LECTURES ON MINIMAL SURFACES

Volume 1

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Introduction, fundamentals, geometry and basic boundary value problems

JOHANNES LECTURES
University of Minnesota, Minneapolis

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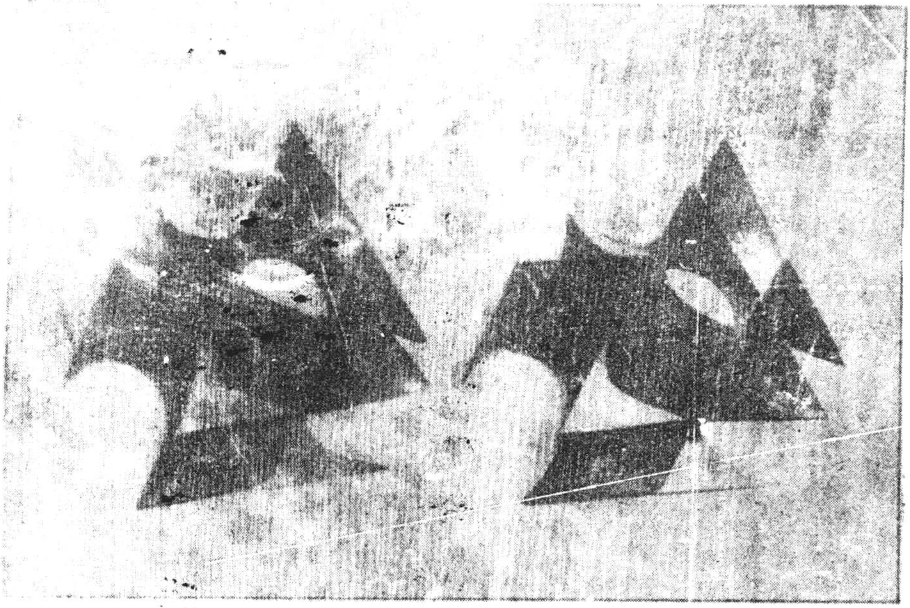
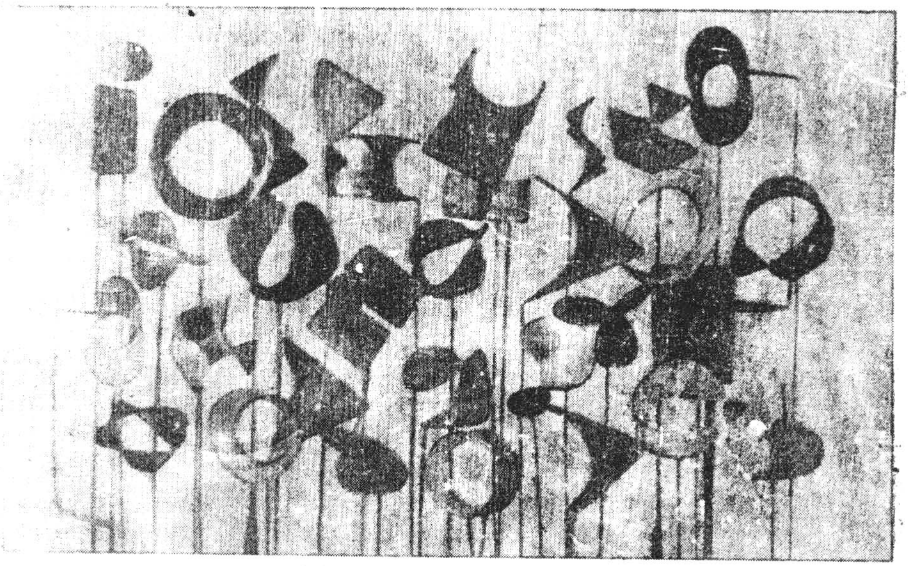


Two parts of Schwarz's triply periodic minimal surface; see §279*



Garden of minimal surfaces*

** Models (and photographs) by the author and C. D. Nitsche*

Dedicated to the memory of my parents
Ludwig Johannes and Irma Nitsche

PREFACE TO THE ENGLISH EDITION

δìς ἐς τὸν αὐτὸν ποταμὸν οὐκ ἄν ἐμβαίης.
You can't step twice into the same river.
(Heraclitus, according to Plato)

This is not the same book. It is, and it is not. I must explain. The original German text which was completed in 1972 and appeared in print in 1975 had been the fruit of extended efforts by a slow author, ever since Professor Heinz Hopf had invited me fifteen years earlier to contribute an exposition of the theory of minimal surfaces for the Yellow *Grundlehren* Series of Springer. Hopf, with whom I enjoyed close contacts, had been intrigued by my proof [2] of Bernstein's theorem which had occurred to me during my visit from Berlin to Stanford University, 1955–6. This was the year when he presented there his inimitable lectures on Differential Geometry in the Large – now volume 1000 in the Springer *Lecture Notes* – which I attended. Also in residence at the time was Erhard Heinz, an old acquaintance from our student days in Göttingen, whose surprising earlier inequality [1] for the Gaussian curvature of a minimal surface $z = z(x, y)$, the first such estimate of a geometric quantity, had given a fresh impetus to the subject which by then, owing to its elegance, its depth and its implications, had again caught the attention of geometers and analysts in many countries. The venture into higher dimensions, regarding Heinz's inequality as well as many other questions, and especially the subsequent discovery [1] of Bombieri–DeGiorgi–Giusti which opened unforeseen vistas onto new territories, not fully explored even today, lay still in the future while Osserman's corresponding investigations of parametric minimal surfaces, to be concluded in a most satisfactory sense by H. Fujimoto and Mo–Osserman in 1987–8, were brand new. What Hopf had in mind was a slender volume, possibly taking Bernstein's ideas as point of departure, to embrace the pearls of minimal surface theory, and comparable in pithiness and gist to his own matchless *Lecture Notes* or to Radó's slim *Ergebnisbericht*. To be sure, thirty years earlier he had coauthored a voluminous tome himself, and I remember several conversations in which he questioned (himself) whether the work on it had been worth the efforts. Regarding the *Topologie*, there is of course a clear answer. In any case, this was a hard road to travel for the novice; my prior research lay in the areas of partial differential equations, differential geometry, numerical analysis and fluid mechanics. But the assignment to expound what I came to think might be called the *Story of Minimal Surfaces* was a challenge indeed, and I set out to make plans for this slender volume, to learn, collect material, delve into the historical sources and get in touch with the contemporary mathematicians. I was fortunate to have had personal relations with many of the great older contributors to minimal surface theory, now passed away, who had kept the fires alive by their brilliant contributions, among them W. Blaschke, R. Courant, J. Douglas, E. Hopf, H. Lewy, E. J. McShane, C. B. Morrey, T. Radó, G. Stampacchia, I. N. Vekua, as I have now contacts with many of the powerful young mathematicians – I think I have motivated some of them: continuity in the transmission of our mathematical heritage at work. Any systematic account leading from the historical origins to the present state of the subject threatened to break the confines of sparing pages. I realized soon that for many of the principals, due to their technical strengths and personal preferences, or to their ignorance, the theme appeared to be a rather narrow one. In truth, however, belying its apparent rank as nothing but one of many subfields of differential geometry or by its identification through the four sparse subsections 49F10, 53A10, 53C42 and 58E12 in the Mathematical Reviews Subject Classification Index, the theory of minimal surfaces, and, more generally but intimately related, the theory of surfaces of constant or otherwise prescribed mean curvature, have a life entirely of their own endowed with overwhelming riches and with often surprising applications as primary tools in numerous important areas of mathematics as well as in the other sciences. Various preparations and attempts at 'streamlining' – graduate courses devoted to certain chapters of minimal surface theory and the calculus of variations which I gave repeatedly at several institutions, the survey lecture [18] presented to the American Mathematical Society in 1964 – helped but did in the end not prevent the *Vorlesungen* running into 775 pages, with a bibliography containing in excess of 1230 precise entries (all of which, with a mere handful of exceptions, had been physically before me for scrutiny!); this even though I had, after long inner struggles, made up my mind to restrict the presentation to only a selection of clearly circumscribed aspects of the subject. I dare believe that this selection allowed the inclusion of the most appealing and important parts of the theory. Nevertheless, the list of topics not included was thus long; but, of course, a book should not be judged by what it does not contain. It had been my chief aim to prepare a complete and up-to-date exposition designed for the scholar and specialist, an exposition, however, which at the same time should be didactically crafted so as to be of benefit to the graduate student, indeed, to all interested scientists using geometry and analysis in their work – physicists, engineers, biologists – who might well derive gain from their acquaintance with the wealth of our subject and the manifold fundamental connections emanating from it. All in all, I thought the *Vorlesungen* might also demonstrate the erroneousness of the captious Blaschke–Reichardt quote of 1960 which I had unearthed and interjected as a stimulus in the introduction of the survey lecture mentioned above (my own translation): 'If one brings into comparison the first and the last of these works (i.e. the surveys by E Beltrami [3] and T. Radó [I]), one can confront the stormy youth of a geometric question with its tired old age.'

The plan more than ten years ago of Cambridge University Press to bring out a translation of the German *Vorlesungen* and the contract signed between the publishing houses, while encouraging and gratifying to the author, did not envisage his participation, nor did it require his signature. This is not to say that the senior editors David Tranah and the late Walter Kaufmann-Bühler, a grandson of Arthur Schoenflies who is represented by two entries in the bibliography, did not visit the author at regular intervals, always pleasant occasions, to converse, inform and consult. The translation project turned out to be a major undertaking prone to protraction. Long sentences lovingly constructed in the German original, extensive use of the subjunctive mood, literate and welcome in the German language but considered archaic by some, etc., not to mention the bulk of the technical details and the vast bibliography, led to occasional confusion and made for a difficult task. Matters were not simplified by the fact that the last decade has been one of extraordinary research activities on all fronts of minimal surface theory, producing striking results which demanded recognition or consideration. Many of the research problems formulated in section IX.2 of the *Vorlesungen* had attracted wide attention over the years and had been solved; new questions had arisen. Various topics, to mention here the concept of Lebesgue area, McShane's theory of 'excrescences' and the momentous theories of Jesse Douglas predating the use of Teichmüller theory, have gone out of vogue – there may well be revivals; new approaches, notably those of geometric measure theory, extensions of functional analysis and higher-dimensional differential geometry, have made their mark and, as it must be, are finding authoritative expositions in independent treatises. Naturally, all this impinged on the translation draft made available to the publisher; it also led eventually to the author's involvement, an involvement which became intensive during the last one and a half years. A fresh start was deemed undesirable by the publisher; rather, the draft was to be the basis for modifications and extensions while, of course, the essence of the *Vorlesungen*, in spirit, scope and expository style, had to be preserved. These were many side conditions, none of my choosing, but I hope that the current text comes at least close to achieving the set goal and that the original German version does survive in it with far more than its mere skeleton. During all the trials, through the arduous proofreading efforts, Dr Tranah and his associates have remained cooperative and patient. Particular praise must go to the printers and draftsmen who converted an often deficient original into its appealing final form, as well as to those who performed valuable yeoman's services (assistance in compiling the index, etc.). On my side, the support of my wife and children, in multifarious ways, has been an immense help for which I am grateful. As a pastime, my sons have developed their own clever computer graphics program which now lets me see the most bizarre surfaces of interest on the screen of my own modest home computer.

It was decided to break the book into two parts. Volume One covering Chapters I–V of the German original is presented here to the reader. Also sections 691–838 of Chapter IX are germane to the material comprising this volume. They will remain in their place and will be included in the forthcoming Volume Two. It should be mentioned that the German text had been designed in hierarchical order; accordingly, the first volume contains a compilation of auxiliary material preparatory for later applications and may for this reason appear to be a bit top-heavy. This will be evened up in due course. There are numerous additions of varying lengths fused into the text at appropriate locations. Appendices A1–A8 have been added to focus on topics of current interest and to account for latest advances. Many of the historical remarks in the original text have been expanded and supplemented. The bibliography has swollen to 1595 items; more will have to be added in the future. In all citations my aim has been to turn consistently also to the primary sources. I am aware that this is somewhat contrary to a widespread, though hardly commendable, current practice where authors seldom go beyond tertiary sources or, for reasons best known to themselves, mainly quote work by the members of their respective schools. Concerning the figures, see the remarks below. One section (§ 18) of the German original describing numerical methods has been excised for fuller treatment later on. Numerical analysis has come a long way; in the era of super-computers, it certainly does far better than Douglas's statement of 1928 implies: 'The spirit of this paper being entirely numerical, we do not concern ourselves with theoretical questions of convergence, which are besides too difficult for us to deal with.' What is less clear, even today, is the question whether the new technological means will prove themselves also in regard to the mathematical questions with which the main body of our work will have to struggle. As for the forward references to sections 475–968, the reader is for the time being asked to look to the second half of the *Vorlesungen*. To account for the research progress of recent years, Volume Two will require substantial reworking and infusion of new material. In fact, the proper impartation of the *Vorlesungen*, with all the extensions necessary to retain their overall character, calls for an additional Volume Three to join its brethren. This volume is now in preparation.

As indicated, one of the vexing aspects of the revision project, and one continuously frustrating the author's attempts at creating a conclusive text, has been the task of keeping abreast of the accelerating developments. Every year new and often major results come into print; there are numerous preprints and even more numerous 'personal communications', mostly hard to verify, if at all, occasionally also unsustainable, not to mention the plethora of conferences. Nothing much has changed in this regard over the years. One would, for instance, like to know more about the circumstances under which earlier mathematicians – Enneper, Riemann, Scherk, Schwarz and others, as well as many of the supporting figures – came to turn their attention at specific times to specific questions in minimal surface theory which seemed often to have been quite removed from their prior interests. Personal communications, with all that the term implies, may have been at work then, too. There must be a sorting of all the material, new or, as it sometimes turns out, not really new, to identify the precious parts and to categorize them according to their importance and suitability for a book of generous but restricted length. This activity is rather like the prospector's search for diamonds: tons of Kimberlite rocks must be fragmented and combed in the quest for the reward. Without doubt, the striking accomplishments of powerful contemporary mathematicians, complementing, generalizing or superseding existing theorems, are on a par with the results of earlier periods. The level of technical complexity to which these accomplishments have risen is noteworthy by itself, for this very reason often virtually excluding them from a self-contained exposition.

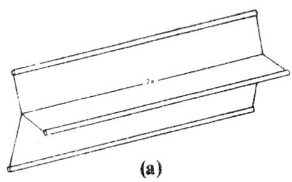
The preface to the German edition expresses the goals and contents of the *Vorlesungen* exhaustively. It fully covers the present text also, even to the point that the one problem singled out as particularly challenging has until now resisted solution so that its repetition at this point seems justified: 'To prove that a reasonable (analytic, polygonal, . . .) Jordan curve cannot bound infinitely many solution surfaces of Plateau's problem, and to estimate the number of such surfaces by the geometric properties of its boundary.' Nevertheless, at the occasion of our embarkment on a larger journey, also to set the stage for the new edition of this work, a few further expounding remarks will be apposite. It is my conclusion that historical references, here of course restricted to the general theme of minimal surfaces, are called for, today more than ever. They provide fascinating glimpses and they alone lend proper perspective to our subject. It is desirable to gauge and chronicle with diligence contemporary developments as we witness them unfolding – in due time some of them will become part of history as well – but equally important not to lose our knowledge of the events of earlier eras. A prime case in point is the birth of our subject in the eighteenth century which bears telling here. This century, with its fertile environment for mathematical thought, brought forth the creation of the calculus of variations at the hands of giants and with it also the dawn of minimal surface theory, first in the form of special examples but soon to be imbued with geometrical content by Meusnier and coming of age. The first such example can be found in Euler's *Methodus inveniendi*, the next and more influential one in Lagrange's *Essai d'une nouvelle méthode*. Euler had been lured from St Petersburg to Berlin in 1741 by Frederick II of Prussia who, competing with the pre-eminent models in other countries, planned to revive Leibniz's old *Societaet* and found the Berlin *Akademie*. It was Frederick's ambition, initially despite his waging of the Silesian Wars (1741–3, 1744–5) and later the Seven Years' War (1756–63), to attract to his capital the princes of science, in particular the great mathematicians, representatives of the new species *Apollon newtonianis*, even though he had continuing doubts about the value of higher mathematics, and discussions concerning use or uselessness of geometry took place during many of his dinner conversations. Euler returned to St Petersburg in 1766, disappointed that the presidency of the academy had been withheld from him on repeated occasions. He was followed in Berlin by the younger Lagrange, 'a great man joining a great king', who resided and created much of his work there for the following two decades. Euler had already sponsored Lagrange for the preceding eleven years ever since the then nineteen-year-old had impressed him with his powerful new δ-calculus. The rest, as the saying goes, is history. Many of the major mathematicians of the last three centuries have fallen under the spell of our subject. Meusnier, a student of the great Gaspard Monge, was twenty-one when he wrote his seminal *Mémoire*; Hermann Amandus Schwarz who, as Bieberbach put it, had retreated to a full professorship (Ordinariat) at Berlin University at age forty-nine and whose one known photograph shows a staid gentleman with a hoary beard, had made plaster models of algebraic surfaces for E. E. Kummer, later his father-in-law, when he was a graduate student, and had just turned twenty-two when his solution of Plateau's problem for a quadrilateral, again accompanied by a model fashioned with wire and gelatin skin, was presented to the Academy. One can speculate, nay, predict what the young Schwarz, who had a clear vision of the global shape of his creations, would have done with the technological tools available today.

The academic scene has undergone drastic changes in the intervening years.

An ever burgeoning body of knowledge makes it increasingly harder for the student to cut through, at an early age, to the frontiers of science. There is also a current trend inducing mathematicians to carry on research in teams or as members of larger organizational units, a trend considered progressive, often even necessary, and one being strongly encouraged, if not demanded, by official agencies. It might therefore be of interest to recall the contrasting opinion of G. Frobenius from 1917 (the year of his death)[53]. Protagonists of our story are prominently referred to in the quote. Reading between the lines, one can discern Frobenius's antipathy for Sophus Lie, Felix Klein and their activities; the last and most striking sentence is borrowed from Euclid: '. . . Organization is of the utmost importance for military affairs, as it is . . . for other disciplines where the gathering process of practical knowledge exceeds the strength of any individual. In mathematics, however, organizing talent plays a most subordinate role. Here weight is carried only by the individual. The slightest idea of a Riemann or a Weierstrass is worth more than all organizational endeavors. To be sure, such endeavors have pushed to take center stage in recent years, but they are exclusively pursued by people who have nothing, or nothing any more, to offer in scientific matters. There is no royal road to mathematics.'

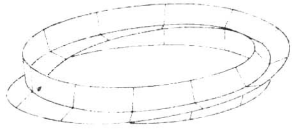
Meusnier had characterized a minimal surface by the vanishing of what we call today its mean curvature H. The eminent physical significance of this quantity was recognized in 1805 and 1806 by T. Young and P. S. Laplace, respectively, in their studies of capillary phenomena which were later the subject of Joseph-Antoine-Ferdinand Plateau's celebrated experimental and theoretical investigations. These have led to the simple macroscopic mathematical model which assigns to an interface separating a liquid from the surrounding medium an energy proportional to the surface area of the interface. The interface considered as a lamina of negligible thickness appears thus as solution of a variational problem $\delta \iint \mathrm{d}A = 0$, i.e. a minimal surface. In the presence of a volume constraint or other forces, the energy functional, and consequently the associated Euler–Lagrange differential equation, may become more complex. For instance, already in 1812 S. D. Poisson introduced and investigated the problem $\delta \iint H^2 \, \mathrm{d}A = 0$, a problem which is associated now with the name T. J. Willmore and enjoys great popularity today. It is thus clear that much of the interest in our surfaces stems from their often striking similarity to the real interfaces and separating membranes which are so abundant in nature, science and even (inspired by nature, but man-made) in architecture: labyrinthic structures found in botany and zoology, sandstone and other porous media, polymer blends, microemulsions and liquid crystals, to mention just a few. One caveat is in order at this point. The material structure and the interaction of forces, on a *microscale*, which must provide the foundations leading to the various macroscopi energy expressions put forth in the literature are not fully understood today[52]. Although it is enticing and exhilarating to recognize in electron microscope plates seemingly familiar shapes, e.g. periodic minimal surfaces, or singularity formations, there is often no firm theoretical basis at all to implicate such shapes and formations with certainty, particularly since specific terms appearing in the various macroscale energy expressions are frequently manipulated quite casually by their creators. Surely, the assertion that 'theoretical' (to boot, mostly numerical) data are in good agreement with specific experimental observations is, by itself, not sufficient justification for the validity of a comprehensive theory.

Be this as it may, while we are anxious to strengthen the relation with the natural sciences, the perspective of our presentation in these *Lectures* must remain primarily oriented on geometry, 'the only science that it hath pleased God hitherto to bestow on mankind', sharing priorities with the viewpoints of the calculus of variations and partial differential equations. Of course, it is not only the basic physical laws governing the shape of the surfaces but also the defining properties of these surfaces themselves, to be regarded as point sets in an ambient space, which challenge the researcher. In fact, several quite different definitions of the surface concept are in broad use today and compete with each other; each is applicable in specific situations. These concepts agree in basic test cases, but differ widely in others. Correspondingly, the fundamental boundary value problems – Plateau's problem and its manifold generalizations – may actually possess as many disaccording solutions as there are definitions. For the text at hand, the choice made by the author favors the time-honored and eminently useful war-horse of differential geometry: parameter surfaces – mappings from a parameter set into space. Still in concordance with the classical concept and not prejudging the ultimate topological type, each piece of such a surface can be taken as the germ of a global entity, an (open) two-dimensional point set in space which is known to be 'surface-like' in a neighborhood of each of its points. To qualify as minimal surface, it must be *locally* area-minimizing. The concept of boundary for such a surface requires an independent careful discussion. The parametric minimal surfaces under scrutiny here, or the surfaces of constant or otherwise prescribed mean curvature to which they are closely related, can very often be taken as suitable models for soap films or other physical interfaces. This will be the case, for instance, whenever these surfaces are embedded in space and stable within their boundary. However, we do not want to rule out self-intersections, nor do we wish to be restricted by a preoccupation with area-minimizing structures or to renounce the study of unstable surfaces, even though the latter is considered to be a sterile exercise by some precisely with reference to physical arguments. Generally, the relationship between geometrical and physical objects is obscure in many situations, and, as pointed out, it is often advisable to enlarge the space of admissible surfaces at the outset so that the desired objects are not excluded *a priori*. For further details, see §§ 289, 476 and 480; but all this leads far beyond the limits imposed on our book. A single example, however, will illustrate the reality. It had already been observed by Plateau and his students, especially the astute E. Lamarle, and will be expounded in more detail in Chapter VI, that soap films, in order to minimize area, tend to arrange themselves in the form of surface systems in which three sheets may meet along a common curve, a branch line, including there an angle of 120° with each other, or six sheets may share a common vertex where the four incoming branch lines form an angle of 109.471°. Points on the branch lines and the vertices represent special singularities often denoted by the symbols **Y** and **T**. Making precise the work of Lamarle, J. E. Taylor proved in 1976 that these are the only possibilities. In the absence of any advance knowledge whether branch lines or vertices will actually occur, it would be a cumbersome, if not impossible, task to force such aggregates into the Procrustean bed of parametric surface theory or, for that matter, the approach of geometric measure theory based on mass-minimizing integral currents, although the author is aware of sophisticated attempts, as yet unsuccessful or incomplete, toward this very goal. To wit: If one bends the surface aggregate depicted in figure (a) so that the curve along which the three sheets meet forms a circle, while twisting it by 120 degrees, then the originally disconnected outside boundaries fit together to form a simple closed curve Γ, in fact, a torus knot of type (3, 1) shown in figure (b). This curve bounds thus an embedded surface aggregate of approximate area $6\pi\varepsilon$; see figure (c). The



(a)

(b)



(c)

solution of Plateau's problem associated with Γ, on the other hand, will have self-intersections. It also must 'cover the hole' three times. Under these circumstances, the arguments of sections 246 and 392 show that its area will be larger than $3\pi(1-\varepsilon)^2$. For small values of ε, this is more than the area of the three-sheeted surface system. An area-minimizing integral current bounded by Γ will be free of self-intersections, but its area will again be too large. For the example at hand, the discrepancies can be overcome if the problem is considered in the framework of $(M, 0, \delta)$-minimal sets developed by F. J. Almgren, Jr since 1976. It is a challenging task to discover facts about the geometrical properties of such sets for concrete boundaries. To mention one particular problem: The 4π-theorem (J. C. C. Nitsche [43]) guarantees the uniqueness of the solution of Plateau's problem for an analytic Jordan curve whose total curvature does not exceed the value 4π. Does the theorem remain true if surfaces of general topological type or $(M, 0, \delta)$-minimal sets are admitted for comparison? Moreover (my old conjecture of 1973): Is the unique solution surface free of self-intersections; see A7.29?

There have been scattered discussions of our subject from the perspective of intuitionism, for instance attempts at isolating the constructive elements in the existence proofs, with the aim of devising, if possible, constructive proof arrangements. This would be of potential value for numerical approaches. The choice of Heraclitus's dictum as head quote for this preface raises coincidentally another age-old epistemological question which certainly goes back to the times of Heraclitus and Plato (not Plateau). In our description of the factual developments we are accustomed to say that a certain mathematician *discovered* a particular minimal surface. Here the use of the word 'discovered' is quite automatic. Consider that minimal surfaces appear to us in the most bizarre and totally dissimilar shapes, and think of the foregoing references to minimal surfaces as physical interfaces, bear in mind also that nobody has actually ever seen a perfect minimal surface; are these surfaces constructs of the intellect, or do they exist in reality, whether we perceive them or not – creations or discoveries?

Interspersed in the German text were 85 figures, including two color plates, I believe a first for the Yellow Series at the time, albeit surely appropriate considering the subject matter. Most of these figures had been computed, designed and photographed by me and subsequently gone over by expert art personnel, six were taken from the existing literature. Some of the more appealing designs have found their way into various later publications by others, normally without attribution. I shall not bring out my evidence here, but one observation is in order: The piracy of graphic material, in mathematical papers and especially in today's numerous conferences where it obviously adds spice to a speaker's presentation, a fact of which he is acutely conscious, has taken reprehensible dimensions. I have always believed in the usefulness and power of graphic additions. It is true that over the years there have been schools whose disciples prided themselves, often to the point of smugness, on withholding from the reader any visual aid, even the simplest figure, which could have provided insight, pleasure or relief. It is reported that in the seminars of a geometer's geometer as Jakob Steiner the use of figures was scorned, although in Steiner's case this seemed to have been conducive to the geometric understanding of the students. The construction of my minimal surface models was a family enterprise. While less than perfect, they are lovable creatures to me, ranging in diameter from one-half to three and one-half inches. Twenty-seven years ago, when I had bought my first Paillard–Bolex movie camera, we put these models on a turntable and filmed them from various angles as they rotated in front of the lens. Later the films were transferred to video tape and set to my favorite classic musical themes. There are close-ups of surfaces which to date have eluded the grasp of computer graphics, because no representation formulas are available, nor have they been computed yet. Of course, their mathematical existence is secured. It is an enchanting experience for us to see the garden of minimal surfaces appear on the screen to the tunes of *Les Troyens* (Queen Dido . . . !). A number of further illustrations has been added to the present Volume One, some drawn by the author, others kindly provided by colleagues: D. M. Anderson, F. Gackstatter, D. Bloss and E. Brandl, I. Haubitz, D. A. Hoffman and J. T. Hoffman, J. M. Nitsche, K. Peters. I also hope that the likenesses of Plateau and Schwarz as well as the adjunction of facsimile pages from the original works of Euler, Lagrange and Meusnier will be of interest to the reader.

Regrettably for me personally, many of my activities concerned with graphic representations took place before the advent of computer graphics (and, still to come and to be utilized for mathematics, computer modelling) which has turned out to be one of the most remarkable developments of recent years. It has influenced mathematical exposition profoundly – a far far cry from the use of colored chalk in the classroom. Today, computer graphics work stations spring up at many locations, leading to the creation of most beautiful illustrations. As far as minimal surfaces are concerned, mention must be made here of the striking pictures of D. M. Anderson in whose work I am happy to have participated as a mathematical consultant, along with my colleagues H. T. Davis and L. E. Scriven who were of course more intimately involved, and particularly of the recent contributions of D. A. Hoffman and his collaborators (M. J. Callahan, J. T. Hoffman, W. H. Meeks III, J. Spruck) at the Geometry, Computation and Graphics Facility of the University of Massachusetts. These authors enumerate the virtues of interactive computer graphics also for the research mathematician: 'Computer-generated images allow new, often unexpected mathematical phenomena to be observed. . . . Richer, more complex examples of known phenomena can be explored. . . . On the basis of exploration of examples and phenomena, new patterns are observed. . . . Easier and more fruitful