Martin M. Guterman
Zbigniew H. Nitecki

DIFFERENTIAL EQUATIONS

A First Course

THE SAUNDERS SERIES

Martin M. Guterman
Zbigniew H. Nitecki

DIFFERENTIAL EQUATIONS

A First Course

THE SAUNDERS SERIES

Address orders to:
383 Madison Avenue
New York, NY *10017*

Address editorial correspondence to:
West Washington Square
Philadelphia, PA 19105

Text Typeface: 10/12 Times Roman
Compositor: Waldman Graphics
Acquisitions Editor: Leslie Hawke
Project Editor: Janis Moore, Diane Ramanauskas
Managing Editor & Art Director: Richard L. Moore
Art/Design Assistant: Virginia A. Bollard
Text Design: Lawrence R. Didona
Cover Design: Lawrence R. Didona
Text Artwork: J & R Technical Services
Production Manager: Tim Frelick
Assistant Production Manager: Maureen Iannuzzi

Library of Congress Cataloging in Publication Data

Guterman, Martin M.
Differential equations.

(The Saunders series)
Bibliography: p.
Includes index.
1. Differential equations. I. Nitecki, Zbigniew.
II. Title.
QA371.G875 1983 515.3′5 83-7530
ISBN 0-03-062502-5

DIFFERENTIAL EQUATIONS:A FIRST COURSE ISBN 0-03-62502-5

Library of Congress catalog card number

3456 032 987654321

CBS COLLEGE PUBLISHING
Saunders College Publishing
Holt, Rinehart and Winston
The Dryden Press

PREFACE

This book is an introduction to standard topics in differential equations for the average sophomore engineering or science student. The fundamental first two chapters treat first-order equations and linear constant-coefficient equations, including a brief view of 2×2 systems via elimination. The five chapters that follow give mutually independent treatments of systems, the Laplace transform, power series solutions, numerical methods, and Fourier series methods for partial differential equations. With an appropriate choice of topics, this book can serve as the basis for various courses to follow a two- or three-semester calculus sequence; the possibilities range from a basic survey of general methods for solving a single differential equation to an integrated introduction to linear differential equations and systems that incorporates the rudiments of linear algebra (without assuming previous exposure to determinants, matrices, or vectors).

Our exposition is aimed at the beginning user of differential equations; it guides the reader through the underlying ideas of the subject while maintaining a hands-on experience of specific problems. The discussion proceeds from the concrete to the abstract by means of many worked-out examples and observation of general patterns. Boxed summaries reiterate the main points to remember, including explicit problem-solving procedures. These serve as handy reference points for the reader. The notes that follow the summaries discuss technical fine points and specific shortcuts or difficulties that arise in practice. Exercises at the end of each section are arranged roughly in order of increasing difficulty and abstraction; especially involved problems are starred. Review problems at the end of each chapter provide an opportunity for the reader to check his or her understanding of the chapter as a whole. The answers to odd-numbered exercises appear at the end of the book.

The introductory section at the beginning of each chapter considers a single class of physical models (e.g., populations, springs, electrical circuits, or heat flow) as a practical motivation for the mathematical discussion that follows. This can be treated as independent reading when no class time is available for modeling or applications. The chapter-by-chapter contents of the mathematical discussion are summarized as follows:

First-Order Equations (Chapter 1) are handled primarily by separation of variables and variations of parameters. The discussion of graphing and exact equations can be skipped or deferred until later.

In Chapter 2, **Linear Differential Equations,** the method of characteristic roots is developed through specific examples. The treatment of completeness tests for lists of solutions allows flexibility concerning which methods to stress. The methods of undetermined coefficients and variation of parameters are treated as natural extensions of earlier techniques. Some further consideration of physical models, and a brief treatment of 2×2 differential systems by elimination, can be used or skipped at the user's discretion. Those wishing to treat 2×2 systems, but nothing larger, may choose to use Section 2.12 and skip Chapter 3. Appendix A supplements Section 2.4 with a treatment of determinants of higher order.

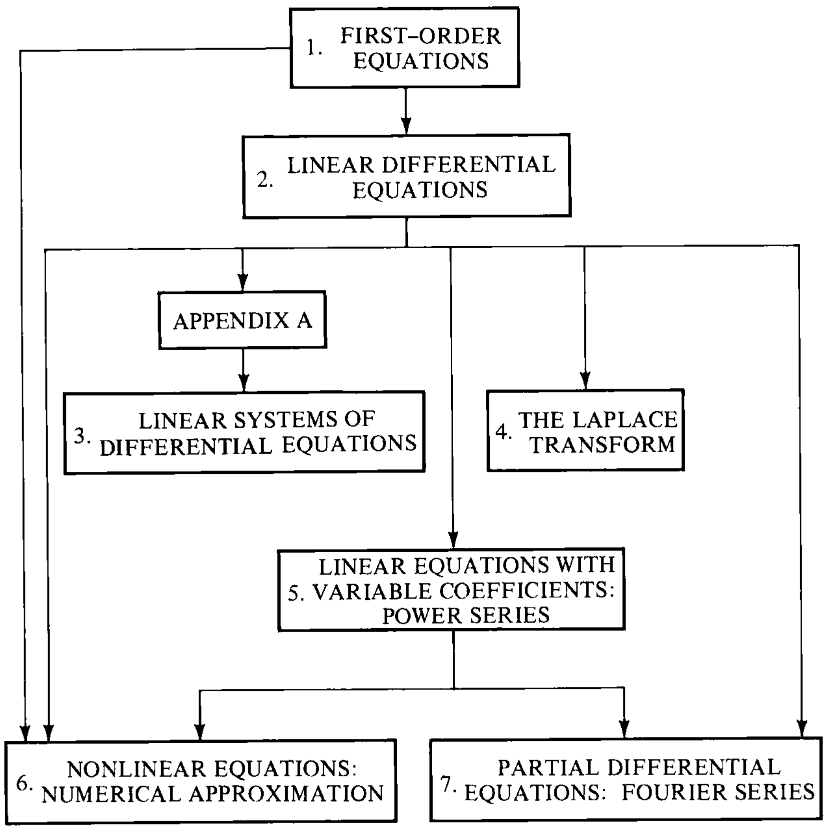
Our discussion of **Linear Systems of Differential Equations** (Chapter 3) occurs earlier than usual because we feel it forms a natural sequel to the discussion in Chapter 2. (The book, however, is written in such a way that the sequence in which topics from Chapters 3 to 7 are covered can be easily altered to suit the user's taste.) Basic tools from linear algebra are developed as the need for them arises. A fundamental computational tool in this chapter is row reduction of matrices (3.6), which in itself is one of the most useful techniques for a student of engineering or science to learn at this stage. The depth and generality of the treatment of systems can be controlled by the extent to which the material on complex eigenvalues (3.8) and generalized eigenvectors (3.9) is covered. The retrospective appendices that end this chapter show the reader how the phenomena and ideas encountered in Chapters 1 to 3 fit into a more general mathematical context.

The motivation for Chapter 4, **The Laplace Transform,** comes from problems with discontinuous forcing terms. The chapter develops an operational calculus for handling initial value problems via the Laplace transform, including the adaptation to systems.

The motivation for Chapter 5, **Linear Equations with Variable Coefficients: Power Series,** comes from the Cauchy-Euler, Legendre, and Bessel equations as they arise in temperature distribution problems. After a review of power series, we develop power series solution of o.d.e.'s with polynomial coefficients, including the Frobenius method.

In Chapter 6, **Nonlinear Equations: Numerical Approximations,** we discuss the Euler, average-slope, and Runge-Kutta methods. We believe that it is both desirable and practicable to give students a hands-on experience of calculating numerical schemes on electronic devices. Our treatment is written to allow the use of computers or programmable hand calculators. Notes at the end of Sections 6.2, 6.4, 6.5, and 6.7 include sample computer (BASIC) and calculator programs. An important section, which should be included in any treatment of numerical methods, is Section 6.6, treating examples of the limitations of numerical methods. The ideas of Chapter 6 are applied in Appendix D, in a sketch of the proof of the Existence and Uniqueness Theorem for o.d.e.'s

Chapter 7, **Partial Differential Equations: Fourier Series,** introduces p.d.e.'s via the heat equation. Separation of variables and Fourier series (including sine series and cosine series) are used to solve various boundary value problems. These techniques are then also applied to the one-dimensional wave equation and the two-dimensional Dirichlet problem. In the final section we show how expansion in other orthogonal families of functions can be needed to solve certain higher-dimensional problems.



POSSIBLE ARRANGEMENT OF TOPICS.

ACKNOWLEDGMENTS

Many people have contributed to this book directly or indirectly by their advice, general interest, and moral and material support.

Our ideas about content evolved from our experience in teaching the o.d.e. courses at Tufts. These ideas were influenced by our students and many of our colleagues at Tufts, including Bill Crochetiere, George Cybenko, Bob Devaney, Gary Goldstein, David Krumme, Michal Krych, Ben Kuipers, John Montgomery, Todd Quinto, Bill Schlesinger, and Kay Whitehead.

The writing of early drafts of this book, and classroom-testing at Tufts, was facilitated by support from the University, which we obtained through the generosity of our Department Chairman, George Leger, and our Dean, Nancy Milburn.

The final form of the book was greatly influenced by the thoughtful comments of many readers, including W. D. Barcus, C. K. Bradshaw, J. A. Burns, W. C. Fox, H. Gaines, R. B. Guenther, P. Herron, R. M. Koch, E. Krych, H. E. Lacey, S. M. Luckawecki, J. Power, C. P. Simon, A. Sutcliffe, and J. Thorpe.

A monumental contribution was made by Elizabeth Branson and Andrea Ventrice, who transformed our scribbled reams of paper into beautiful, readable text.

For their indispensable role in transforming our manuscript into this book, we are indebted to the editorial and production staff at Saunders College Publishing, including Leslie Hawke, Elizabeth Czyszcon, Janis Moore, Jay Freedman, Richard Moore, and Diane Ramanauskas.

Finally, for the inevitable errors we both thought had been eliminated, each of us has only the other to thank.

M. G.
Z. N.

CONTENTS

First-order Equations

ONE

1.1 INTRODUCTION

In the late seventeenth century, Isaac Newton in England (1665, 1687) and Gottfried Wilhelm Leibniz in Germany (1673) synthesized several centuries of mathematical thought to create a language and method for describing and predicting the motion of bodies in various physical situations. The invention of the calculus was immediately followed by a period of intense mathematical activity, and the effect of these ideas on the development of mathematics, science, and technology makes this event surely one of the most important in the history of western thought. During the development of calculus, differential equations and their solutions played the central role. They arose as mathematical formulations of physical problems, and attempts at their solution motivated much of the mathematical development of calculus.

The role of differential equations in the modeling of physical phenomena is well illustrated by **Newton's second law** of motion, familiar to all physics students in the mnemonic form

$$F = ma.$$

In the situation that most interested Newton (gravity), the force F is the weight of the body, the constant m is its mass, and a is its acceleration. Although we are really interested in the position of the body, the equation tells us about neither the position nor its rate of change, but rather about the rate of change of the rate of change of the position. In the language of calculus, if $x = x(t)$ represents the position at time t, then the velocity is the derivative of x, $v = dx/dt$, and the acceleration is the derivative of velocity, $a = dv/dt = d^2x/dt^2$. Thus Newton's second law

$$F = m\frac{d^2x}{dt^2}$$

is an equation involving a derivative of the interesting variable—that is, it is a **differential equation.**

It was Newton's brilliant observation that in many physical situations the relation between rates of change of observable quantities is simpler than the relation between the quantities themselves. This is at the same time the source of the power of differential equations and the central problem in using them to predict physical phenomena. For if Newton's second law is to lead to useful physical predictions, we must translate this statement about the second derivative of position into a prediction of the position of the body at some time in the future; that is, we must express x as a function of time

$$x = \phi(t).$$

Any such prediction (function) that is consistent with a given law (differential equation) is called a **solution** of the differential equation. The problem of obtaining solutions to a given differential equation is a purely mathematical one and forms the subject of this book.

Some important features of the solution of differential equations can be illustrated by a special case of Newton's law. When the force is constant, it is easy to solve the equation by integrating both sides twice. We first recall that $v = dx/dt$ and write the law in the form

$$\frac{dv}{dt} = \frac{F}{m}.$$

Integrating both sides with respect to t

$$\int \frac{dv}{dt}\, dt = \int \frac{F}{m}\, dt$$

gives

$$\frac{dx}{dt} = v = \frac{F}{m} t + c_1.$$

Now, we integrate again

$$\int \frac{dx}{dt}\, dt = \int \left(\frac{F}{m} t + c_1\right) dt$$

$$x = \frac{F}{2m} t^2 + c_1 t + c_2.$$

Note that the solution involves two "arbitrary constants," c_1 and c_2, which resulted from taking two indefinite integrals. The physical significance of these constants becomes clearer when we think of a specific instance of this equation, the motion of a falling ball under the force of gravity. Whereas the differential equation takes into account the Earth's gravity and the mass of the ball, this is hardly enough to predict the ball's position. We need to know where it started from, and whether it was dropped or thrown. Without such information, we can make only a **general** prediction, vague enough to apply to all possible circumstances of the ball. To make a **specific** prediction without ambiguity, we need to know the initial position (the value of x when $t = 0$) and the initial velocity (the value of $v = dx/dt$ when $t = 0$). If we pick specific numerical values for the constants c_1 and c_2 in the general solution above, we find upon substituting $t = 0$ that the initial position is

$$x(0) = \frac{F}{2m}(0)^2 + c_1(0) + c_2 = c_2$$

while the initial velocity is

$$v(0) = \frac{F}{m}(0) + c_1 = c_1.$$

We see that in this case the values of the two "arbitrary constants" in our general solution are numerically equal to the **initial conditions** that determine a specific solution. We shall consider the role of initial conditions in determining specific solutions as we study various kinds of differential equations.

Of course, the process of finding the general solution in the preceding case was very easy. Some of the difficulties of the subject become clearer if we consider Newton's law in a context closer to the problems that motivated it in the first place. When the distance between bodies reaches interplanetary scale, the gravitational force depends on position according to an inverse-square law. Fixing masses and the constant g appropriately, this leads to the differential equation

$$\frac{-g}{x^2} = \frac{d^2x}{dt^2}.$$

If we try to solve this equation by integrating both sides, we run into trouble on the left. Remember, we want to take "$\int(\quad)dt$" of both sides. To evaluate

$$\int \frac{-g}{x^2}\,dt$$

we need to express x as a function of t. But if we knew *that*, we wouldn't need to integrate, since the equation would already be solved.

This shows that, even when our final goal is a practical one, we need a certain amount of theory to handle the differential equations that arise in physical models. We will consider specific theoretical questions as they come up in our study of solution methods for differential equations. For the moment we consider some population models, with an eye toward understanding the different kinds of differential equations that can arise in modeling various phenomena. Other physical models leading to similar differential equations are considered in the exercises that follow. We will solve the equations of these examples later in the text.

Example 1.1.1

A population grows at the rate of 5% per year. If $x = x(t)$ stands for the number of individuals in the population after t years, then the rate of change of x is numerically equal to 5% of x. Written as an equation this is

$$\frac{dx}{dt} = \frac{5}{100}x.$$

Whereas Newton's law involves the second derivative of the interesting variable, the equation in Example 1.1.1 involves only the first derivative. In general, we refer to the highest order of differentiation as the **order** of the equation. Thus, Newton's law is a second-order differential equation, while the population equation is of first order.

Example 1.1.2

Our first population model assumed a constant growth rate. This will not always be realistic. For example, the growth rate of the United States rose sharply after World War II but recently has been decreasing slowly. One function that exhibits this phenomenon (although it does not accurately portray the U. S. population) is

$$g(t) = \frac{t}{t^2 + 1}.$$

Some calculation shows that starting from $g(0) = 0$, $g(t)$ rises to a maximum value of $g(1) = 1/2$ (i.e., 50% annual growth rate), then falls off gradually, approaching 0 in the distant future (see Figure 1.1). A model based on this changing growth rate would give the equation

$$\frac{dx}{dt} = \frac{t}{t^2 + 1}x.$$

This equation, like the previous one, is of first order. Note that here the variable t appears explicitly in the coefficients of the equation.

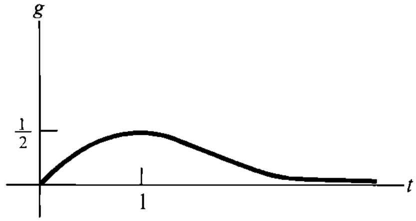


FIGURE 1.1

Example 1.1.3

The rates of change of the populations in the previous examples were multiples of the populations. In some cases there may be an additional component (immigration) that depends only on time.

Suppose a disease-causing organism reproduces in its host by dividing once a day (on the average). Suppose also that its presence causes the host's resistance to deteriorate, so that on the tth day after the initial infection, t thousand organisms are able to enter the host from the surrounding environment. Let $x = x(t)$ be the number of organisms in the host, measured in thousands. Then the rate of change of x has two components: reproduction contributes x to dx/dt, and new organisms entering from the surrounding environment contribute t. The total rate of change is

$$\frac{dx}{dt} = x + t.$$

Example 1.1.4

Suppose a population consisting of $x = x(t)$ thousand organisms would, in an unlimited environment, have a growth rate of 5% per year. Assume the environment is limited and can support at most a population of 10 thousand. Then as the population approaches 10 thousand, we would expect the growth rate to decline. The simplest way to take account of this fact is to multiply the unlimited growth rate by a factor that approaches zero as x approaches 10; the simplest such factors are constant multiples of $10 - x$. Thus, we expect our population to satisfy

$$\frac{dx}{dt} = .05\ \alpha(10 - x)x.$$

Since a small population experiences little competition, the limited-environment growth rate should approach the unlimited-environment growth rate as x approaches zero. Then $\alpha = 1/10$. Our model has the equation

$$\frac{dx}{dt} = .005(10 - x)x.$$

Note that this equation, like our second gravitational example, involves an x^2 term.

Example 1.1.5

Suppose two neighboring countries, with populations $x_1(t)$ and $x_2(t)$, have natural growth rates (birth rate minus death rate) of 15% and 10%, respectively. Suppose that 4% of the first population moves to the second country each year, while 3% of the second population moves to the first country each year. Then the rate of change of each of the populations is made up of three components, the natural growth rate, emigration, and immigration:

$$\frac{dx_1}{dt} = .15\, x_1 - .04\, x_1 + .03\, x_2 = .11\, x_1 + .03\, x_2$$

$$\frac{dx_2}{dt} = .10\, x_2 - .03\, x_2 + .04\, x_1 = .04\, x_1 + .07\, x_2$$

In this case we have a **system** of two differential equations, each involving both variables x_1 and x_2 in an unavoidable way.

Each of our examples so far has involved only ordinary derivatives (as opposed to partial derivatives). We refer to such equations as **ordinary differential equations** (abbreviated **o.d.e.'s**). An equation like

$$\frac{\partial u}{\partial t} = c\,\frac{\partial^2 u}{\partial x^2}$$

which involves partial derivatives, is called a **partial differential equation (p.d.e.).**

In this book we shall concentrate primarily on a special class of o.d.e.'s and systems (linear ones) for which a systematic solution procedure can be formulated and which are used in a broad variety of physical models. The precise delineation of this class will occur piecemeal as we study various specific instances.

In this chapter we will look at first-order equations, like the ones in Examples 1.1.1 through 1.1.4. These are the simplest from the point of view of calculus since they involve only first derivatives. Yet a large variety of phenomena can be described by using such equations.

We close this section with a summary of the basic definitions that will play a large role in the first part of the book.

SOME BASIC DEFINITIONS

An **ordinary differential equation** (abbreviated **o.d.e.**) is an equation whose unknown x is a function of one independent variable t. The equation relates values of x and its derivatives to values of t.

The **order** of an o.d.e. is the highest order of differentiation of x appearing in the equation.

A **solution** of an nth-order o.d.e. is a function $x = \phi(t)$, with derivatives at least up to order n, which when substituted into the o.d.e. yields an identity on the domain of definition of $\phi(t)$.

The **general solution** of an o.d.e. of order n is a formula (usually involving n "arbitrary constants") that describes all **specific solutions** of the equation. A specific solution (or, equivalently, the value of each of the constants in the general solution) is determined by certain **initial conditions**, such as the starting point and the starting velocity.

EXERCISES

1. Determine the order of each of the following o.d.e.'s.

   a. $t^4 \dfrac{d^3x}{dt^3} + t\dfrac{dx}{dt} - x = t^7$ b. $\left(\dfrac{dx}{dt}\right)^5 + \dfrac{d^4x}{dt^4} - t^3x^7 + t^7 = 0$

   c. $x^8 \dfrac{dx}{dt} + \dfrac{d^7x}{dt^7} = x + t^9$ d. $(x')^2x''' = x^4x'' + t^5x'$

In Exercises 2 to 6, check to see whether the given function $x = \phi(t)$ is a solution of the given o.d.e.

2. $\phi(t) = t^5$; $\dfrac{d^2x}{dt^2} - \dfrac{20x}{t^2} = t^3$

3. $\phi(t) = e^{3t}$; $\dfrac{d^3x}{dt^3} - 9\dfrac{d^2x}{dt^2} = 0$

4. $\phi(t) = te^{3t}$; $\dfrac{d^2x}{dt^2} - 9\dfrac{dx}{dt} = 6e^{3t}$

5. $\phi(t) = \ln(-t)$, $t < 0$; $tx' = 1$

6. $\phi(t) = \begin{cases} t^2, & t > 0 \\ 3t^2, & t < 0 \end{cases}$; $tx' - 2x = 0$

In Exercises 7 to 12, find all values of the constant k for which the given function $x = \phi(t)$ is a solution of the given o.d.e.

7. $\phi(t) = t^k$, $t > 0$; $t^2x'' - 6x = 0$

8. $\phi(t) = e^{kt}$; $\dfrac{d^2x}{dt^2} - x = 0$

9. $\phi(t) = k$; $\dfrac{d^7x}{dt^7} + \dfrac{dx}{dt} - x = 7$