Lecture Notes in Mathematics

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Semigroups Theory and Applications

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Proceedings of a Conference held in Oberwolfach, FRG, Feb. 23 – Mar. 1, 1986

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PREFACE

During the week of February 23rd to March 1st, 1986, a conference on semigroups was held at Oberwolfach, Germany, at the Mathematisches Forschungsinstitut. It was organized by H. Jürgensen (The University of Western Ontario), G. Lallement (Pennsylvania State University), and H. J. Weinert (Technische Universität Clausthal). It was the third conference on semigroups held at Oberwolfach, this time with an emphasis on combinatorial semigroups and their applications. The previous ones were held in 1978 and 1981. Their proceedings have been published as volumes 855 and 998 of these Lecture Notes in Mathematics.

The conference was attended by 53 participants from 15 countries: 11 from Germany; 25 from the countries of Czechoslovakia, Finland, France, Hungary, the Netherlands, Poland, Portugal, the Soviet Union, the United Kingdom, and Yugoslavia; 15 from Canada and the United States; 1 from each of Australia and Taiwan. The conference program included 42 lectures, most of which are presented in this volume.

The organizers would like to express their gratitude to the staff at Oberwolfach for creating excellent conditions for the meeting, and to the editors of the Lecture Notes in Mathematics for publishing these proceedings. They also thank all authors and the referees for the work they contributed to the publication of this volume. Special thanks are due to Dr. U. Hebisch (Technische Universität Clausthal) for his continued and indispensable assistance in the preparation of the conference itself and of this volume.

H. Jürgensen, G. Lallement, H. J. Weinert
London (Ontario), University Park (Pennsylvania), and Clausthal-Zellerfeld,
December 1987.

INTRODUCTION

The papers gathered in this volume reflect various trends of research activity over the past several years in pure algebraic semigroup theory, in some areas of theoretical computer science related to semigroup theory (languages, automata, rewriting rules, systems of equations), and in areas of ring theory, universal algebras, and category theory where the objects of interests do have some direct connections with semigroups.

The following brief analysis of the papers regroups them under somewhat artificial headings. This is essentially intended to help the reader gain a better understanding of the general aims of researchers in the various fields mentioned above.

1. Congruences

Unlike in group theory or ring theory, congruences on a semigroup are somewhat difficult to apprehend. In general, subobjects replacing the kernels are not available. Inverse and regular semigroups have proven to offer the best grounds of approach, and the paper by *B. P. Alimpić and D. N. Krgović*, where some classes of congruences on regular semigroups are studied, illustrates perfectly this point.

In the sixties the work of Rhodes on complexity of finite semigroups led him to consider sequences of morphisms collapsing a semigroup to a singleton, each individual morphism of the sequence collapsing as little as possible. The corresponding notion is that of minimal congruence. This is the object of the article by *M. Demlová and V. Koubek* which provides a classification of minimal congruences, and studies their relationship to the extension problem. In the same context, subdirectly irreducible semigroups (i. e. semigroups with a finest congruence distinct from equality) are of interest. An example of structural investigation of this kind for a special class of semigroups is provided by *A. Nagy's* article.

Structural properties of the lattice of all congruences have also been studied. It is well-known, for example, that the lattice of congruences of a completely simple semigroup is semimodular. Here *P. R. Jones* determines almost all varieties of semigroups having a semimodular lattice of congruences and his paper contains results relevant to both congruences and varieties.

2. Varieties and pseudovarieties

Besides the paper by *P. R. Jones* mentioned above, another one by *P. G. Trotter* concentrates on varieties of completely regular semigroups (formerly called unions of groups). These varieties have been vigorously investigated in recent years, e. g. by Petrich, Gerhardt, Jones, and Pollák. Here *P. G. Trotter* determines the injective objects ('injective' means that any morphism $S \to I$ extends to $T \to I$ where T is an extension of S) in several completely regular varieties.

Pseudo-varieties of finite semigroups and monoids are classes closed under sub, quotient, and finite direct products (while for varieties there are no finiteness restrictions). Following Eilenberg's correspondence theorem between varieties of rational languages and pseudo-varieties of monoids, a wealth of activity has been devoted to make this correspondence more precise in special cases. Talks illustrating this were given at the conference by J. Sakarovitch and by H. Straubing and D. Thérien. In the same vein the paper by *J. Almeida* deals with the problem of the connection between a pseudo-variety V of semigroups and the pseudo-variety MV generated by the monoids S^1 for all S in V.

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3. Languages

The relationship between star-free languages and first order logic was established by McNaughton in 1971 (see Counterfree Automata, MIT Press). The connection has been investigated further more recently, especially when similarities were detected between the dot-depth hierarchy of Brzozowski and Knast, and the quantifier alternating depth of first order sentences. The paper by *D. Lippert and W. Thomas*, which clarifies the differences between the dot operation in languages and the existential quantifier in first order formulas, is a contribution to this line of work.

In recent years the Western Ontario school has produced many new results on languages and free semigroups dealing with properties of disjunctive languages, various conditions on codes, and properties of partial orders on free semigroups. The papers by *M. Petrich and G. Thierrin* and by *M. Katsura and H. J. Shyr* illustrate this original approach to the study of languages.

The paper by *G. Pollák* dealing with infima in the power set of a free monoid is more set theoretically oriented but it can also be viewed as a contribution to language theory. I should also mention an interesting lecture by D. Perrin (not reported here) where he uses classical semigroup theory results to investigate properties of infinite words.

4. Presentations, equations in free monoids

R. V. Book gave an overview of results on presentations of semigroups and monoids with the so-called Church-Rosser property. The paper by *K. Madlener and F. Otto* contains numerous results on groups having such presentations. In my own paper I survey most of the known results on the decidability of the word problem for one-relator semigroups, concentrating mostly on results of the Russian school.

The paper by *K. Culik II and J. Karhumäki* deals with a problem related to the Ehrenfeucht conjecture proved in 1985 (Each system of equations over a free monoid A^*, A finite, with finitely many variables, is equivalent to a finite subsystem). The question they consider here is when such a finite subsystem can effectively be found. In another paper on equations, *J.-C. Spehner* uses an earlier result of his on presentations of submonoids of free monoids, to give a classification of certain systems of equations in three variables.

Other important recent developments were presented at the Conference but are not reported in this volume: The plactic monoid and its connections with Young tableaux by M. P. Schützenberger; the study of presentations of inverse semigroups by S. W. Margolis and J. C. Meakin.

5. Inverse semigroups and generalizations

The papers by *N. R. Reilly* and by *G. A. Freiman and B. M. Schein* present problems of interest either directly in the area of inverse semigroups or inspired by inverse semigroups. In her paper, *M. B. Szendrei* studies certain classes of semigroups with involutions and shows that the free objects in these classes admit descriptions that are quite similar to the well-known descriptions of free inverse semigroups e. g. by Scheiblich and Munn. Similarly, *J. Fountain* studies certain free right adequate semigroups (S is right adequate if each $\mathcal{L}^*$-class has an idempotent, where $a\mathcal{L}^*b$ iff $a\mathcal{L}b$ in an oversemigroup, and the idempotents commute). Again the free objects Fountain considers do have descriptions extending those of free inverse semigroups.

6. Semigroups of endomorphisms

V. Fleischer and U. Knauer prove that the endomorphism monoid of an act (i. e. of a monoid acting on a set) has a nice representation as a wreath-product of a monoid and a small category. *S. M. Goberstein* studies more generally correspondences. A correspondence on a universal algebra A is simply a subalgebra of $A \times A$. A survey of known results on correspondences on universal algebras and groups is made, and new results on semigroup correspondences are announced.

7. Semigroups and other algebraic structures

a) In the theory of partial semigroups an extension of $(S_1, \circ_1)$, where $\circ_1$ denotes the partial operation on S_1, is defined as $(S_2, \circ_2)$ such that $S_1 \subseteq S_2$ and $a \circ_1 b = c$ implies $a \circ_2 b = c$. In his paper *E. S. Ljapin* develops a number of conditions for the existence of a semigroup extension for a partial semigroup.

b) A typical example of a "transfer" theorem in the theory of semigroup rings is as follows: The monoid ring $R[M]$ is Artinian if and only if the ring R is Artinian and M is a finite monoid (Zelmanov). *J. Okniński* studies here similar types of transfer theorems with respect to the Krull dimensions of rings.

Based on semimodules over semirings *H. J. Weinert* extends the notion of (generalized) algebras over rings by introducing (generalized) semialgebras over semirings including those where infinite sums are used.

c) A semiring is said to be a weak p. o. semiring if it has a partial order compatible with its addition only. The paper by *U. Hebisch and L. C. A. van Leeuwen* contains results on embeddings, and on weak p. o. semirings S such that $(S,+)$ or $(S,\cdot)$ are idempotent semigroups.

d) *K. D. Schmidt* introduces a new class of partially ordered semigroups called minimal clans, and shows how their properties allow to retrieve properties of both Boolean rings and lattice-ordered groups, thereby solving a problem posed about 20 years ago by Birkhoff.

e) A category is called universal if it contains the category of graphs as a full subcategory. *P. Goralčik and V. Koubek* prove here the following interesting result: The category of all extensions of a semigroup S is universal if and only if S has no idempotents.

f) The object of the paper by *W. Lex* are acts in the general meaning of semi-automata, especially lattices of torsion theories of acts as proposed by him and Wiegandt. In this context a new characterization of the non- trivial abelian groups is obtained.

g) Is it possible to get machines to prove theorems for you? Not quite. The machines still need assistance from the operator, as shown in *R. B. McFadden's* paper, using several problems in the theory of semigroups, the last of which I liked particularly.

As these short analyses show, a large variety of topics have been the object of lectures at the Conference. It is a clear sign that the algebraic theory of semigroups is steadily growing over the years, both in strength and in depth. It also appears that semigroups are increasingly connected to more and more distinct areas of Mathematics. This is perhaps the most important warrant of the future vitality of the field.

Gerard Lallement
University Park (Pennsylvania), November 1987

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SOME CONGRUENCES ON REGULAR SEMIGROUPS

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A congruence ρ on a regular semigroup S is uniquely determined by its kernel $\ker \rho = \{x \in S \mid (\exists e \in E) x \,\rho\, e\}$ and trace $\mathrm{tr}\, \rho = \rho|_{E(S)}$ [2] . Let Con S be the congruence lattice of S, K and T equivalences on Con S defined by $\rho \,K\, \xi \iff \ker \rho = \ker \xi$ and $\rho \,T\, \xi \iff \mathrm{tr}\, \rho = \mathrm{tr}\, \xi$. It is known that K-classes $[\rho_K, \rho^K]$ and T-classes $[\rho_T, \rho^T]$ are intervals on Con S ([13],[15]). In this paper K-classes with $\mathrm{tr}\, \rho^K = \omega_E$ and T-classes with $\ker \rho^T = S$ are considered. It turns out that such a K-class consists exactly of E-unitary congruences on S, and such a T-class consists exactly of band of groups congruences on S. Similarly, K-classes for which ρ^K is a Clifford congruence consist of E-reflexive congruences. These results generalize corresponding results for inverse semigroups [14].

Throughout this paper, S stands for an arbitrary regular semigroup. For $X \subseteq S$, E(X) denotes the set of idempotents of X. If ρ is a relation on S, then ρ^* denotes the least congruence on S which contains ρ. If ρ is an equivalence on S then ρ^0 denotes the greatest congruence on S contained in ρ. Let $X \subseteq S$. A congruence ρ on S <u>saturates</u> X if for any $a \in X$, the ρ-class $a\rho$ is contained in X. In particular, a congruence ρ on S is <u>idempotent pure</u> if ρ saturates E(S). If Θ_X is the equivalence on S induced by the partition $\{X, S \setminus X\}$ of S, then Θ_X^0 is the greatest congruence which saturates X. We write τ instead of Θ_E^0. If ρ is a congruence on S and α is an equivalence on S/ρ, then the equivalence $\bar{\alpha}$ on S is defined by

$$a \,\bar{\alpha}\, b \iff (a\rho) \,\alpha\, (b\rho) \qquad (a,b \in S).$$

Obviously, $\bar{\alpha}$ is a congruence on S if and only if α is a congruence on S/ρ.

For undefined notations or terminology see [3] or [14].

RESULT 1. [9]. *For any congruences* ρ *and* ξ *on* S,

$$\rho \; T \; \xi \iff \bar{\mathcal{H}}_{S/\rho} = \bar{\mathcal{H}}_{S/\xi}.$$

COROLLARY 1. *Let* ρ *and* ξ *be congruences on* S *such that* $\rho \; T \; \xi$. *Then* $\mathcal{H}_{S/\rho} \in \mathrm{Con}(S/\rho) \iff \mathcal{H}_{S/\xi} \in \mathrm{Con}(S/\xi)$. *Moreover, if* $\mathcal{V}$ *is a variety of bands then* $\mathcal{H}_{S/\rho}$ *is a* $\mathcal{V}$*-congruence if and only if* $\mathcal{H}_{S/\xi}$ *is such one.*

RESULT 2. ([15],[13],[9]). *Let* ρ *and* ξ *be congruences on* S. *Then*

(*i*) $\rho T = [\rho_T, \rho^T]$ *and* $\rho K = [\rho_K, \rho^K]$ *are intervals of* Con S.

(*ii*) $\rho_T = (\mathrm{tr}\rho)^*$, $\rho^T = \bar{\mathcal{H}}^0_{S/\rho}$,

$\rho_K = \{(x,x^2) \mid x \in \ker \rho\}^*$, $\rho^K = \Theta^0_{\ker \rho}$.

(*iii*) $\mathrm{tr}\,\rho \subseteq \mathrm{tr}\,\xi \Rightarrow \rho_T \subseteq \xi_T$ and $\rho^T \subseteq \xi^T$,

$\ker \rho \subseteq \ker \xi \Rightarrow \rho_K \subseteq \xi_K$.

(*iv*) $\ker \rho \subseteq \ker \xi$ and $\mathrm{tr}\,\rho \subseteq \mathrm{tr}\,\xi \Rightarrow \rho \subseteq \xi$.

Using this result and Theorem [19] it is easy to prove the following lemma.

LEMMA 1. *Let be a nonempty family of congruences on* S. *Then*

$$\bigcap_{\rho\in\mathcal{F}} \rho^T = \left(\bigcap_{\rho\in\mathcal{F}} \rho\right)^T \quad \text{and} \quad \bigvee_{\rho\in\mathcal{F}} \rho_T = \left(\bigvee_{\rho\in\mathcal{F}} \rho\right)_T.$$

REMARK. This result is a part of Theorem 4.13[10].

RESULT 3. [18]. *For* $X \subseteq S$,

$$a \; \Theta^0_X \; b \iff (\forall x,y \in S^1)(xay \in X \iff xby \in X) \qquad (a,b \in S).$$

COROLLARY 2. *Let* ρ *be a congruence on* S *and let* $\tau_{S/\rho}$ *be the greatest idempotent pure congruence on* S/ρ. *Then* $\rho^K = \bar{\tau}_{S/\rho}$. *Consequently*

$$\rho \; K \; \xi \iff \bar{\tau}_{S/\rho} = \bar{\tau}_{S/\xi} \qquad (\rho,\xi \in \mathrm{Con}\; S).$$

Proof. $a \; \bar{\tau}_{S/\rho} \; b \iff (a\rho)\,\tau_{S/\rho}\,(b\rho)$

$\iff (\forall x,y \in S^1)((xay)\rho \in E(S/\rho) \iff (xby)\rho \in E(S/\rho))$

$\iff (\forall x,y \in S^1)(xay \in \ker \rho \iff xby \in \ker \rho)$

$\iff a \; \Theta^0_{\ker \rho} \; b$ (by Result 3)

$\iff a \; \rho^K \; b$ (by Result 2).

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If ω denotes the universal congruence on S then $\sigma = \omega_T$ $[\beta = \omega_K]$ is the least group [band] congruence on S. Similarly, if ε denotes the equality on S then $\mu = \varepsilon^T [\tau = \varepsilon^K]$ is the greatest idempotent separating [idempotent pure] congruence on S.

Using Result 2 we obtain

PROPOSITION 1. The following inclusions are valid for any congruence $\rho \in$ Con S.

(*i*) $\rho \cap \tau \subseteq \rho_T \subseteq \rho \cap \sigma$

(*ii*) $\rho \cap \mu \subseteq \rho_K \subseteq \rho \cap \beta$

(*iii*) $\rho \vee \mu \subseteq \rho^T \subseteq \rho \vee \beta$

(*iv*) $\rho^K \subseteq \rho \vee \sigma$

Proof. (*i*) Since $\rho \subseteq \omega$ implies $\rho_T \subseteq \omega_T = \sigma$ it follows $\rho_T \subseteq \rho \cap \sigma$. From $\mathrm{tr}(\rho \cap \tau) \subseteq \mathrm{tr}\ \rho = \mathrm{tr}\ \rho_T$ and $\ker(\rho \cap \tau) = E \subseteq \ker \rho_T$ we have $\rho \cap \tau \subseteq \rho_T$.

(*ii*) The argument here is similar to that in the proof of (*i*) and is omitted.

(*iii*) Since $\varepsilon \subseteq \rho$ implies $\mu = \varepsilon^T \subseteq \rho^T$ it follows $\rho \vee \mu \subseteq \rho^T$. From $\ker \rho^T \subseteq S = \ker(\rho \vee \beta)$ and $\mathrm{tr}\ \rho^T = \mathrm{tr}\ \rho \subseteq \mathrm{tr}(\rho \vee \beta)$ we have $\rho^T \subseteq \rho \vee \beta$.

(*iv*) From $\mathrm{tr}\ \rho^K \subseteq \omega_E = \mathrm{tr}(\rho \vee \sigma)$ and $\ker \rho^K = \ker \rho \subseteq \ker(\rho \vee \sigma)$ we have $\rho^K \subseteq \rho \vee \sigma$.

The following example shows that the analoque of the first inclusion of (*iii*), i.e. the inclusion $\rho \vee \tau \subseteq \rho^K$, does not hold in general.

EXAMPLE 1. [14;III.4.11]. Let S be a semilattice of two groups G and H of order 2 determined by an isomorphism $\varphi: G \to H$. Let ρ be the Rees congruence on S relative to H. Then $\rho \vee \tau = \omega$ and $\rho^K = \rho \neq \omega$.

By [14] a semigroup in which $\mathcal{H}$ is a congruence is cryptic. A completely regular cryptic semigroup (i.e. a band of groups) is a cryptogroup.

The next theorem characterizes T-classes with $\ker \rho^T = S$.

THEOREM 1. The following statements concerning a congruence ρ on S are equivalent.

(*i*) ρ is a cryptogroup congruence.

(*ii*) ρ^T is a band congruence.

(*iii*) $\rho^T = \rho \vee \beta$.

(*iv*) $\mathrm{tr}\ \rho = \mathrm{tr}\ (\rho \vee \beta)$.

Proof. (*i*) => (*ii*). Since S/ρ is a cryptogroup, $\mathcal{H}_{S/\rho}$ is a congruence on S/ρ which together with Result 2 (*ii*), shows that $\rho^T = \overline{\mathcal{H}}_{S/\rho}$. Since $\mathcal{H}$-classes of S/ρ are groups we have

$$(\forall a \in S)(\exists e \in E(S))\ a\rho\, \mathcal{H}_{S/\rho}\, e\rho \qquad \text{(by Lallement's Lemma)}$$

$$\Longleftrightarrow (\forall a \in S)(\exists e \in E(S))\ a\, \overline{\mathcal{H}}_{S/\rho}\, e$$

$\Longleftrightarrow$ ρ^T is a band congruence.

(*ii*) => (*iii*). The hypothesis implies that $\rho^T \supseteq \rho \vee \beta$, and thus by Proposition 1 (*iii*), we have $\rho^T = \rho \vee \beta$.

(*iii*) => (*iv*). This is obvious.

(*iv*) => (*i*). This is immediate from Corollary 1.

COROLLARY 3. On a regular semigroup S the following conditions are equivalent.

(*i*) S is a cryptogroup.

(*ii*) $\mu = \beta$.

(*iii*) For every $\rho \in$ Con S, $\rho^T = \rho \vee \beta$.

(*iv*) For every $\rho \in$ Con S, $\rho_K = \rho \wedge \mu$.

Proof. (*i*) <=> (*ii*) is a consequence of Theorem 1.

(*ii*) => (*iii*) and (*ii*) => (*iv*) follow immediately from Proposition 1.

(*iii*) => (*ii*). $\mu = \varepsilon^T = \varepsilon \vee \beta = \beta$.

(*iv*) => (*ii*). $\beta = \omega_K = \omega \wedge \mu = \mu$.

REMARK. Equivalences (*i*) <=> (*ii*) <=> (*iii*) are implicitly in [17] and [12].

The next simple result describes the least cryptogroup congruence.

PROPOSITION 2. The congruence $\kappa = \beta_T$ is the least cryptogroup congruence on S.

Proof. Since $\mathrm{tr}(\beta_T) = \mathrm{tr}\,\beta$, Theorem 1 implies that β_T is a cryptogroup congruence on S. If ρ is any cryptogroup congruence on S, ρ^T is a band congruence, so $\beta \subseteq \rho^T$. Hence by Result 2 (*iii*), $\beta_T \subseteq (\rho^T)_T = \rho_T \subseteq \rho$.

Let η denotes the least semilattice congruence on S. Similarly one proves the following series of results.

THEOREM 2. *The following statements concerning a congruence* ρ *on* S *are equivalent.*

(*i*) ρ *is a Clifford congruence.*

(*ii*) ρ^T *is a semilattice congruence.*

(*iii*) $\rho^T = \rho \vee \eta$.

(*iv*) $\mathrm{tr}\,\rho = \mathrm{tr}(\rho \vee \eta)$.

COROLLARY 4. *On a regular semigroup* S *the following conditions are equivalent.*

(*i*) S *is a Clifford semigroup.*

(*ii*) $\mu = \eta$.

(*iii*) For every $\rho \in \mathrm{Con}\, S$, $\rho^T = \rho \vee \eta$.

PROPOSITION 3. *The congruence* $\eta_T = \nu$ *is the least Clifford congruence on* S.

Following R.Feigenbaum [1], for any non-empty subset H of S the *closure* $H\omega$ of H is defined by $H\omega \overset{def}{=} \{x \in S \mid (\exists h \in H) hx \in H\}$. H is *closed* if $H\omega \subseteq H$. If H is a subsemigroup of S or if it is full $(E(S) \subseteq H)$, then $H \subseteq H\omega$.
A regular semigroup S is E-*unitary* if the set E(S) is closed.Any E-unitary semigroup is orthodox [4].
A subset H of S is called *self-conjugate* if $x'Tx \subseteq T$ for every x of S and every inverse x' of x. Let U be the least full self-conjugate subsemigroup of S, and let σ be the least group congruence on S. According to [1], $\ker \sigma = U\omega$. If the semigroup S is orthodox, $U = E(S)$.

For a subset H of S, and any congruence ρ on S, let
$H\rho = \{x \in S \mid (\exists h \in H) x \;\rho\; h\}$.

RESULT 4 [6]. *For any congruence* ρ *on* S

$$\ker(\rho \vee \sigma) = (U\rho)\omega.$$

RESULT 5 [6]. *Let* S *and* T *be regular semigroups and* $\Phi : S \to T$ *a homomorphism of* S *onto* T. *If* U *is the least self-conjugate full subsemigroup of* S, $U\Phi$ *is the least such subsemigroup of* T.

Now we shall consider K-classes of Con S with $\mathrm{tr}\,\rho^K = \omega_E$ where ω_E is the universal congruence on E(S).

THEOREM 3. *The following statements for a congruence ρ on S are equivalent.*

(*i*) ρ *is* E-*unitary*.

(*ii*) ker ρ *is closed*.

(*iii*) $\ker \rho = \ker(\rho \vee \sigma)$.

(*iv*) $\rho^K = \rho \vee \sigma$.

(*v*) ρ^K *is a group congruence*.

Proof. (*i*) <=> (*ii*). ρ is E-unitary

$\iff (\forall a,h \in S)((ha)\rho,\ h\rho \in E(S/\rho) \Rightarrow a\rho \in E(S/\rho))$

$\iff (\forall a,h \in S)(ha,h \in \ker \rho \Rightarrow a \in \ker \rho)$

<=> ker ρ is closed.

(*i*) => (*iii*). Let $x \in S$. Then

$x \in \ker(\rho \vee \sigma) \iff x \in (U\rho)\omega$ (by Result 4)

$\iff (\exists s \in S)(s \in U\rho$ and $sx \in U\rho)$

$\Rightarrow (\exists s \in S)(s\rho \in U(S/\rho)$ and $(sx)\rho \in U(S/\rho))$ (by Result 5)

$\Rightarrow (\exists s \in S)(s\rho \in E(S/\rho)$ and $(sx)\rho \in E(S/\rho))$

(since S/ρ is orthodox)

$\Rightarrow x\rho \in E(S/\rho)$ (since S/ρ is E-unitary)

$\Rightarrow x \in \ker \rho$.

Thus $\ker(\rho \vee \sigma) \subseteq \ker \rho$. Since the opposite inclusion is obvious, (iii) follows.

(*iii*) => (*iv*). From $\ker \rho^K = \ker \rho = \ker(\rho \vee \sigma)$ it follows $\rho^K \supseteq \rho \vee \sigma$. By Proposition 1 (*iv*) we have $\rho^K = \rho \vee \sigma$.

(*iv*) => (*v*). This is obvious.

(*v*) => (*i*). The hypothesis implies that ρ^K is E-unitary and by (*i*) <=> (*ii*) it follows $\ker \rho = \ker \rho^K$ is closed. Thus ρ is E-unitary.

COROLLARY 5. *On a regular semigroup S, the following conditions are equivalent*

(*i*) S *is* E-*unitary*.

(*ii*) $\sigma = \tau$.

(*iii*) *For every* $\rho \in \mathrm{Con}\, S$, $\rho_T = \rho \cap \tau$.

(*iv*) *Every idempotent pure congruence on* S *is* E-*unitary*.

(*v*) *There exists an idempotent pure* E-*unitary congruence on* S.

REMARK. Equivalence (*i*) <=> (*ii*) is proved also in [16].

The proof of the following proposition is similar to the proof of the Proposition 2.

PROPOSITION 4. <u>The congruence</u> $\pi = \sigma_K$ <u>is the least</u> E-<u>unitary congruence on</u> S.

Using the Corollary 5 and Lemma 1 one can prove that the following holds.

PROPOSITION 5. <u>Let</u> S <u>be an</u> E-<u>unitary regular semigroup</u>. <u>The mapping</u>

$$\varphi: \rho \to \rho \cap \tau$$

<u>is a complete lattice homomorphism of</u> Con S <u>onto the lattice of idempotent pure congruences on</u> S.

Let S be an orthodox semigroup and let Y be the least inverse congruence on S. Then we have

PROPOSITION 6. <u>For an orthodox semigroup</u> S <u>the following conditions are equivalent</u>.

(*i*) S <u>is</u> E-<u>unitary</u>.

(*ii*) Y <u>is</u> E-<u>unitary</u>.

(*iii*) ba Y a => $b \in E$ $(a, b \in S)$.

<u>Proof</u>. (*i*) <=> (*ii*) follows from Corollary 5.

(ii) <=> S/Y is E-unitary
<=> (baYa ⇒ bY $\in$ E(S/Y)) (by Proposition III 7.2.[14])
<=> (iii) (since Y is idempotent pure).

REMARK. The equivalence (*i*) <=> (*ii*) is also proved in [8] and [11].

In the remainder of the paper we consider K-classes which consist of E-reflexive congruences. A semigroup S is E-<u>reflexive</u> if $exy \in E(S)$ => $eyx \in E(S)$ for every $x, y \in S$ and $e \in E(S)$. We observe that every E-unitary semigroup is E-reflexive [4].

RESULT 6 [7]. <u>On a regular semigroup the following conditions are equivalent</u>

(*i*) $\nu \subseteq \tau$.

(*ii*) <u>Every</u> η-<u>class of</u> S <u>is</u> E-<u>unitary</u>.

(*iii*) S <u>is</u> E-<u>reflexive</u>.

We can now prove an analogue of Theorem 3.

THEOREM 4. *The following statements concerning a congruence* ρ *on* S *are equivalent.*

(*i*) ρ *is* E-*reflexive*.

(*ii*) $\ker \rho \cap N$ *is closed in* N *for every* η-*class* N *of* S.

(*iii*) ρ^K *is a Clifford congruence*.

(*iv*) $\ker \rho = \ker(\rho \vee \nu)$.

Proof. (*i*) => (*ii*). Let N be an η-class of S and let $a \in N$. Then we have

$$
\begin{aligned}
a \in (\ker \rho \cap N)\omega_N &\Rightarrow (\exists x)(xa \in \ker \rho \cap N \text{ and } x \in \ker \rho \cap N) \\
&\Rightarrow (\exists x)(xa, x \in \ker \rho \text{ and } a\ \eta\ x) \\
&\Rightarrow (\exists x)((x\rho),(xa)\rho \in E(S/\rho) \text{ and } (a\rho)\ \eta_{S/\rho}\ (x\rho)) \\
&\Rightarrow a\rho \in E(S/\rho) \qquad \text{(by Result 6)} \\
&\Rightarrow a \in \ker \rho.
\end{aligned}
$$

(*ii*) => (*i*). $(\forall N) \ker \rho \cap N$ is closed in N

$$
\begin{aligned}
&\Rightarrow (\forall N)\rho|_N \text{ is an E-unitary congruence on } N \\
&\Rightarrow S/\rho \cap \eta \text{ is a semilattice of E-unitary semigroups} \\
&\Rightarrow S/\rho \cap \eta \text{ is E-reflexive} \qquad \text{(by Result 6)} \\
&\Rightarrow S/\rho \text{ is E-reflexive} \qquad \text{(since } \ker \rho = \ker(\rho \cap \eta)\text{)}. \\
&\Rightarrow \rho \text{ is an E-reflexive congruence.}
\end{aligned}
$$

(*i*) <=> (*iii*). S/ρ is E-reflexive <=> $\tau_{S/\rho}$ is a Clifford congruence
(by Result 6)

<=> ρ^K is a Clifford congruence
(by Corollary 2)

(*iii*) => (*iv*). Since ρ^K is a Clifford congruence we have $\nu \leq \rho^K$ which yields $\rho \vee \nu \subseteq \rho^K$, so $\ker(\rho \vee \nu) \subseteq \ker \rho^K = \ker \rho$. But $\ker \rho \subseteq \ker(\rho \vee \nu)$ and therefore $\ker \rho = \ker(\rho \vee \nu)$.

(*iv*) => (*iii*). From $\ker \rho^K = \ker \rho = \ker(\rho \vee \nu)$ it follows that $\rho^K \supseteq \rho \vee \nu \supseteq \nu$, hence ρ^K is a Clifford congruence on S.

The following proposition is an analogue of Proposition 4.

PROPOSITION 7. *The congruence* $\lambda = \nu_K$ *is the least* E-*reflexive congruence on* S.

One may ask whether the equivalence (*i*) <=> (*ii*) of the Result 6 would remain true if ν and η were replaced by κ and β respectively. It can be proved that $\kappa \leq \tau$ implies that E(N) is closed in N for every β-class N

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