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Locally Semialgebraic Spaces

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continued on page 335

To Christl and Gisela

Preface

The primary occupation of real algebraic geometry, or better "semialgebraic geometry", is to study the set of solutions of a finite system of polynomial inequalities in a finite number of variables over the field $\mathbb{R}$ of real numbers. One wants to do this in a conceptual way, not always mentioning the polynomial data, similarly as in algebraic geometry, say over $\mathbb{C}$, where one most often avoids working explicitely with the systems of polynomial equalities (and non-equalities $f \neq 0$) involved.

But a semialgebraic geometry which deserves its name should be able to work - at least - over an arbitrary real closed field R instead of the field $\mathbb{R}$. Such fields are useful and even unavoidable in semialgebraic geometry for much the same reason as algebraically closed fields of characteristic zero - at least - are unavoidable in algebraic geometry over $\mathbb{C}$, as soon as one tries to avoid transcendental techniques or even then.

In order to illustrate this we give a somewhat typical example. Let $f: V \to W$ be an algebraic map between irreducible varieties over $\mathbb{R}$. This yields, by restriction, a continuous map $f_{\mathbb{R}}: V(\mathbb{R}) \to W(\mathbb{R})$ between the sets of real points. We assume that $W(\mathbb{R})$ is Zariski dense in W which means that $W(\mathbb{R})$ contains non singular points or, equivalently, that the function field $\mathbb{R}(W)$ is formally real. The generic fibre X of f, i.e. $X = f^{-1}(\eta)$ with η the generic point of W (regarding V and W as schemes), is an algebraic scheme over the function field $\mathbb{R}(W)$ of W, which contains a lot of information about f and $f_{\mathbb{R}}$. But it may be too difficult to study X, since the field $\mathbb{R}(W)$ is usually very complicated. In algebraic geometry one often replaces X by the algebraic variety $\overline{X}$ obtained from X by extension of the base field $\mathbb{R}(W)$ to the algebraic closure C of $\mathbb{R}(W)$. It is much easier to study the "geometric generic fibre" $\overline{X}$

instead of X, and still one may hope to extract relevant information about f from $\overline{X}$. But in semialgebraic geometry this procedure is not advisable, since most real phenomena in X will be destroyed in $\overline{X}$. Instead of $\overline{X}$ one should study the varieties X_α, obtained from X by base extension from $\mathbb{R}(W)$ to the real closures R_α of $\mathbb{R}(W)$ with respect to the various orderings α of the function field $\mathbb{R}(W)$, and the sets of rational points $X_\alpha(R_\alpha)$. For every such α we have $R_\alpha(\sqrt{-1}) = C$. Thus the R_α are "as near as possible" to C and nevertheless we may hope to detect some of the real phenomena of X, and ultimately of f, in the sets $X_\alpha(R_\alpha)$.

The variety X is the projective limit of the schemes $f^{-1}(U) = V\times_W U$ with U running through the Zariski-open subsets of W, since these U are the Zariski neighbourhoods of the generic point η in W. Similarly $\overline{X}$ is the projective limit of the fibre products $V\times_W U$, with respect to the etale morphisms $\varphi: U \to W$ from arbitrary varieties U over $\mathbb{R}$ to W ($U \neq \emptyset$, but $U(\mathbb{R})$ may be empty), since these morphisms φ are the etale neighbourhoods of η. How about the X_α? An ordering α of $\mathbb{R}(W)$ corresponds uniquely to an ultrafilter F in the Boolean lattice $\gamma(W(\mathbb{R})$ of semialgebraic subsets of $W(\mathbb{R})$ such that every $A \in F$ has a non empty interior $\mathring{A}$ in the strong topology (= classical topology on $W(\mathbb{R})$), which means that A is Zariski dense in W, cf. [B, 8.11],[Br, §4]. (A rational function $h \in \mathbb{R}(W)$ is positive with respect to α if and only if h is defined and positive on some set $A \in F$). It turns out that X_α is the projective limit of the fibre products $V\times_W U$ with respect to those etale morphisms $\varphi: U \to W$ such that $\varphi(U(\mathbb{R})) \in F$. (N.B. $\varphi(U(\mathbb{R}))$ is semialgebraic.) This is due to the fact that R_α can be interpreted as the union of the rings of Nash functions $\mathcal{N}_W(U)$ on the various smooth open sets $U \in F$, cf. [Ry].

Much more can be said about a geometric interpretation over $\mathbb{R}$ of the fields R_α, the varieties X_α and the points in $X_\alpha(R_\alpha)$. But this would take us too far afield. We only mention that the real spectra of commutative rings invented by M. Coste and M.F. Coste-Roy provide exactly the right language to understand all this, cf. [CR], [Ry], and the literature cited there and, for an introduction to real spectra , also [L, §4, §7], [Br, §3, §4], [K], [BCR, Chap. 7].

We have been somewhat vague above. In particular we did not make precise the various direct systems which yield the projective limits $\overline{X}$ and X_α. We only wanted to indicate that in semialgebraic geometry over $\mathbb{R}$ real closed fields may come up in a natural and geometric way.

The present lecture notes give a contribution to a basic but rather modest aspect of semialgebraic geometry: the topological phenomena of semialgebraic sets in V(R) for V a variety over a real closed field R. There is a difficulty with the word "topological" here. Of course, V(R) is equipped with the strong topology coming from the topology of the ordered field R. But, except in the case $R = \mathbb{R}$, the topological space V(R) is totally disconnected.

These pathologies can be remedied by considering on V(R) a topology in the sense of Grothendieck, where only open semialgebraic subsets U of V(R) are admitted as "open sets", and for such a set U essentially only coverings by finitely many open semialgebraic subsets of U are admitted as "open coverings".

It seems that the category of semialgebraic spaces and maps over a real closed field R, which has been introduced in our paper [DK_2], provides the right framework for this "semialgebraic topology". Already in that paper and later in other ones ([D], [D_1], [DK_3], [DK_4], [DK_5]) we found analogues of many results in classical topology. Sometimes things are even nicer here. This is not astonishing since, in the case $R = \mathbb{R}$, the semialgebraic sets are rather tame from a topological viewpoint.

In the case $R = \mathbb{R}$ the category of semialgebraic spaces can be compared with the category of topological spaces, and this affords us a new perspective concerning the two branches of mathematics involved, semialgebraic geometry and algebraic topology, cf. the introduction of [B]. For example, a long journey along this road should give a thorough understanding of why so many spaces occuring in usual algebraic topology are semialgebraic sets.

Nevertheless the category of semialgebraic spaces is too restrictive for some purposes. A good instance where this can be seen is the theory of <u>semialgebraic coverings</u>. If M is a connected affine semialgebraic space over R, and x_o is some point in M, we can define the fundamental group $\pi_1(M,x_o)$ in the usual way as the set of semialgebraic homotopy classes of semialgebraic loops with base point x_o (cf. III,§6)*). This is an honest to goodness group, generated by finitely many elements satisfying finitely many relations. On the other hand we evidently have the notion of an (unramified) covering $p: N \to M$ of M, p being a locally trivial semialgebraic map with discrete (= zero-dimensional) fibres. Of course, one would like to classify the coverings of M by subgroups of $\pi_1(M,x_o)$. But a zero-dimensional semialgebraic space is necessarily a finite set. Thus every semialgebraic covering has finite degree. It can be shown that indeed the isomorphism classes of semialgebraic coverings of M correspond uniquely to the conjugacy classes of subgroups of finite index in $\pi_1(M,x_o)$ in the usual way. But there should also exist coverings of a more general nature which correspond to the other subgroups of $\pi_1(M,x_o)$. In particular there should exist a universal covering of M. These more general coverings can be defined in

*) This means §6 in Chapter III of this book.

the category of "locally semialgebraic" spaces and maps.

After several years of experimenting with locally semialgebraic spaces we are convinced that these spaces exist "in nature". The coverings of affine semialgebraic spaces are regular paracompact locally semialgebraic spaces, to be defined in I, §4. Regular paracompact spaces seem to be the "good" locally semialgebraic spaces, analogous to the affine spaces in the semialgebraic category. For instance, for these spaces there exists a satisfactory cohomology theory of sheaves, based on flabby and soft sheaves, which parallels the classical theory for topological paracompact spaces. We will not deal with these matters here, except for some brief remarks in Appendix A, but they are quite important for defining homology and cohomology groups of various kinds for these spaces, cf. [D], [D_1], [D_2].

Although regular paracompact spaces are a very satisfying subclass of locally semialgebraic spaces one has to face the fact that there exist many locally semialgebraic spaces in nature which are not paracompact. (It seems that regularity may be assumed in most applications.) For example, studying open subsets of quite innocently looking real spectra may lead to regular spaces which are not paracompact, cf. Appendix A. Thus it is not just for fun or for systematic reasons that we study in Chapter I more general spaces. In the later chapters we are forced to restrict to paracompact spaces, since otherwise our deeper techniques break down.

There is one phenomenon in our theory which may seem somewhat unusual for a reader of our previous papers. In a semialgebraic space M it is strictly forbidden to work with subsets of M other than the semialgebraic subsets [DK_2, §7]. But in a locally semialgebraic space M there exist two natural classes of admissible subsets, the class $\mathcal{T}(M)$ of locally semialgebraic subsets of M and the smaller class $\gamma(M)$ of semialgebraic subsets of M. The interplay between $\gamma(M)$ and $\mathfrak{T}(M)$ is a theme which recurs througout the whole theory.

The goal of the first volume of our lecture notes is to establish the category of locally semialgebraic spaces and maps over an arbitrary real closed field R on firm grounds, and to prove enough results about these spaces and maps, that the reader will feel well acquainted with them and will regard them as concrete and accessible objects. The next topics, to be covered in the second volume, are the theory of locally semialgebraic fibrations and fibre bundles (Chapter IV) and the theory of coverings (Chapter V).

As background material we assume our papers [DK_2], [DK_4], [DK_5], some sections of [DK_3], and Robson's paper [R]. Here you find nearly everything which we need about semialgebraic spaces, written up in a systematic way compatible with the spirit of these lecture notes. Of course, it would have been more comfortable for the reader if we had started the lecture notes with a review of the results of those papers. But this is not really necessary and would have made the lecture notes too long. Of course, the book [BCR] of Bochnak and the Costes - as soon as it has appeared - will contain most basic facts which are necessary for an understanding of these lecture notes and much more.

A survey on some basic results about semialgebraic spaces has been given in [DK]. Another survey on basic results about locally semialgebraic spaces, which, of course, all will be covered by the two volumes of these lecture notes, has been given in [DK_6] and [DK_7].

We hope that these lecture notes, designed in first place for the needs of semialgebraic geometry, are also of interest for topologists. The main results are usually non trivial also in the case $R = \mathbb{R}$ and not much easier to be proved in this special case. The category of locally semialgebraic spaces over $\mathbb{R}$ lies somewhat "in between" the category TOP of topological Hausdorff spaces and the category PL of piecewise linear spaces, being less rigid than PL and, in some respects, less pathological than TOP.

The central result of the whole volume seems to be Theorem 4.4 in Chapter II, §4, which states that every regular paracompact locally semialgebraic space M can be triangulated, and moreover a given locally finite family of locally semialgebraic subsets of M can be triangulated simultaneously. Thus we may regard every regular paracompact space as a locally finite polyhedron with some open faces missing (cf. the definition of strictly locally finite simplicial complexes, in I, §2, which is slightly different from the classical definition). But in contrast to PL-theory, we may subdivide simplices not only linearly but "semialgebraically". Nevertheless, in the special case that $R = \mathbb{R}$ and M is partially complete, Shiota and Yokoi have recently proved that any two PL structures on M which refine the given semialgebraic structure are isomorphic ([SY, Th. 4.1], they prove this more generally for suitable locally subanalytic spaces). This remarkable theorem can be extended to partially complete regular paracompact spaces over any R, as we hope to explain in the second volume.

If S is a real closed field containing R then, as a consequence of Tarski's principle, we can associate with every locally semialgebraic space M over R a locally semialgebraic space M(S) over S by "<u>extension of the base field R to S</u>", cf. I.2.10. This yields a very good natured functor $M \mapsto M(S)$ from the category of regular paracompact spaces over R to the category of regular paracompact spaces over S, which is of crucial importance for our whole theory. The homotopy groups (cf. III, §6), the homology groups (cf. III, §7) and also the various K-groups of M (orthogonal, unitary, symplectic, cf. Chapter IV in the second volume) are preserved under base field extension from R to S. These are examples of the main message of our whole theory, that over a complicated real closed field the locally semialgebraic spaces are in many respects not more complicated than over a simple field, as the field $\mathbb{R}$ or the field R_o of real algebraic numbers. We believe that this message is by no means trivial. It may be regarded as a vast generalization of Tarski's principle for topological statements. As soon as one leaves the cadre of semialgebraic topology and works, say with algebraic functions then the analogue of our message seems to hold only under severe restrictions. For example, it is well known that, in general, semialgebraic functions on the unit interval [0,1] in R cannot be approximated uniformly by polynomials, in contrast to the Stone-Weierstraß theorem for $R = \mathbb{R}$.

The book has two appendices. Appendix B (to Chapter I) contains some easy but fundamental results in the theory of base extension. They have not been included into Chapter I since some of the techniques needed to derive them seem to have their natural place in Chapter II. Appendix A is of different kind. Here we draw the connections between our theory and "abstract" semialgebraic geometry which, starting from the notion of the real spectrum, now is in a process of rapid development. Appendix A is not needed for our theory in a technical sense, but there we will find the occasion to explain some more points of our philosophy about the "raison d'être" of locally semialgebraic spaces.

We thank the members of the former Regensburger semialgebraic group, in particular Roland Huber and Robby Robson, for stimulating discussions and criticism about the contents of these lecture notes. Special thanks are due to José Manuel Gamboa and R. Huber for a penetrating (and very successful) search for mistakes in the final version of the manuscript.

We thank Marina Richter for her patience and excellence in typing the book and R. Robson for eliminating some of the most annoying grammatical mistakes. We are well aware that we could have written a better book

in our native language, but since the book is designed as a "topologie générale" for semialgebraic geometry which should be useful as a widely accepted reference, we have written in that language which will be understood by the most.

Regensburg, July 1985

Hans Delfs, Manfred Knebusch

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TABLE OF CONTENTS