gamma t \mathbb{E}\left(|x_0|^2\right) + 2\int_0^t \left[s\int_0^s \left(\mathbb{E}\left(W(y_0)\right) + \frac{d}{2}u\right) du + 2P(s)\right] ds,$$

which is a polynomial $P_4(t)$ of degree 4 in t. Collecting the estimates given in (3.4) and (3.13) we find

$$\mathbb{E}\left(\int_0^t |\gamma x(s) + K(x(s))|^2 ds\right) \leq 2\int_0^t \left(P_4(s) + P_3(s)\right) ds < \infty. \tag{3.14}$$

This finishes the proof.

■

Remark 3.4

It looks tempting to try to use the estimate in Lemma 3.3, and the assumptions on K to check Novikov's condition $\mathbb{E}(exp(\frac{1}{2}\int_0^t |\gamma x(s) + K(x(s))|^2)) < \infty$, sufficient for Girsanov's theorem below, to hold. In reality, this is not possible in our situation. However, it is possible to check that other types of sufficient (and necessary) conditions for Girsanov's theorem [Lip-S] hold under the conditions of Theorem 2.1, see [AHZ],[H] for a detailed proof.

Theorem 3.5

Let Y with initial data y_0 be the global solution of the nonlinear stochastic differential equation (2.2) with K satisfying the assumptions of the existence and uniqueness Theorem 2.1, and the growth condition of Lemma 3.2.