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an Application of Stokes' Theorem

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PREFACE

The following pages deal with *"processes" on manifolds*.
A process is a solution of differential equations. These equations are supposed to depend on parameters, called control-variables, which parameters control the process.
The main topic in the following pages is dealt with in Chapter II, where a *maximum-principle*, which has to be satisfied by the control of an *"optimal process"* is established. This maximum-principle is of the first importance since it reduces a problem of optimal control to a problem of mathematical programming. For solving problems of mathematical programming a lot of relevant work has been done.
A maximum-principle for processes described by ordinary differential equations was proved by *Pontryagin* and his coworkers in 1958. Therefore this principle is known as Pontryagin's maximum-principle, though *Hestenes* proved this principle as early as 1950 in a report published by Rand Corporation.
We shall follow a general approach by considering a set of differential equations on a manifold. Applying variational methods we arrive at an expression where *Stokes' theorem* turns out to be pivotal. Application of this theorem leads to the "adjoint equations". We then obtain a fundamental inequality, from which we arrive at the maximum-principle by introducing a boundary-condition, which essentially implies a way for finding the appropriate additional conditions for the "adjoint functions", satisfying the adjoint equations.
Chapter I is introductory; a summing up concerning linear alternating functions, integration on manifolds and Stokes' theorem is

given. For a more complete treatment we refer to the book *"Analysis, Manifolds and Physics"*, written by *Yvonne Choquet-Bruhat, Cécile de Witt-Morette* and *Margaret Dillard-Bleick* (North Holland Publishing Company, 1977).

In Chapter II processes defined by a set of differential equations on a manifold are dealt with. By applying Stokes'theorem we obtain a fundamental inequality, from which we start to find the maximum-principle for an optimal process.

In the next three Chapters we specialize, after having made some introductory observations, to ordinary differential equations and to first and second order partial differential equations. Application of the results of Chapter II requires verification of the assumption concerning a *"local variation"*, made in the general approach. Therefore the main problem in these Chapters is to examine the effect of a local variation.

CONTENTS.

I. INTRODUCTION

1. *Manifolds.*

A *manifold* may be considered as a generalization of a surface in euclidean 3-space, which surface is represented locally bij a parametrization

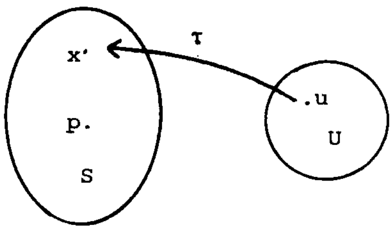
$$x = x(v_1, v_2)$$

$$y = y(v_1, v_2)$$

$$z = z(v_1, v_2)$$

depending on two parameters v_1, v_2. The derivatives of these functions, which functions are supposed to be sufficiently smooth, define two tangent vectors B_1 and B_2. If these vectors are independent at a point p then the surface has at p locally the topology of euclidean 2-space, which means that p has an environment on the surface, which environment is homeomorphic to an open subset of euclidean 2-space.
Let X be a subset of euclidean n-space E_n. If each point p of X has a (relatively open) neighbourhood S in X, which neighbourhood is homeomorphic to an open subset U of euclidean m-space E_m then X is called an *m-dimensional manifold*. Notice that every point of X has a neighbourhood in X, which neighbourhood is homeomorphic to E_m.

So there exists a bijective correspondence τ between points $u \in U$ and points $x = \tau(u) \in S$. Let $x_1, x_2, \ldots, x_m$ denote coördinates of a point u of U. These coördinates are called *local coördinates* of the point x; the mapping τ furnishes a *parametrization* of X.



We shall be concerned with *differential manifolds*, which means that τ and its inverse τ^{-1} are both continuously differentiable. Therefore continuous linear mappings $d\tau$ and $d\tau^{-1}$ exist such that

$$\| \tau(u+h) - \tau(u) - d\tau h \| = o(\|h\|), \qquad u \in U, u+h \in U$$

and

$$\| \tau^{-1}(x+h) - \tau^{-1}(x) - d\tau^{-1} h \| = o(\|h\|), \qquad x \in S, x+h \in S$$

where the norm denotes the euclidean norm. So

$$\lim_{\|h\| \to 0} \frac{\| \tau(u+h) - \tau(u) - d\tau h \|}{\|h\|} = 0$$

and

$$\lim_{\|h\| \to 0} \frac{\| \tau^{-1}(x+h) - \tau^{-1}(x) - d\tau^{-1} h \|}{\|h\|} = 0$$

Such a mapping τ is called a *diffeomorphism*.

Let $e_1, e_2, \ldots, e_m$ denote a basis of E_m. If $\tau_1, \tau_2, \ldots, \tau_n$ denote the coördinates of $x = \tau(u)$ in E_n and if we put

$$\tau_i(u+h) - \tau_i(u) = \sum_{j=1}^{m} B_j^i h_j + R_i , \qquad i=1,2,\ldots,n$$

where $\sum B_j^i h_j = d\tau_i h = \sum h_j d\tau_i e_j$, h_j denoting components of h and where R_i is such that

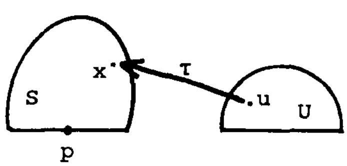
$$\|R_i\| = o(\|h\|)$$

then clearly

$$B_j^i = \frac{\partial \tau_i}{\partial x_j}$$

So the functions τ_i have continuous partial derivatives. The vectors $B_j = d\tau e_j$ with components B_j^i in E_n constitute the columns of a matrix B, which matrix represents the linear mapping $d\tau$. Because τ is a diffeomorphism B has rank m. The vectors in E_m, which vectors are imagined to originate from the point u are mapped by $d\tau$ onto vectors of the *tangentspace*, denoted by $T_x(X)$. The vectors $B_1, B_2, \ldots, B_m$ form a basis for this linear space. Therefore the dimension of the tangentspace $T_x(X)$ equals m.

Thusfar we considered manifolds without boundary. Let now X be a subset of euclidean n-space E_n such that each point p of X has a neighbourhood S in X, which neighbourhood is diffeomorphic to an open subset U of euclidean halfspace $H_m = \{u \in E_m | x_m \geq 0\}$. Then X is an m-dimensional *manifold with boundary* ∂X. So to every point $u \in U$ corresponds bijectively a point $x = \tau(u) \in S$. Since τ is a diffeomorphism a continuous linear mapping $d\tau$ exists such that

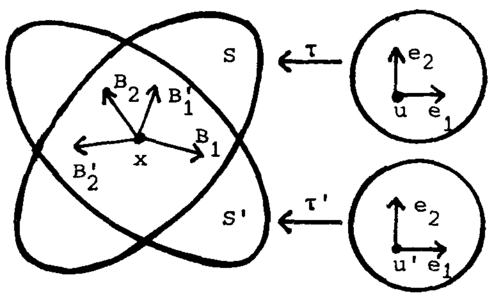


$$||\tau(u+h) - \tau(u) - d\tau h|| = o(||h||) , \qquad u \in U,\ u+h \in U$$

By this mapping $d\tau$ the vectors h in E_m, which vectors are imagined to originate from the point u are mapped onto the *tangentspace* $T_x(X)$. So $d\tau$ maps the vectors h of E_m and is not restricted to vectors of H_m, even if u is a boundary-point of H_m.

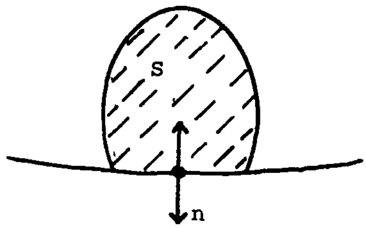
A *boundary-point* of X is the image of a boundary-point of H_m. So x is a boundary-point of X if $x_m = 0$. The *boundary* ∂X of X, *which boundary is an (m-1) - dimensional manifold* consists of the boundary-points of X.

Let us consider an m-dimensional manifold X with boundary ∂X and let $e_1, e_2, \ldots, e_m$ denote a basis of E_m, to which basis we assign a positive orientation. Another basis $Ae_1, Ae_2, \ldots, Ae_m$ of E_m is said to be equivalently oriented if the determinant of the linear transformation A is positive. So we distinguish two orientations, positive orientations and negative orientations . Let $x = \tau(u)$ be a point of X. So x is a point of an open subset S of X, which subset is diffeomorphic to an open subset U of H_m. By $d\tau$ the basis $e_1, e_2, \ldots, e_m$ is mapped onto a basis $B_1, B_2, \ldots, B_m$ of $T_x(X)$. Now suppose that $x = \tau'(u')$, where τ' is another diffeomorphism of the parametrization of X. By $d\tau'$ the basis $e_1, e_2, \ldots, e_m$ is mapped onto a basis $B'_1, B'_2, \ldots, B'_m$ of $T_x(X)$. If the bases $B_1, B_2, \ldots, B_m$ and $B'_1, B'_2, \ldots, B'_m$ are equivalently oriented (for any point x of X) then X is said to be oriented. This means that both bases of $T_x(X)$ are connected bij a linear transformation with positive determinant.



A manifold is *orientable* if it may be oriented. So a manifold is orientable if it admits a parametrization such that on an intersection $S \cap S'$ the Jacobian determinant $\det \frac{\partial(x_1',x_2',\ldots,x_m')}{\partial(x_1,x_2,\ldots,x_m)}$ is positive, where $x_1,x_2,\ldots,x_m$ are coördinates on S and $x_1',x_2',\ldots x_m'$ are coördinates on S'.

Let x be a boundary-point of X. We distinguish two unit vectors in $T_x(X)$, which vectors are perpendicular to $T_x(\partial X)$. One of these unit vectors is mapped bij $d\tau^{-1}$ into H_m and the other of these unit vectors, which vector is called the *outer normal* n is mapped by $d\tau^{-1}$ into $-H_m$. Now if we declare the orientation of a basis $b_1,b_2,\ldots,b_{m-1}$ of $T_x(\partial X)$ to have the same sign as the orientation of the basis n, $b_1,b_2,\ldots,b_{m-1}$ of $T_x(X)$ then an orientation of X clearly induces an orientation of $T_x(\partial X)$, the *"boundary-orientation"*.



2. *Linear Alternating Functions*

Let V_m denote an m-dimensional linear space over the real numbers. On V_m a function F is said to be a k-*tensor*, depending on k arguments $v_1 \in V_m, v_2 \in V_m, \ldots, v_k \in V_m$, if F is a real valued function on $V_m^k = V_m \times V_m \times \ldots \times V_m$, which function is linear in each argument v_j. So

$$F(v_1,v_2,\ldots,v_j+\lambda w_j,\ldots,v_k) = F(v_1,v_2,\ldots,v_j,\ldots,v_k) + \lambda F(v_1,v_2,\ldots,w_j,\ldots,v_k),$$

where $w_j \in V_m$ and where λ denotes a real number.

A k-tensor F is said to be a k-*form* or *form of degree* k if it is alternating, which means that

$$F(v_{i_1},v_{i_2},\ldots,v_{i_k}) = \mathrm{sgn}(i)\ F(v_1,v_2,\ldots,v_k)$$

where sgn(i) is the sign of the permutation $i_1,i_2,\ldots,i_k$ of the numbers $1,2,\ldots,k$. From this definition it follows that if F is a k-form and if $v_1,v_2,\ldots,v_k$ are linearly dependent then $F(v_1,v_2,\ldots v_k) = 0$. The space of all k-forms is denoted by Ω^k.

The *sum* $F_1 + F_2$ ot two k-forms F_1, F_2 is the k-form, defined by

$$(F_1+F_2)(v_1,v_2,\ldots,v_k) = F_1(v_1,v_2,\ldots,v_k) + F_2(v_1,v_2,\ldots,v_k)$$

The *tensor-product* $F_1 \otimes F_2$ of a k_1-form F_1 and a k_2-form F_2 is the (k_1+k_2)-form, defined by

$$F_1 \otimes F_2(v_1,v_2,\ldots v_{k_1+k_2}) = F_1(v_1,v_2,\ldots,v_{k_1})\,F_2(v_{k_1+1},v_{k_2+2},\ldots,v_{k_1+k_2})$$

Generally the tensor-product $F_1 \otimes F_2$ of a k_1-form F_1 and a k_2-form F_2 is not alternating. Therefore we make use of the product $F_1 \wedge F_2$ *("wedge-product", "Grassmann-product")* defined by

$$F_1 \wedge F_2(v_1,v_2,\ldots,v_{k_1+k_2}) = \frac{1}{k_1!k_2!} \sum \mathrm{sgn}(i)\ F_1 \otimes F_2(v_{i_1},v_{i_2},\ldots,v_{i_{k_1+k_2}}) ,$$

where sgn(i) is the sign of the permutation $i_1, i_2, \ldots, i_{k_1+k_2}$ and where summation extends over all permutations of the numbers $1, 2, \ldots, k_1 + k_2$. This product is

- distributive over addition:

$$F \wedge (G_1+G_2) = F \wedge G_1 + F \wedge G_2$$

- not commutative:

$$F_1 \wedge F_2 = (-1)^{k_1 k_2} F_2 \wedge F_1$$

- associative :

$$(F_1 \wedge F_2) \wedge F_3 = F_1 \wedge (F_2 \wedge F_3)$$

Next we will consider special k-forms. Let $e_1, e_2, \ldots, e_m$ be a basis of V_m and let $v = \sum \alpha_i e_i$ be an element of V_m. We introduce the 1-forms ϕ_i, defined by

$$\phi_i : \quad v \rightarrow \alpha_i \ , \qquad\qquad i=1,2,\ldots,m$$

From these 1-forms we obtain k-forms by applying the wedge-product; by

$$\phi_i = \phi_{i_1} \wedge \phi_{i_2} \wedge \ldots \wedge \phi_{i_k} \ ,$$

where i denotes the index-sequence $i_1, i_2, \ldots, i_k$ a k-form is defined, which k-form vanishes if i contains two equal indices.

Now suppose F to be a k-form. Then F is uniquely expressed by

$$F = \sum a_i \phi_i \ , \qquad 1 \le i_1 < i_2 < \ldots < i_k \le m$$

where summation extends over all strictly increasing index-sequences $i_1, i_2, \ldots, i_k$ and where

$$a_i = F(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$$

The k-forms ϕ_i form a basis for the space Ω^k. Therefore the dimension of Ω^k equals $\binom{m}{k}$. If $k = m$ then Ω^k is a one dimensional space; an m-form is the determinant of the matrix with rows $v_1, v_2, \ldots, v_m$ except for a scalar factor.

A transformation

$$e' = Ae \ ,$$

which connects another basis $e'_1, e'_2, \ldots, e'_m$ denoted by e' with the basis $e_1, e_2, \ldots, e_m$ denoted by e, causes a change of the components a_i. Let with respect to e'

$$F = \sum a'_j \phi'_j$$

where

$$\phi'_j = \phi'_{j_1} \wedge \phi'_{j_2} \wedge \ldots \wedge \phi'_{j_k}$$

then

$$a'_j = F(e'_{j_1}, e'_{j_2}, \ldots, e'_{j_k}) = \sum a_i \frac{\partial e'_j}{\partial e_i} ,$$

where $\frac{\partial e'_j}{\partial e_i}$ indicates the determinant of the matrix obtained from A by omitting rows and columns that have numbers which are not contained in j or i respectively.

3. *Differential Forms*

Let X be an m-dimensional manifold. If F is a function that assigns to each point x of X a k-form on $T_x(X)$ then F is said to be a *k-form on X*. A vector v of $T_x(X)$ is a linear combination $\sum h_j B_j$ of the basis vectors $B_1, B_2, \ldots, B_m$. If dx_i denotes the 1-form

$$dx_i : \quad v \longrightarrow h_i , \qquad i=1,2,\ldots,m$$

then $dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$, where the index-sequence $i_1, i_2, \ldots, i_k$ is strictly increasing, denotes a basis element of the space of all k-forms on $T_x(X)$. Hence a k-form F is uniquely expressed by a linear combination

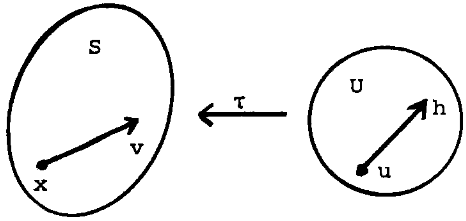
$$\sum a_i(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k} , \qquad 1 \le i_1 < i_2 < \ldots < i_k \le m$$

where summation ranges over all strictly increasing index-sequences $i_1, i_2, \ldots, i_k$, indicated by i. In view of this expression F is said to be an (exterior) *differential form of degree k*. In fact such a form "pulls back" the k-form F into E_m. To make this clear we observe that by the linear mapping $d\tau$ a vector $v = \sum h_j B_j = d\tau \sum h_j e_j = d\tau h$ of $T_x(X)$

corresponds to a vector h in E_m. So a k-form $\tau^* F$ on U is defined by

$$\tau^* F\ (u,h) = F\ (x,v)$$

where $x = \tau(u)$ and where v stands for vectors



$v_1, v_2, \ldots, v_k$, which vectors correspond to vectors $h_1, h_2, \ldots, h_k$, indicated by h in E_m. Thus a differential form $\tau^* F$ on U is induced from F by τ. The mapping τ^* is said to *pull back* the k-form F onto U.

Now suppose τ' to be another diffeomorphism, which maps an open subset U' of H_m onto a neighbourhood S' of $x = \tau'(u')$ in X. Let

$dx'_j : \sum h'_i B'_i = d\tau' \sum h'_i e'_i \longrightarrow h'_j$,then

$$\tau'^* F = \sum a'_j(x) dx'_{j_1} \wedge dx'_{j_2} \wedge \ldots \wedge dx'_{j_k}, \quad 1 \le j_1 < j_2 < \ldots < j_k \le m$$

where

$$a'_j(x) = \sum a_i(x) \frac{\partial e'_j}{\partial e_i} = \sum a_i(x) \det \frac{\partial (x_{i_1}, x_{i_2}, \ldots, x_{i_k})}{\partial (x'_{j_1}, x'_{j_2}, \ldots, x'_{j_k})}$$

in which expression $\det \frac{\partial (x_{i_1}, x_{i_2}, \ldots, x_{i_k})}{\partial (x'_{j_1}, x'_{j_2}, \ldots, x'_{j_k})}$ denotes Jacobian determinants. Hence

$$\tau'^* F = \sum \sum a_i(x) \det \frac{\partial (x_{i_1}, x_{i_2}, \ldots, x_{i_k})}{\partial (x'_{j_1}, x'_{j_2}, \ldots, x'_{j_k})} dx'_{j_1} \wedge dx'_{j_2} \wedge \ldots \wedge dx'_{j_k}.$$

which differential form is obtained from $\sum a_i(x) dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$ by substituting