RUDIMENTS OF RAMSEY THEORY

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Preface

It is no exaggeration to say that within the past 10 years there has been a veritable explosion of activity in the general field of combinatorics. Within this domain, one particular subject has enjoyed even more remarkable growth. This subject is Ramsey theory, the topic of these lecture notes. The notes are based rather closely on lectures given at a Regional Conference at St. Olaf College in June, 1979. It was the purpose of the lectures to develop the background necessary for an understanding of these recent developments in Ramsey theory. In keeping with the style of the lectures, the notes are informal. However, complete proofs are given for most of the basic results presented. In addition, many useful results may be found in the exercises and problems.

I wish to thank all the participants at the Conference for providing the stimulating atmosphere so beneficial to meetings of this sort. In particular, special thanks go to Fred Abramson, Lowell Beineke, Tom Brown, Stefan Burr, Fan Chung, Linda Lesniak, George Mills, Mel Nathanson, Jerry Paul and John Selfridge for their illuminating auxiliary lectures on various topics in Ramsey theory. The critical comments of Tom Brown, Stefan Burr and Mark Haiman on portions of the text have been most helpful. Finally, without the beautiful organization and gracious hospitality of Rich Allen and Cliff Corzatt (and the support of the National Science Foundation) the meeting would not have been possible.

Contents

Introduction

Loosely speaking, Ramsey theory is that branch of combinatorics which deals with structure which is preserved under partitions. Typically one looks at the following kind of question: If a particular structure (e.g., algebraic, combinatorial or geometric) is arbitrarily partitioned into finitely many classes, what kinds of substructures must always remain intact in at least one of the classes?

For example:

(i) In any partition of the integers into finitely many classes, some class always contains arbitrarily long arithmetic progressions (van der Waerden's theorem);

(ii) For any partition of the k-element subsets of an infinite set S into finitely many classes, there is always an infinite subset of S with all *its* k-element subsets in a single class (Ramsey's theorem);

(iii) For any partition of the points of the plane into finitely many classes, some class always contains three points forming a right triangle of area 1.

During the past few years, a number of spectacular advances have been made in the field of Ramsey theory. These include, for example, the work of Szemerédi and Furstenberg settling the venerable conjecture of Erdös and Turán (that a set of integers with no k-term arithmetic progression must have density zero), the Nešetřil-Rödl theorems on induced Ramsey properties, the results of Paris and Harrington on "large" Ramsey numbers and undecidability in first-order Peano arithmetic, Deuber's solution to the old partition regularity conjecture of Rado, Hindman's surprising generalization of Schur's theorem, and the resolution of Rota's conjecture on Ramsey's theorem for vector spaces by Graham, Leeb and Rothschild. It has also become apparent that the ideas and techniques of Ramsey theory span a rather broad range of mathematical areas, interacting in essential ways with parts of set theory, graph theory, combinatorial number theory, probability theory, analysis and even theoretical computer science.

It is the purpose of these lecture notes to lay the foundation on which much of this recent work is based. Most of what is covered here is treated in considerably more detail in the recent monograph *Ramsey theory* by Graham, Rothschild and Spencer. Indeed, I have borrowed freely from several sections of this book when I felt it was appropriate. On the other hand, a number of the results and proofs given here have not appeared before in the literature.

Chapter 1. Three views of Ramsey theory

There are a number of viewpoints which can be taken when studying various classes of Ramsey theorems. We mention several of these now.

Let $(S, \prec)$ be a (finite) set partially ordered by $\prec$ and having a unique minimal element 0. We say that S is *graded* if all maximal chains from any element $x \in S$ to 0 have the same length. In this case we call this length the *rank* of x, and denote it by $\rho(x)$. We usually denote the set of rank k elements of S by $\left[{S \atop k}\right]$. Examples of this are:

(a) $S = 2^{[n]}$, the collection of subsets of the set $[n] = \{1, 2, \ldots, n\}$ partially ordered by inclusion, and for $x \in S$, $\rho(x) \equiv |x|$, the cardinality of x;

(b) $S =$ the lattice of subspaces of a given n-dimensional vector space V over a fixed finite field $GF(q)$ partially ordered by inclusion, and for $x \in S$, $\rho(x) \equiv$ dimension of x;

(c) $S =$ collection of partitions of $[n]$ partially ordered by refinement, and for the partition x: $B_1 \cup B_2 \cup \cdots \cup B_k$ of $[n]$, $\rho(x) \equiv n - k$.

Let $S = (S_n, \prec)$, $n \in \omega$, be a sequence of graded partially ordered sets. We say that S has the Ramsey property if for any[1] $k, l, r \in \omega$ there is an n such that if the rank k elements of S_n are arbitrarily partitioned into r classes, there is always a rank l element $y \in S_n$ such that all rank k elements x with $x \prec y$ belong to a single class. More symbolically:

For all $k, l, r \in \omega$ there exists n such that for

all λ: $\begin{bmatrix} S_n \\ k \end{bmatrix} \longrightarrow [r]$ there exists $y \in \begin{bmatrix} S_n \\ l \end{bmatrix}$ and

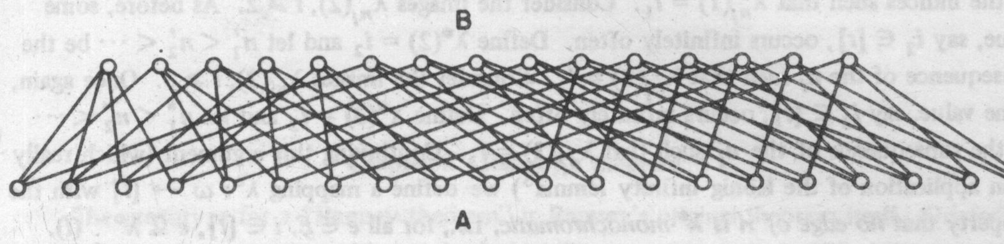
$$i \in [r] \text{ so that } \left\{ x \in \begin{bmatrix} S_n \\ k \end{bmatrix} : x \prec y \right\} \subseteq \lambda^{-1}(i).$$

The reader is invited to try this statement out for various families S, for example, with S_n taken to be sets S of maximum rank n in (a), (b), (c) (as well as for other families). We will see proofs for these particular cases in later sections.

For another viewpoint, let us consider a bipartite graph G with vertex sets A and B and edge set $E \subseteq A \times B$. We say that G is *r-Ramsey* if for all mappings $\lambda: B \longrightarrow [r]$ there is an $x \in A$ such that, for some $i \in [r]$, $\{y \in B: (x, y) \in E\} \subseteq \lambda^{-1}(i)$. Much of Ramsey theory can be reduced to determining whether particular graphs are r-Ramsey. However, while conceptually simple, this formulation has not (so far) contributed very much to the

[1] Normally we treat the case that k, l or r is 0 as holding vacuously.

solution of specific questions in Ramsey theory. The reason may be simply that it is so general that it is usually not able to take advantage of the special structure of the particular problem at hand. For example, consider the bipartite graph G shown in Figure 1.1.



G

FIGURE 1.1

It is a fact that G is 2-Ramsey. However, a direct verification of this could involve checking each of the 2^{20} mappings $\lambda: B \longrightarrow [2]$. In fact, G is exactly the graph obtained by (suitably) identifying A and B with the 3-sets and 2-sets, respectively, of [6] and forming an edge between $x \in A$ and $y \in B$ if $y \subset x$. With this interpretation the fact that G is 2-Ramsey is immediate.

EXERCISE 1.1. (a) Verify the claim that G is 2-Ramsey. (b) Does G remain 2-Ramsey if a vertex of A is deleted? What about deleting two vertices of A?

A final point of view we mention is that of hypergraphs. By a hypergraph $H = H(V, E)$ we mean a set V together with a family E of subsets of V, each containing at least two elements. The *chromatic number* of H, denoted by $\chi(H)$, is defined to be the least integer t such that there is a mapping $\lambda: V \longrightarrow [t]$ so that there is no $e \in E$ and $i \in [t]$ with $e \subseteq \lambda^{-1}(i)$. The term "chromatic" comes from the following interpretation. We imagine the mapping λ to be an assignment of *colors* to the points of H. If all the points of some edge $e \in E$ are assigned the same color, we say that e is *monochromatic*[2] (or monoχ). Thus, $\chi(H) = t$ if t is the least integer for which there is a t-coloring of V forming no monochromatic edge of E.

It is not difficult to see the connection between a t-chromatic hypergraph H and the corresponding (appropriately constructed) $(t-1)$-Ramsey bipartite graph $G = G(H)$.

A fundamental tool which is used quite often in Ramsey theory is (some version of) the compactness theorem of deBruijn and Erdös **[BrE]**. Before stating it we need one more bit of terminology. A hypergraph $G = G(V', E')$ is said to be a ***subhypergraph*** of $H = H(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

COMPACTNESS THEOREM. *If $\chi(H) > t$ and all edges of H are finite then there is a finite subhypergraph G of H with $\chi(G) > t$.*

PROOF. (*Countable case.*) Without loss of generality we can take $V = \omega$. Define H_n to be the subhypergraph with vertex set $V_n = [n]$ and edge set $E_n = \{e \in E: e \subseteq [n]\}$.

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[2] A synonym for this in common use (especially by set theorists) is *homogeneous*.

Suppose $\chi(H_n) \leq t$ for all $n \in \omega$. Thus, there exists λ_n: $[n] \longrightarrow [t]$, such that no monoχ edge is formed. Consider the images $\lambda_n(1)$, $n = 1, 2, 3, \ldots$. By the pigeonhole principle, some value, say $i_1 \in [t]$, occurs infinitely often. Define $\lambda^*(1) = i_1$ and let $n_1 < n_2 < \cdots$ be the indices such that $\lambda_{n_i}(1) = i_1$. Consider the images $\lambda_{n_i}(2)$, $i \geq 2$. As before, some value, say $i_2 \in [t]$, occurs infinitely often. Define $\lambda^*(2) = i_2$ and let $n'_1 < n'_2 < \cdots$ be the subsequence of the n_i such that $\lambda_{n'_i}(2) = i_2$. Consider the images $\lambda_{n'_i}(3)$, $i \geq 3$. Once again, some value, say $i_3 \in [t]$, occurs infinitely often. Define $\lambda^*(3) = i_3$ and let $n''_1 < n''_2 < \cdots$ be the subsequence of the n'_i such that $\lambda_{n''_i}(3) = i_3$. Continuing this argument (which really is an application of the König infinity lemma[3]) we define a mapping λ^*: $\omega \longrightarrow [t]$ with the property that *no edge of* H *is* λ^**-monochromatic*, i.e., for all $e \in E$, $i \in [t]$, $e \not\subseteq \lambda^{*-1}(i)$. This follows immediately from the construction of λ^* since for any n there is an n' (in fact infinitely many) such that

$$\lambda^*(i) = \lambda_{n'}(i), \qquad i \in [n].$$

However, this contradicts the assumption that $\chi(H) > t$ and the proof (for the countable case) is completed. The proof in the general case requires the use of the Axiom of Choice or something equivalent, such as Tychonoff's theorem, and will not be given here. Of course, this result also follows the compactness thorem for propositional calculus.

We remark in closing that it is possible to develop many of the concepts of Ramsey theory in terms of category theory, and indeed, a number of very strong results have recently been obtained using this point of view (e.g., see the papers of Leeb [L], and Nešetřil-Rödl [NR1], [NR3]). However, for the purposes of concreteness and in order not to (unnecessarily) limit the readership we will not pursue this line of development here.

[3] Which asserts that an infinite tree with all vertices having finite degree contains an infinite path.

Chapter 2. Ramsey's theorem

The prototype for a "Ramsey theorem" is Ramsey's original theorem itself. Originally proved for use in connection with certain decidability questions in logic [R], it has proved to be a remarkably fertile seed from which much of Ramsey theory can be traced.

We first consider (as Ramsey did) the infinite version.

RAMSEY'S THEOREM (INFINITE VERSION). *For all k, $r \in \omega$ and any r-coloring χ: $[{\omega \atop k}] \longrightarrow [r]$ of the k-element subsets of ω, there is always an infinite subset $S \subseteq \omega$ with all its k-element subsets having the same color.*

Note. Statements of this type are very often written in the literature with the so-called arrow notation of Erdös and Rado. For example, the above statement can be written simply as $[{\omega \atop k}] \longrightarrow [{\omega \atop k}]_r$ or as $(\omega)^k \longrightarrow (\omega)^k_r$ if the convention is used of denoting the k-subsets of ω by $(\omega)^k$ rather than $[{\omega \atop k}]$. We will occasionally use this arrow notation unless there is danger of no confusion.

PROOF. We first treat the case $k = 2$ since it is easy to visualize. The case $k = 1$ is nothing more than an infinite version of the pigeonhole principle: If an infinite number of pigeons occupy a finite number of pigeonholes then some pigeonhole contains an infinite number of pigeons.

For $k = 2$, we can identify the pairs $[{\omega \atop 2}]$ with *edges* of the complete graph K_ω on ω points. Let χ: $[{\omega \atop 2}] \longrightarrow [r]$ be an arbitrary r-coloring of the edges of K_ω.

(1) Consider the edges of the form $\{0, x\}$, i.e., incident to the point 0. Some color, say c_1, must occur infinitely often. Let $\chi = \{x_i: i \in \omega\}$ be the set of those $x > 0$ with $\chi(\{0, x\}) = c_1$.

(2) Consider the edges of the form $\{x_0, x_i\}$ where $x_i \in X$. Some color, say c_2, must occur infinitely often. Let $Y = \{y_i: i \in \omega\}$ be the set of those $y_i > x_0$ in X with $\chi(\{x_0, y_i\}) = c_2$.

(3) Consider the edges $\{y_0, y_i\}$. Some color, say c_3, must occur infinitely often. Let $Z = \{z_i: i \in \omega\}$ be the set of $z_i > y_0$ in Y with $\chi(\{y_0, z_i\}) = c_3$, etc.

It is clear we can continue in this manner indefinitely. Now, form the infinite set $T = \{0, x_0, y_0, z_0, \ldots\}$.

BASIC FACT. The color of any pair $\{t, t'\} \in [{T \atop 2}]$ depends only on the *value* of $\min\{t, t'\}$. Thus, we can associate to each integer $t \in T$ a new color $\chi^*(t) \in [r]$ (well-) defined by

$$\chi^*(t) = \chi(\{t, t'\}) \quad \text{for } t' > t \text{ in } T.$$

By the pigeonhole principle some *infinite* subset $S \subseteq T$ must be monochromatic under χ^*, i.e., all $\chi^*(s)$, $s \in S$, are the same. But by the definition of χ^* this just means that all $\{s, s'\}$ have the same color under χ. This proves Ramsey's theorem for $k = 2$.

We can summarize the preceding argument as follows.

(i) From the given coloring $\chi: [{\omega \atop 2}] \longrightarrow [r]$ define a new "induced" coloring χ^*: $[{\omega \atop 1}] \longrightarrow [r]$ having rather regular structure on a (large) subset of ω.

(ii) Apply (by induction) the corresponding Ramsey theorem for $[{\omega \atop 1}]$ to χ^*.

This general format occurs over and over in Ramsey theory. It is probably the single most useful approach in proving the existence of a Ramsey theorem for a class of structures. The reader can expect to see this technique occurring in a variety of guises throughout this monograph.

As an example, we continue our proof of Ramsey's theorem by considering the case $k = 3$. Given the initial r-coloring $\chi: [{\omega \atop 3}] \longrightarrow [r]$, we define an induced r-coloring χ_1 of the pairs of $X = \omega - \{0\}$ by $\chi_1(\{x, x'\}) = \chi(\{0, x, x'\})$. By Ramsey's theorem for $k = 2$, X contains an infinite monoχ_1 set $X' = \{x_i'; i \in \omega\}$ (i.e., all the values $\chi_1(\{x_i', x_j'\})$ are the same), say having color c_1. Next, define an induced r-coloring χ_2 on $[{Y \atop 2}]$ where $Y = X' - \{x_0'\}$ by

$$\chi_2(\{y, y'\}) = (\{x_0', y, y'\}).$$

As before, by Ramsey's theorem for $k = 2$, Y contains an infinite monoχ_2 set $Y' = \{y_i': i \in \omega\}$, say having color c_2. Of course, we next define an induced r-coloring χ_3 on $[{Z \atop 2}]$ where $Z = Y' - \{y_0'\}$ by $\chi_3(\{z, z'\}) = \chi(\{y_0', z, z'\})$, etc.

As before we finally form a set $T = \{0, x_0', y_0', \ldots\}$ which by construction has the property that the color of any triple $\{t, t', t''\}$ depends only on $\min\{t, t', t''\}$. From T we can then form the desired set S with all its triples having a single color.

The proof for general k follows exactly the same lines. This proves Ramsey's theorem. □

The finite version of Ramsey's theorem usually has the following form.

RAMSEY'S THEOREM (FINITE VERSION). *For all $k, l, r \in \omega$ there exists $n(k, l, r) \in \omega$ such that if $n \geq n(k, l, r)$ and $\chi: [{n \atop k}] \longrightarrow [r]$ is any r-coloring of the k-subsets of $[n]$ then some l-subset of $[n]$ has all its k-subsets with the same color.*

(More briefly: For all $k, l, r \in \omega$ there exists $n(k, l, r) \in \omega$ such that $n \geq n(k, l, r)$ implies $[{n \atop k}] \longrightarrow [{l \atop k}]_r$.)

The finite version of Ramsey's theorem follows at once from the infinite version by the Compactness Theorem. Unfortunately, this type of proof gives no estimate for the minimum possible values for the numbers $n(k, l, r)$. These numbers, known as the (classical) Ramsey numbers, and denoted by $R(k, l, r)$, have been studied extensively during the past 50 years. In spite of this effort, however, there is relatively little known about them.

All known (nontrivial) values are listed in Table 2.1.

k	l	r	$R(k, l, r)$
2	3	2	6
2	4	2	18
2	3	3	17
2	5	2	42–55

TABLE 2.1

Of course,

$$R(1, l, r) = (l - 1)r + 1, \qquad R(k, k, r) = k.$$

The strongest general bounds on $R(k, l, r)$ known are obtained by use of the probabilistic method of Erdös. Not surprisingly, it is also to him that the earliest (and still among the best) bounds are due.

We illustrate a simple version of the method in the following result: We abbreviate $R(2, k, 2)$ by $R(k)$.

THEOREM (ERDÖS [E1]).

(2.1) $$R(k) > ck2^{k/2}$$

for some fixed constant $c > 0$.

PROOF. Consider the complete graph K_n on the vertex set $[n]$. Let us call a 2-coloring of the edges of K_n *good* if it contains a monoχ copy of K_k. For each choice of k points X of K_n there are $2 \cdot 2^{\binom{n}{2}-\binom{k}{2}}$ ways to 2-color the edges of K_n so that X spans a monoχ K_k. Since there are just $\binom{n}{k}$ ways of choosing X then there are at most $\binom{n}{k}2^{\binom{n}{2}-\binom{k}{2}+1}$ good colorings. Thus, if

$$\binom{n}{k}2^{\binom{n}{2}-\binom{k}{2}+1} < 2^{\binom{n}{2}} \quad \text{(the total number of 2-colorings of } K_n\text{)}$$

i.e., if

(2.2) $$\binom{n}{k} < 2^{\binom{k}{2}-1}$$

then there exists a 2-coloring of K_n which is *not* good. But this means that $R(k)$ must exceed this value of n since by definition any 2-coloring of $K_{R(k)}$ must be good. Since (2.2) holds for $n > ck2^{k/2}$ then (2.1) is proved. □

The most precise bounds known for Ramsey numbers are (not surprisingly) those for $r(k, 3)$, defined to be the least m such that any 2-coloring of K_m contains either a color 1 K_k or a color 2 K_3. In this case we have

$$\frac{ck^2}{(\log k)^2} < r(k, 3) < \frac{c'k^2}{\log k}$$

for suitable positive constants c, c'. The upper bound is a very recent result of Ajtai, Komlos and Szemerédi which at the time of this writing has not yet appeared. The best *constructive* lower bound known for $R(k)$ is

$$R(k) > \exp(c(\log k)^{4/3}(\log\log k)^{1/3})$$

due to Frankl [Fr] and Chung [Ch]. Erdös is currently offering (U.S.) \$250 for a constructive proof that $R(k) > (1+\epsilon)^k$ for a fixed $\epsilon > 0$.

We conclude this chapter with a typical application of Ramsey's theorem.

PROPOSITION. *Every finite semigroup S contains an idempotent, i.e., an element x such that $x^2 = x$.*

PROOF. Consider an arbitrary sequence $\bar{x} = (x_0, x_1, x_2, \ldots, x_t)$ where $x_i \in S$ and $s = |S|$. Define an s-coloring of K_t with vertex set $[t]$ by $\chi(\{i, j\}) = x_{i+1} \cdot x_{i+2} \cdot \cdots \cdot x_j \in S$ for $i < j$. Taking $t = R(s)$, K_t contains a monoχ triangle, say $\chi(\{i, j\}) = \chi(\{i, k\}) = \chi(\{j, k\}) = x \in S$, $i < j < k$. But this means

$$\prod_{i<\alpha\leq j} x_\alpha = \prod_{j<\beta\leq k} x_\beta = \prod_{i<\gamma\leq k} x_\gamma = \left(\prod_{i<\alpha\leq j} x_\alpha\right)\left(\prod_{j<\beta\leq k} x_\beta\right) = x,$$

i.e., $x = x^2$. □

EXERCISE 2.1. Define the *off-diagonal* Ramsey number $R(k, l)$ to be the least integer m such that if the edges of K_n are arbitrarily 2-colored, say using red and blue, then there is always either a red K_k or a blue K_l formed.

(a) Show that $R(k, l) \leq R(k-1, l) + R(k, l-1)$.

(b) Deduce an upper bound for $R(k, l)$ and $R(k)$ from this.

EXERCISE 2.2. (a) Show that for each n there exists $f(n)$ such that any set of $f(n)$ points in the plane in general position always contains n points which form a convex n-gon. Estimate f from below and above. (Hint: use Ramsey's theorem.)

(b) What if we require that the convex n-gon contain no other points of the set as interior points?

EXERCISE 2.3. By a *tournament* we mean a complete graph in which every edge is directed. A tournament is *transitive* if $a \to b$ and $b \to c$ implies $a \to c$. Show that for each n there exists $g(n)$ such that any tournament on $g(n)$ points contains a transitive subtournament on n points.

EXERCISE 2.4. Let $A = \{a_1 < a_2 < \cdots\}$ consist of the set $\{4^i + 4^j : 0 \leq i < j\}$. Show:

(i) Every $n \in \omega$ has at most three representations as $a_i + a_j$, $i < j$.

(ii) For any partition of A into finitely many sets, say $A = A_1 \cup \cdots \cup A_r$, for some $A_i = \{a_1' < a_2' < \cdots\}$ infinitely many $n \in \omega$ can be written as $a_i' + a_j'$, $i < j$, in at least three ways. (Hint: Ramsey's theorem, of course.)

EXERCISE 2.5. Let P_n denote the graded set of partitions of $[n]$ partially ordered by refinement, where the rank of a partition $[n] = B_1 \cup \cdots \cup B_k$ is defined to be $n - k$. Show that the family $\{P_n : n \in \omega\}$ has the Ramsey property. (Hint: Consider the sublattice of P_n which is generated by a partition which has largest block size 2.)

Chapter 3. van der Waerden's theorem

In 1927, B. L. van der Waerden published [V1] a proof of the following unexpected theorem:

If the positive integers are partitioned into two classes, then at least one of the classes must contain arbitrarily long arithmetic progressions.

This result is often attributed (as a conjecture) to the Dutch mathematician P. J. H. Baudet, probably because of the title of van der Waerden's paper which first proved it. However, there seems to be strong evidence that it was actually first conjectured by I. Schur in connection with his work on the distribution of quadratic residues modulo p. (The reader can consult A. Brauer's preface to Schur's collected works [Sc2] for an account.) It has turned out to be the genesis of a number of very interesting developments in combinatorics and number theory, some of which we will encounter in later chapters. In this chapter we will examine several proofs of this classical theorem of van der Waerden.

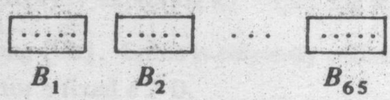
The essential ideas. There are two rather harmless-looking modifications we shall make in the statement of van der Waerden's theorem, each of which affects the proof in a substantial way. First, for each k we shall allow only a *finite* initial segment (depending on k) of Z^+ (the set of positive integers) to be partitioned in order that at least one class always contains a k-term arithmetic progression (A.P.). This modification is attributed to O. Schreier (see [V2]) and is seen to be equivalent to the original assertion by the König infinity lemma. Second, we shall allow the sets to be partitioned into r classes rather than just two. This idea was suggested by E. Artin and is essential to all known proofs of van der Waerden's theorem. With these modifications the "new" version of van der Waerden's theorem becomes:

For $k, r \in \omega$, there exists an integer $W(k, r)$ so that if $[W(k, r)]$ is partitioned into r classes then at least one class contains a k-term A.P.

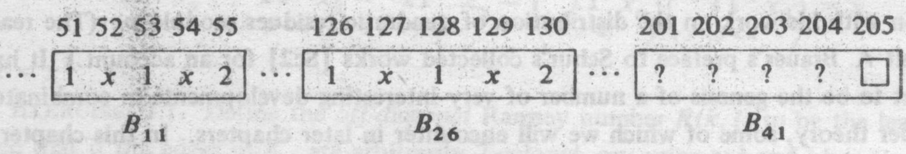
Or, more chromatically:

For all $k, r \in \omega$, there exists $W(k, r) \in \omega$ so that any r-coloring of $[W(k, r)]$ always has a monoχ k-term A.P.

In order to motivate the proof of the general theorem, we first examine a few small cases. To begin with, for $k = 2$ and any r, the result is immediate (in fact, we may choose $W(2, r) = r + 1$). Let us consider the case $k = 3$, $r = 2$. We claim we can take $W(3, 2) = 325$. To see this, assume χ: [325] $\longrightarrow$ [2] is an arbitrary 2-coloring. Let us think of [325] as consisting of 65 consecutive blocks of length 5, i.e., $[325] = [1, 5] \cup [6, 10] \cup \cdots \cup [321, 325]$ which we can write symbolically as



Each B_i has 5 points and therefore can be colored in one of $2^5 = 32$ ways. Thus, among the first 33 blocks, some pair must be colored in exactly the same way (by the pigeonhole principle), say, for example, B_{11} and B_{26}. Let us examine the 2-coloring of $B_{11} = \{51, 52, 53, 54, 55\}$. Among the first *three* elements of B_{11}, at least two must have the same color, say, $\chi(51) = \chi(53) = 1$. If $\chi(55)$ is also 1 we are done; hence, we may assume $\chi(55) = 2$. Let us consider the situation up to this point.



We claim we are finished! For if $\chi(205) = 2$ then 55, 130, 205 is a color 2 3-term A.P. On the other hand, if $\chi(205) = 1$ then 51, 128, 205 is a color 1 3-term A.P.

What we have really done is to "focus" two 2-term A.P.'s having *different* colors on the integer 205 so that no matter which of the two colors it was assigned, it must form the third term of some monoχ A.P.

Let us use the same theme to show that $W(3, 3)$ exists. This time, however, we start with a 3-coloring χ of the first $7(2 \cdot 3^7 + 1)(2 \cdot 3^{7(2 \cdot 3^7 + 1)} + 1)$ integers!

We first divide these integers into $2 \cdot 3^{7(2 \cdot 3^7 + 1)} + 1$ consecutive blocks B_i, each of length $7(2 \cdot 3^7 + 1)$. Now, since there are just $3^{7(2 \cdot 3^7 + 1)}$ ways a block B_i can be 3-colored by χ, then among the first $3^{7(2 \cdot 3^7 + 1)} + 1$ of them, at least two, say B_{i_1} and $B_{i_1 + d_1}$, must be 3-colored by χ in exactly the same way. (The reason we use $2 \cdot 3^{7(2 \cdot 3^7 + 1)} + 1$ blocks is so that the block $B_{i_1 + 2d_1}$ is well defined; we shall soon need it.)

Next, for each i, partition the integers in B_i into $2 \cdot 3^7 + 1$ subblocks $B_{i,j}$ of 7 integers each. Since there are just 3^7 ways of 3-coloring each $B_{i,j}$, then among the first $3^7 + 1$ blocks $B_{i_1,j}$, at least two, say B_{i_1,i_2} and B_{i_1,i_2+d_2}, have exactly the same 3-colorings.

Finally, in the first four elements of B_{i_1,i_2}, some color must occur at least twice, say $\chi(i_3) = \chi(i_3 + d_3) = 1$ where $i_3, i_3 + d_3 \in B_{i_1,i_2}$. Since $i_3 + 2d_3$ is also in B_{i_1,i_2} (this is why we chose it to have length 7) then we may assume without loss of generality that $\chi(i_3 + 2d_3) = 2$ (if $\chi(i_3 + 2d_3) = 1$ we would be done). The current situation is shown in Figure 3.1.

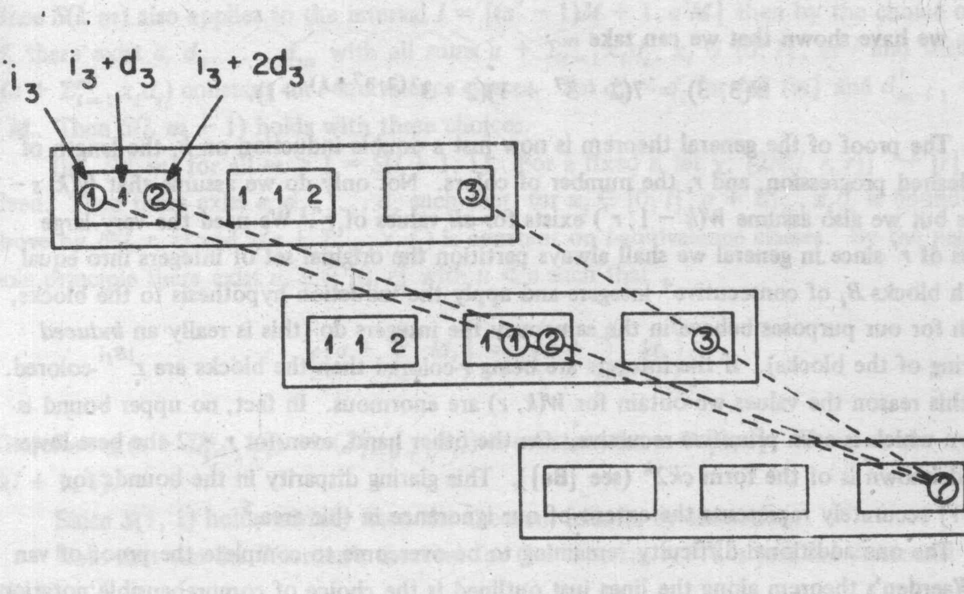


FIGURE 3.1

Consider the subblock B_{i_1,i_2+2d_2}. By the choice of i_2 and d_2, this is a subblock of B_{i_1}. Also since B_{i_1,i_2} and B_{i_1,i_2+d_2} have the same 3-colorings then the integers $i_3 + 7d_2$ and $i_3 + d_3 + 7d_2$ must have color 1 and the integer $i_3 + 2d_3 + 7d_2$ must have color 2. Therefore, the corresponding element $i_3 + 2d_3 + 14d_2 \in B_{i_1,i_2+2d_2}$ must have color 3, because of the potential monochromatic A.P.'s $i_3 + 2d_3$, $i_3 + 2d_3 + 7d_2$, $i_3 + 2d_3 + 14d_2$ and i_3, $i_3 + d_3 + 7d_2$, $i_3 + 2d_3 + 14d_2$. Of course, since B_{i_1} and $B_{i_1+d_1}$ have the same colorings then $x \in B_{i_1}$ and $x + 7(2 \cdot 3^7 + 1)d_1 \in B_{i_1+d_1}$ always have the same color. In particular,

$$\chi(i_3 + 7(2\cdot 3^7 + 1)d_1) = \chi(i_3 + d_3 + 7(2\cdot 3^7 + 1)d_1) = \chi(i_3 + 7d_2 + 7(2\cdot 3^7 + 1)d_1) = 1,$$

$$\chi(i_3 + 2d_3 + 7(2 \cdot 3^7 + 1)d_1) = \chi(i_3 + 2d_3 + 7d_2 + 7(2 \cdot 3^7 + 1)d_1) = 2,$$

and

$$\chi(i_3 + 2d_3 + 14d_2 + 7(3^7 + 1)d_1) = 3.$$

Now, consider the integer $m = i_3 + 2d_3 + 14d_2 + 14(3^7 + 1)d_1$. We claim there are *three* monoχ 2-term A.P.'s focussed on m, all having different colors, so that no matter what the value of $\chi(m)$ is, m will be the third term of a monoχ 3-term A.P. To see this, we simply extract the appropropriate subset from Figure 3.1:

i_3 . . .
$i_3 + d_3 + 7d_2 + 7(3^7 + 1)d_1$ . . . .
$i_3 + 3d_3$ . . . .
$i_3 + 2d_3 + 7d_2 + 7(3^7 + 1)d_1$ . . . . .
$i_3 + 2d_3 + 14d_2$ $\cdots$ $i_3 + 2d_3 + 14d_2 + 7(3^7 + 1)d_1$ $\cdots$ $i_3 + 2d_3 + 14d_2 + 14(3^7 + 1)d_1 = m.$

Thus, we have shown that we can take

$$W(3, 3) = 7(2 \cdot 3^7 + 1)(2 \cdot 3^{7(2 \cdot 3^7 + 1)} + 1).$$

The proof of the general theorem is now just a double induction on k, the length of the desired progression, and r, the number of colors. Not only do we assume that $W(k, s-1)$ exists but we also assume $W(k-1, r')$ exists for *all* values of r'. We need the very large values of r' since in general we shall always partition the original set of integers into equal length blocks B_i of consecutive[4] integers and apply the induction hypothesis to the blocks, which for our purposes behave in the same way the integers do (this is really an *induced* coloring of the blocks). If the integers are being r-colored then the blocks are $r^{|B_i|}$-colored. For this reason the values we obtain for $W(k, r)$ are enormous. In fact, no upper bound is known which is even primitive recursive. On the other hand, even for $r = 2$ the best lower bound known is of the form $ck2^k$ (see [Be]). This glaring disparity in the bounds for $W(k, r)$ accurately represents the extent of our ignorance in this area.[5]

The one additional difficulty remaining to be overcome to complete the proof of van der Waerden's theorem along the lines just outlined is the choice of comprehensible notation.

EXERCISE 3.1. Complete this proof of van der Waerden's theorem (no cheating!).

EXERCISE 3.2. $W(k, r)$ is usually defined to be the *least* value for which van der Waerden's theorem with k-term A.P.'s and r-colors holds. With this convention:

(a) What is $W(3, 2)$? $W(3, 3)$?

(b) What is an upper bound for $W(10, 10)$?

A SHORT PROOF. It is perhaps not surprising that by strengthening the conclusions of van der Waerden's theorem we obtain a somewhat stronger result which at the same time is a bit easier to prove (see [GR2]). We now give such a result. We should point out that the structure of this proof is essentially the same as van der Waerden's original proof.

Let us call two m-tuples $(x_1, \ldots, x_m), (x'_1, \ldots, x'_m) \in [0, l]^m$ *l-equivalent* if they agree up through their last occurrences of l. (Thus, any two l-tuples not containing l are l-equivalent.) For any $l, m \geq 1$, consider the statement:

For any r, there exists $N(l, m, r)$ so that for any r-coloring

$S(l, m)$ $\quad \chi\colon [N(l, m, r)] \longrightarrow [r]$ there exist $a, d_1, \ldots, d_m \in \mathbf{P}$ such that

$\chi(a + \sum_{i=1}^{m} x_i d_i)$ is constant on each l-equivalence class of $[0, l]^m$.

THEOREM. *$S(l, m)$ holds for all $l, m \geq 1$.*

PROOF. (i) $S(l, m)$ for some $m \geq 1 \Rightarrow S(l, m+1)$. For a fixed r, let $M = N(l, m, r)$, $M' = N(l, 1, r^M)$ and suppose $\chi\colon [MM'] \longrightarrow [r]$ is given. Define the induced coloring $\chi'\colon [M'] \longrightarrow [r^M]$ so that $\chi'(k) = \chi'(k')$ iff $\chi(kM - j) = \chi(k'M - j)$ for $0 \leq j < M$. By the inductive hypothesis, there exist a' and d' such that $\chi'(a' + xd')$ is constant for $x \in [0, l-1]$.

[4]In fact, the blocks do not have to be disjoint—just equally spaced.

[5]G. Mills has pointed out that the known values $W(2, 2) = 3$, $W(3, 2) = 9$, $W(4, 2) = 34$ and $W(5, 2) = 178$ are remarkably close to $(3/2)k!$.