

# Discrete and System Models

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With 133 Illustrations



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## Preface

The purpose of this four volume series is to make available for college teachers and students samples of important and realistic applications of mathematics which can be covered in undergraduate programs. The goal is to provide illustrations of how modern mathematics is actually employed to solve relevant contemporary problems. Although these independent chapters were prepared primarily for teachers in the general mathematical sciences, they should prove valuable to students, teachers, and research scientists in many of the fields of application as well. Prerequisites for each chapter and suggestions for the teacher are provided. Several of these chapters have been tested in a variety of classroom settings, and all have undergone extensive peer review and revision. Illustrations and exercises are included in most chapters. Some units can be covered in one class, whereas others provide sufficient material for a few weeks of class time.

Volume 1 contains 23 chapters and deals with differential equations and, in the last four chapters, problems leading to partial differential equations. Applications are taken from medicine, biology, traffic systems and several other fields. The 14 chapters in Volume 2 are devoted mostly to problems arising in political science, but they also address questions appearing in sociology and ecology. Topics covered include voting systems, weighted voting, proportional representation, coalitional values, and committees. The 14 chapters in Volume 3 emphasize discrete mathematical methods such as those which arise in graph theory, combinatorics, and networks. These techniques are used to study problems in economics, traffic theory, operations research, decision theory, and other fields. Volume 4 has 12 chapters concerned with mathematical models in the life sciences. These include aspects of population growth and behavior, biomedicine (epidemics, genetics and bio-engineering), and ecology.

These four volumes are the result of two educational projects sponsored by The Mathematical Association of America (MAA) and supported in part by the National Science Foundation (NSF). The objective was to produce needed material for the undergraduate curriculum. The first project was undertaken by the MAA's Committee on the Undergraduate Program in Mathematics (CUPM). It was entitled *Case Studies and Resource Materials for the Teaching of Applied Mathematics at the Advanced Undergraduate Level*, and it received financial support from NSF grant SED72-07370 between September 1, 1972 and May 31, 1977. This project was completed under the direction of Donald Bushaw. Bushaw and William Lucas served as chairmen of CUPM during this effort, and George Pedrick was involved as the executive director of CUPM. The resulting report, which appeared in late 1976, was entitled *Case Studies in Applied Mathematics*, and it was edited by Maynard Thompson. It contained nine chapters by eleven authors, plus an introductory chapter and a report on classroom trials of the material.

The second project was initiated by the MAA's Committee on Institutes and Workshops (CIW). It was a summer workshop of four weeks duration entitled *Modules in Applied Mathematics* which was held at Cornell University in 1976. It was funded in part by NSF grant SED75-00713 and a small supplemental grant SED77-07482 between May 1, 1975 and September 30, 1978. William F. Lucas served as chairman of CIW at the time of the workshop and as director of this project. This activity led to the production of 60 educational modules by 37 authors.

These four volumes contain revised versions of 9 of the 11 chapters from the report *Case Studies in Applied Mathematics*, 52 of the 60 modules from the workshop *Modules in Applied Mathematics*, plus two contributions which were added later (Volume 2, Chapters 7 and 14), for a total of 63 chapters. A preliminary version of the chapter by Steven Brams (Volume 2, Chapter 3), entitled "One Man,  $N$  Votes," was written in connection with the 1976 MAA Workshop. The expanded version presented here was prepared in conjunction with the American Political Science Association's project *Innovation in Instructional Materials* which was supported by NSF grant SED77-18486 under the direction of Sheila K. Mann. The unit was published originally as a monograph entitled *Comparison Voting*, and was distributed to teachers and students for classroom field tests. This chapter was copyrighted by the APSA in 1978 and has been reproduced here with its permission.

An ad hoc committee of the MAA consisting of Edwin Beckenbach, Leonard Gillman, William Lucas, David Roselle, and Alfred Willcox was responsible for supervising the arrangements for publication and some of the extensive efforts that were necessary to obtain NSF approval of publication in this format. The significant contribution of Dr. Willcox throughout should be noted. George Springer also intervened in a crucial way at one point. It should be stressed, however, that any opinions or recommendations

are those of the particular authors, and do not necessarily reflect the views of NSF, MAA, the editors, or any others involved in these project activities.

There are many other individuals who contributed in some way to the realization of these four volumes, and it is impossible to acknowledge all of them here. However, there are two individuals in addition to the authors, editors and people named above who should receive substantial credit for the ultimate appearance of this publication. Katherine B. Magann, who had provided many years of dedicated service to CUPM prior to the closing of the CUPM office, accomplished the production of the report *Case Studies in Applied Mathematics*. Carolyn D. Lucas assisted in the running of the 1976 MAA Workshop, supervised the production of the resulting sixty modules, and served as managing editor for the publication of these four volumes. Without her efforts and perseverance the final product of this major project might not have been realized.

July 1982

W. F. LUCAS

## Preface for Volume 3

This volume contains a rather broad variety of problems and approaches which illustrate the nature of mathematical modeling applied to current problems. The main emphasis is on discrete models, but not exclusively so. The problems presented arise in economics, traffic theory, operations research, decision theory, and other areas. Techniques from graph theory, combinatorics, and optimization are frequently introduced and employed. In several cases, detailed algorithms for obtaining optimal solutions are presented. Some chapters provide concise introductions to basic theories in discrete mathematics, whereas other chapters discuss quite novel new approaches to mathematical modeling which should prove to be fundamental methods in the future. The wealth of material in this volume should prove most useful for courses in discrete mathematics as well as mathematical modeling courses at various levels.

The initial chapter by Frauenthal and Saaty provides some thirty rather elementary problems which can be solved by either quick mental insights or else by more involved calculations or lengthy argument. This interesting collection illustrates the need for imagination and creativity in modeling and problem solving.

Chapter 2 through 5 present several elementary models or approaches which are not normally found in the standard text books. Baker and Marrero use techniques from analytic geometry and calculus to deal with five nautical models and an inventory problem, respectively. Simulation methods are employed by Packel to analyze the best form of traffic control at a street intersection. Greenspan provides a discrete approach to study the fall of a body due to gravity. Such arithmetical models can frequently provide alternatives to the more continuous methods usually employed in the physical

sciences, and in light of modern digital computers they may well have major impact on future analytical methods and modeling.

Chapters 6 through 9 provide a selection of recent applications of modern finite mathematics. Topics from combinatorics, graph theory and network flows are applied to a broad variety of contemporary modeling problems. Tucker and Bodin discuss methods for obtaining efficient routes for sweeping a city's streets. They provide algorithms and computational procedures as well. Prather presents a survey of covering problems which arise frequently and in various fields, and he provides both branching and algebraic methods for solution. Perry makes use of pulse processes through directed graphs to analyze athletic financing at a major university. Peterson describes recently developed techniques for predicting the level of traffic congestion during urban and metropolitan rush hours.

Chapters 10 and 14 by Zahavi are concerned with efficiency in electric power generation systems. These units consider an optimal mix problem in capacity expansion, and cost and reliability calculations given supply and demand uncertainty. Some knowledge of nonlinear programming is presumed in Chapter 10, and some basic probability theory is used in Chapter 14.

Chapters 11 to 13 describe in some detail three fundamental approaches to recently developed theories which are important in a broad spectrum of applications. Saaty describes his new theory of analytical hierarchies which is designed to obtain a ratio scale for a decisionmaker's preferences. Techniques for obtaining such measures of intensity are necessary in modeling many problems arising throughout the social, decisional and managerial sciences. Weber discusses multiple-choice testing and considers different weighting and scoring schemes as well as optimal response strategies and systems for eliciting subjective probabilities. Todd gives an introduction to the recently developing field of path following algorithms for computing fixed points of mappings. He provides several applications of these algorithms, including the determination of equilibrium points in economic models.

July, 1982

WILLIAM F. LUCAS

FRED S. ROBERTS

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## CHAPTER 1

# Foresight—Insight—Hindsight<sup>1</sup>

James C. Frauenthal\*

Thomas L. Saaty\*\*

## Introduction

Frequently, mathematical models describing problems stated in words appear trivial in hindsight. This is because the crucial insight needed to formulate the model is divulged by the structure of the model.

To be able to teach this critical inductive step in the same fashion as the deductive steps in mathematical analysis would be desirable but is probably impossible. However, to see this step detailed for a number of apparently difficult and confusing problems may be instructive. The problems here are chosen in most cases so that, once the insight is gained, the model and its solution are trivial and can be analyzed without pencil and paper. At best, the examples provided will illustrate an incisive way of thinking which can be carried over to more difficult problems; at worst, the student will be left with insight into several unimportant problems.

The format of the chapter is as follows. First, a simple problem is stated in words. Next, what appears to be a natural way of looking at the problem is given with whatever hidden constraining or unnecessary assumptions it may have. The next step is to give the critical insight. Finally, the model and its solution are described. The process of modeling may be explained in terms of a simple picture as depicted in Figure 1.1. The cycle can then be made concrete by using some of the problems in the module as illustrations. Finally, some of the unused problems can be assigned to students as home-

<sup>1</sup> An adaptation of this article has been published in the *Two-Year College Mathematics Journal*, vol. 10, pp. 245–254, 1979.

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