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Mathematics as a Service Subject

Udine 1987

ICMI Study Series Editors A.G. Howson and J.-P. Kahane

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FOREWORD

The present study, the third of the ICMI series, is the result of a cooperation between the Committee on the Teaching of Science of the International Council of Scientific Unions (ICSU-CTS) and the International Commission on Mathematical Instruction (ICMI). It is based on the work of a Symposium held in Udine (Italy), from 6 to 10 April, 1987, at the International Centre for Mechanical Sciences (Centre International des Sciences Mécaniques = CISM).

The study began by a careful investigation about the way mathematics is taught to students in another major subject in a few typical universities: Eindhoven Technical University in the Netherlands, Jadavpur University of Calcutta, India, Eötvös Lorand University and several other institutions in Budapest, Hungary, Florida Agricultural and Mechanical University in the USA, University College, Cardiff, U.K., University of Southampton, U.K., Université de Paris-Sud à Orsay, France. The past and current presidents of ICSU-CTS (the physicist Charles Taylor and the biologist Peter Kelly) took part in the program committee, which included also the president and the secretary of ICMI, the mathematicians Tibor Nemetz and Fred Simons, the statistician Elisabeth de Turckheim, and the physicist Pierre Lauginie. The Program committee issued a discussion document, which was circulated to all national representatives of ICMI, and to various institutions. It was published in the journal L'Enseignement Mathématique, tome 31 (1986), pp. 159-172, and it also appeared in French, Italian and Spanish versions. Abstracts or quotations appeared in other scientific or vocational journals, it was discussed among members of several scientific institutions (including the Académie des Sciences de Paris) and among professionals, for example the Fondation Bernard Grégory. Contributions to the discussion were received from many countries; some are reprinted in this text, others, including the discussion document, in the volume of Selected Papers to be published by Springer Verlag (see p. 92).

The meeting in Udine was attended by 37 participants, on invitations issued by the program committee. The generous hospitality of CISM - located in a beautiful historical mansion - and the working atmosphere made this symposium pleasant and profitable. The main reports - by Bony, Murakami, Pollak, Simons - are in the present book, and are preceded by a survey article written by the four editors of this volume.

Financial help was received from UNESCO, ICSU, IMU (International Mathematical Union), CISM, the Royal Society, the Ministére de l'Education Nationale of France, IBM-Europ, IBM-France, and many universities or institutions which contributed to the expenses of participants. We sincerely thank all of them, and we hope that the success of this study will prove, once again, that international actions of this type meet a real need and have an important effect.

A.G. Howson
J-P. Kahane

November 1987

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ON THE TEACHING OF MATHEMATICS AS A SERVICE SUBJECT

> La science est continuellement mouvante dans son bienfait. Tout remue en elle, tout change, tout fait peau neuve. Tout nie tout, tout détruit tout, tout crée tout, tout remplace tout. La colossale machine science ne se repose jamais; elle n'est jamais satisfaite. .... Cette agitation est superbe. La science est inquiète autour de l'homme; elle a ses raisons. La science fait dans le progrès le rôle d'utilité. Vénérons cette servante magnifique.
>
> Victor Hugo, L'art et la science, in <u>William Shakespeare</u> (1864).

The title of the study may shock. Mathematics is the most ancient of the sciences. Why should it be in the service of others, or worse still, in the service of technical activities? In reducing mathematics to a service rôle, does one not belittle its contents, its image? Let us immediately state that in our view 'mathematics as a service subject' does not imply some inferior form of mathematics or mathematics limited to particular fields. We mean mathematics in its entirety, as a living science, able - as history has ceaselessly shown - to be utilised in, and to stimulate unforeseen applications in very varied domains. The teaching of mathematics to students of other disciplines must now be accepted as a fact, a social need and, also, a relatively new problematic issue. In this introduction we shall try to show the extent of the phenomenon, the social needs which it expresses, foreseeable developments and likely results in terms of choice of subject matter and teaching methods, and finally the size of the new problems which confront students and teachers alike.

I <u>Everywhere there is a need for mathematics</u>

1 As is common knowledge, the chance fact that there was personal enmity between Nobel and Mittag-Leffler resulted in there being no Nobel prize for mathematics. But there are still Nobel prize-winners who are mathematicians. The Nobel prize for chemistry was awarded in 1985 to two mathematicians, H.H. Hauptman and J. Karle, for the development of methods for the determination of crystalline structures, based on Fourier analysis and probability. To quote W. Lipscomb, who presented the prize: "The Nobel prize for chemistry is all about changing the field of chemistry. And this work changed the field." Before that, G. Debreu and L.V. Kantorovič had been awarded the Nobel prize for economics for work which was also of a mathematical nature. Mathematics <u>cannot</u> be excluded from the family of sciences. It is an integral part of scientific thought, a necessary component of contemporary advances in all scientific fields.

In physics the link with mathematics extends to Galileo and before: mechanics, optics, electro-magnetism, relativity, quantum theory are inseparable from the calculus, the geometry of surfaces, partial differential equations, non-euclidean geometries, Hilbert spaces, ... The use of computers has recently given physics an intermediate component between theory and experience, simulation, which is very close to a purely mathematical activity, experimentation on a computer. Simulation in physics and mathematical experimentation often operate on the same objects in the same way. They create new subjects of interest and cause old ones to be resurrected, for example, fractal geometry.

Informatics has a profound influence on all sciences. Yet the link with mathematics is essential. The first computers were the realisation of Turing's ideal machine. Mathematical problems are a testing ground for informatics. The algebraic and algorithmic aspects of mathematical theories are benefitting as a result. Discrete mathematics takes on a new significance. Parallel processing already suggests research avenues involving combinatorics and differential geometry.

Chemistry is a source of many difficult mathematical problems - as Hauptman and Karle's success shows.

Medicine uses sophisticated tools which necessitate cooperation between physicians, physicists and engineers. Mathematics can form a common reference point.

Biology, like economics, makes use of statistical models. Linguistics, geography and geology all use concepts and techniques which demand a solid grounding in mathematics for true mastery.

Engineers, whatever their special branch of activity, have to calculate, to construct models, to test hypotheses. Many technical problems, ranging from coding (for bank or military purposes) to geological prospecting, lead to, and draw upon, important research areas in 'pure' mathematics: the divisibility of integers, the search for prime factors, the theory of 'wavelets' in signal analysis.

2 That is not all. Mathematical concepts now surface at all levels of social life. Let us take three simple examples.

Each individual is now confronted with an avalanche of numerical data and constant changes of scale (from the price of goods and other purchases to the National Budget, from atoms to galaxies, from nanoseconds to geological time spans). The conceptual tools needed to master such data and changes of scale exist: they are, at a fundamental level, numbers written in standard (exponential) notation and, at a higher level, geometric representations and data analysis. Changes of scale in the exploration of figures - a kind of intellectual zoom - correspond to modern notions of measure and dimension. To understand our different environments, modern ways of 'calculating', and modern views of geometry and analysis offer remarkably well-adapted tools.

For the individual, for groups, and for humanity as a whole, the evaluation of risks (car accidents, nuclear accidents, geological catastrophes) has become a necessity and it alone enables us to make rational decisions. Probability is the means appropriate to such evaluations. The concept of probability allows us critically to examine data and suppositions. Acquiring an understanding of probability should be - and how far we are from attaining this goal - a key element in the development of general critical faculties.

In the fields of production and the service industries, information technology and automation have caused programming and 'control' to become essential activities. The conceptual tool adapted to programming is the algorithm, that is to say, a systematic procedure enabling us to solve a whole class of problems. Thus, an algorithm is a means of governing thought which is well adapted to the governing of machines.

Arithmetic and geometry, analysis, probability, algebra and in particular algorithmics have a totally different meaning today, and offer far more, than they did two centuries ago - a time when discussion still occurred on whether -2 could really be called a number. Some modern concepts must now become part of common consciousness, and they must be integrated into higher and professional education if space cannot yet be found for them in the general school curriculum.

3 In higher education mathematics is now taught to a wide range of students - diverse on account of their backgrounds and also because of their specialisms and aspirations. At the beginning of the century one could, as ICMI did in 1911, confine attention to the teaching of mathematics to engineers. Today, however, one must take into account the needs of all future professionals: architects, doctors, managers, etc. Whatever the major subject studied, mathematics has become a necessary adjunct: that is what we call 'mathematics as a service subject'.

The topic is important - in respect of a general concept of mathematics education - for at least three reasons.

Numerically, it involves many mathematicians in higher education (in some institutions in Canada up to 80% of mathematics teaching is to students studying other disciplines).

Socially, it corresponds to the impact of mathematics on all aspects of everyday life.

Intellectually, it forces us to look at things from a new angle - for instance, to perceive that there are many routes by which one can come to mathematics.

Nevertheless, the variety of situations is extremely large. Not only because of the diversity of students already remarked upon, but also because of different national traditions and variations in the structure of institutions of higher education. A general description becomes almost impossible. The papers which follow, and more particularly those in the volume of *Selected Papers*, will give the reader some idea of this wide variety.

Although important and varied, the field is relatively ill-known. In general, professional mathematicians do not see the service-teaching in which they participate as particularly rewarding. It is a part of their activity which is often hidden, in particular from students of mathematics and from future teachers of mathematics. As a result, secondary school teaching does not benefit from innovations introduced into universities as part of service teaching.

4 Let us stress once again the social importance of this service teaching - its importance goes well beyond meeting an explicit social demand. Nowadays this social demand is expressed mainly through the voices of employers and through those of colleagues in other disciplines.

Some indications of the demands of employers is to be found in the paper provided by the Bernard Grégory Association (*Selected Papers*). We note one surprising fact: when engineers from the French Electricity Board were asked which discipline they felt nearer to, mathematics or physics, 90% chose the former and only 10% the latter. The paper by Henry Pollak in this volume illustrates the enormous need for mathematical training which arose in Bell Laboratories, and the magnitude of the present demand.

Demands from sundry disciplines are clearly very varied and depend not only on the disciplines studied but also on the level at which studies are taking place. One can distinguish two types of subject. In those disciplines of the first type - physics, astronomy, theoretical chemistry, parts of engineering science - certain essential concepts are mathematical in nature, data are treated in a quantitative way, numerical solutions are obtainable to given problems through the use of mathematics: it can be said that mathematics permeates the whole of the discipline. Within the disciplines of the second type - biology, economics, etc. - mathematics sheds light on certain concepts and is used to set up or exploit quantitative models, often far removed from reality. The attitudes of students towards mathematics tends to differ greatly according to the type of their major discipline. This gives rise to equally different pedagogical problems.

5 To sum up:

(a) More than ever, and increasingly so, mathematics interacts with other sciences and with technical activities in which science is strongly represented.

(b) A part - changing - of mathematics represents an integral part of the general culture of each age. In our time no individual should be deprived of this component.

(c) Mathematics as a service subject represents a very important activity within institutions of higher education, a very varied, very interesting and ill-understood activity.

(d) Explicit demands for mathematics to be taught as a service subject are already important and they are growing. According to career aspirations and choice of major discipline, mathematics appears sometimes as indispensable, sometimes as useful but of secondary importance. Ways of teaching must be adapted to match these different types of demand.

II <u>What is changing, what is to be done, and why?</u>

1 We live in a rapidly changing world, the sciences are advancing, technologies are changing, societies are being modified. The result is new problems, new means and possibilities, and new needs. Let us quickly review what implications this has for mathematics.

As in other sciences, the output in mathematics (as measured by published papers) is increasing exponentially: it has regularly doubled every decade since the beginning of this century. In 1987 we are talking of more than 100,000 papers. It may very well be that this trend will continue, but in other forms (for otherwise where will all the paper come from!) as an increasing number of countries get involved in mathematics research.

Is the social assimilation of new knowledge on this scale possible? That is a big question which can also be posed for other sciences. Yet true scientific progress is the concurrent development of knowledge, its dissemination, and its assimilation by the public at large. The question then is to pass from development to progress. It concerns society as a whole and each individual in particular. It is not possible for everyone to know everything; but we should not believe that even today's specialised knowledge will remain permanently out of reach of most people.

Information technology has come into being at a most opportune time, for through it new ways of storing, processing and disseminating data have become possible. We therefore have new means of conserving and communicating acquired knowledge. Yet <u>consultation</u> of documents stored on disks and cassettes, and the <u>diffusion</u> of new knowledge necessitate improved means of catologuing and listing, and the production of abstracts, syntheses, manuals and guides. These intellectual tools, which emerge slowly but surely, are the indispensable accompaniment of purely technical tools. They will quite likely continue to appear in printed form; they will be used in research, in production and in teaching. Contrary, perhaps, to a generally accepted notion, new technologies are not going to make recourse to books redundant in the process of teaching. They will make such recourse more necessary than ever.

On the other hand, new technologies give rise both to new possibilities and new demands for mathematical research and for the teaching of mathematics. The first volume in the ICMI study series dealt with the influence of computers and informatics on mathematics and its teaching. Let us extract from that a few ideas which directly concern the teaching of mathematics as a service subject.

New possibilities: the use of the computer allows one to illustrate concepts and methods (for example, differential equations), to experiment (an essential phase of research), to assist memory and replace technical virtuosity (in particular, with the rise of symbolic manipulation), to adapt learning to the potential and rhythm of an individual student (CAL).

New demands: in their professional life, computer-users must know what to ask of computers and how to interpret the results they obtain. Those users must, therefore, have at their fingertips knowledge and concepts which are more varied than hitherto. Computers can free the users from most mechanical drudgery related to the learning of mathematics (memorisation, execution of algorithms), but they demand more imagination, creativity, critical faculties (conception of algorithm, stability, sensitivity to initial conditions, detection of errors, control and exploitation of results). In particular, statistical software is becoming a more and more familiar object and the demand for an understanding of statistical methods is becoming more explicit.

Fortunately, when confronted with these new demands, mathematics has produced and is continuing to produce a flow of general concepts and powerful methods. The development of science is not merely an accumulation of knowledge but a permanent restructuring. It is this restructuring which enables mathematicians not to get lost in the mass of contemporary output, and enables students to assimilate a non-negligible part of mathematics rapidly and deeply. To put it more precisely, it is when general concepts and powerful methods are brought into being - which are the big intellectual tools for the world of machines and the world of men - that the problem of their assimilation begins. Choices become necessary in the subjects to be taught and new methods of teaching have to be introduced.

2 Let us examine how the choice of subjects presents itself. It obviously depends upon the future profession of the students and on the teaching they receive in their major disciplines.

There are two possible criteria. The first is to choose the subjects that one imagines will be those most useful in the course of the students' future professional life. The second is to teach what is immediately usable by students in their learning of their major discipline.

The second criterion is often what colleagues teaching the major discipline spontaneously demand: the necessary mathematics is that which we need, and it should be supplied at a speed to match the demands of our teaching. It can also correspond to the demands of students who seek a certain coherence between the mathematics teaching and those courses which are utilising mathematics. Such demands can, in many cases, induce mathematicians to improve the choice of topics which they teach, the order of presentation and the way in which they introduce or illustrate mathematical concepts. It can provoke a requestioning of certain habits and of traditional curricula. Never-the less, it often leads to the formulation of impossible demands (for example, the chemist or physicist may wish to use functions of several variables long before the mathematician has been able to introduce them). Above all, its essential weakness is to ignore the first criterion.

It is this first criterion which should be the fundamental one. But it means that choice must depend upon a future of which we are ignorant. It is, therefore, hazardous and it necessitates, even much more than in attempting to satisfy the second criterion, that mathematicians work in close cooperation with colleagues working in the major discipline. Very often those colleagues demand of mathematics something other than a justification of the use they make of it. They wish their students to learn mathematical modes of thought and the mathematician's various modes of expression: abstract exploration, geometric representation, an intuition into the calculus, then logical deduction and formal rigour. The reader, for example, can find in the Selected Papers volume, the views of physicists from the Paris Academy of Sciences who demand that geometry should once more be given a prominent place in mathematics for physics students, since geometric intuition is essential for the physicist. There is an equally explicit demand from the engineers (as is shown, for instance, by Pollak, Aillaud, Roubine and Sinha).

Confronted with these two criteria, the mathematician can legitimately take the initiative. Very often, what one can and must teach nowadays depends upon discoveries or formulations made in the last thirty years, and, therefore, unknown when many colleagues in other departments were students. The mathematicians are in a position, therefore, where it is up to them to formulate proposals.

3 Let us begin by proposing an exclusion from existing courses, that of most exercises on differentiation and integration, partial fractions, inverse trigonometric functions, etc. These even today represent a large part of many service courses for first year students. They do not aid the learning of analysis (in which we include ODEs, PDEs, numerical and harmonic analysis); indeed, they often obscure it. The good way of performing this type of calculus is as a branch of algebra and it is desirable that students should be more conversant with underlying principles in order that they should better understand the origins and the use of software for symbolic manipulation which is now replacing calculations done by hand. It can also be argued that the repetition of such 'computing by hand'

when the answers are readily available on a machine is simply a waste of time.

One single example, well discussed and analysed, can be more instructional than a host of repetitive exercises. The time saved could be used to familiarise students with notions fundamental to analysis (for instance, vector fields and line integrals should be substituted for repetitive exercises on first order differential equations) and/or for learning some discrete mathematics.

The study on the influence of computers (ICMI Study 1) highlighted the rapid development of discrete mathematics and proposed its introduction into the curriculum. Recommendations to this effect have also been made by an ad hoc committee of the American Mathematical Society (see the paper by Martha Siegel). Jack van Lint succinctly presents in this volume a stimulating vision of what discrete mathematics can mean and how it can contribute to the solution of problems with very varied origins. It is a mathematical field which has never had more than a foothold in the curriculum yet van Lint's examples show how essential such knowledge now is for engineers. It offers a new and interesting way in which to approach certain algebraic topics and aspects of the theory of numbers (in particular, permutation groups and finite fields).

The introduction of discrete mathematics may seem quite alien to the desire expressed by physicists, which we mentioned earlier, to see greater emphasis given to geometry. In fact, geometry - if we understand the term in its broad sense (see, for example, G. Chatelet's contribution to the Selected Papers) - applies to the discrete as well as the continuous. Its importance in physics, and in many other human activities, proceeds, classically, from what physicists call symmetries and mathematicians see as invariants under a group of transformations. According to Chatelet - who makes reference to very remarkable texts by Hamilton and Maxwell - fundamental geometric concepts express actions rather than visions. Thus, vectors, arrows, diagrams express actions as also do fibrations and parallel transports. The significance of geometric intuition is that it represents thought in action. Whatever the choice of geometric concepts to be taught, and this cannot be the same in physics, engineering and architecture, the active aspect of geometrical thought must be preserved.

So far as the choice of subjects to teach to physicists is concerned, in particular analysis, the meeting clearly showed that merely enumerating desirable topics leads nowhere and that, on the other hand, it is possible to hold a constructive debate around fundamental questions: disparate subjects or unifying concepts, ad hoc procedures or powerful methods, fidelity to tradition or a modern approach. J.L. Bony openly pleads for unifying concepts and powerful modern methods. The examples he quotes are excellent, but they are only examples. The interest in this approach is not to determine a particular choice, but to establish a method by which one can choose.

Without pretending to cover the whole spectrum of mathematics, we must, nevertheless, make a special mention of probability. Probability comes in, or should do so, at all four levels of mathematical need identified by Pollak: everyday life, intelligent citizenship, professional work and general culture. It is not reasonable that a student should leave university - whatever his field of studies - without ever having learned of probability.

4 It is a striking fact that non-mathematicians - even more than most mathematicians - insist on the power and value of a mathematical mode of thought. The idea is expressed equally forcefully by biologists ("never mind what you teach: teach students to reason well") and by engineers (see, for example, Aillaud, Pollak, Roubine). Let us mention, however, the reservation expressed by Tonnelat: 'Mathematical thinking is a good servant, but a bad master'. The ways of thinking acquired in the course of studies will, however, serve strictly to determine an individual's ability to update his/her knowledge in the years of professional life. By this we mean a kind of continuous retraining. Let us borrow an example from G. Aillaud: an engineer trained in combinatorial arguments will easily adapt to operations research, programming, expert systems, but he would be totally blocked should he wish to move from combinatorics to numerical analysis.

The consequence of this is that in the choice of subjects one must think not only of the knowledge we wish our students to acquire, but also of the modes of thought associated with those topics.

5 Again it is the experience of engineering departments which particularly attracts our attention to the other side of the coin (cf. Pollak, Siegel, Aillaud): the importance of knowledge itself, as distinct from the ability to make use of it. In the course of his professional life an engineer will rarely have to solve a mathematical problem, but he will frequently have to recognise whether a question confronting him is capable, or not, of being modelled, of being treated mathematically. As in any other science, the important thing for him is to know enough mathematics to be able to consult a mathematician and to derive the most benefit from this.

The consequence of this is that in the choice of subjects to be taught one must think not only of mathematical modes of thought, but of the large range of knowledge required to permit a professional to know what might be mathematically tractable.

6 Each professional activity demands a particular type of mathematical culture (mathematical literacy) which enables one to be an intelligent user of mathematics. This means an ability (i) to read the mathematics used in the literature of one's profession, (ii) to express oneself using mathematical concepts, (iii) to consult references or competent mathematicians should the need arise. In biology and the human sciences, for example, a need frequently ex-

perienced is to be able to use mathematics as a language to express the problems of the discipline. This concept of a mathematical culture or a type of familiarity with mathematics peculiar to each discipline or each profession seems to us better suited to present needs than that, frequently used, of a knowledge of a 'fundamental' range of techniques. Indeed, this knowledge of a range of basic techniques must be modified as mathematical culture is acquired: they are only fundamental with respect to a particular goal, and this end seems to us to be the mathematical culture in itself, varied and variable in the same way as activities and technologies.

Mathematical culture must unite these two distinct aspects: mathematical modes of thought and a range of essential knowledge.

7 We have chosen to insist on what is changing and what, as a consequence, forces us to modify curricula. In the same way that societies, technologies, sciences, mathematics are not going to stop changing, it would appear that, in the future, curricula will constantly have to be modified. Will this upheaval in curriculum design result in the developing countries being left permanently behind? This anxiety was expressed at the Udine meeting and it must be given serious consideration.

Even, and above all, in developing countries sclerosis of the curriculum will prove a catastrophe. Everywhere, then, we must be on the alert to track down what is - and what needs - changing. But that is not to say that teaching programmes must change everywhere in the same way. It would be absurd to attempt to teach new topics if there is no one fitted to teach them. The choice of topics, at university level, must be made by the teaching staff bearing in mind their levels of competence, their fields of interest, and other circumstances peculiar to their actual situation.

8 Let us end with the underlying idea which has provided cohesion to this study. Much more than in the past, and more and more so, thanks to the influence of computers, users need to understand mathematics, to assimilate concepts rather than techniques. Let us stress that this demand is expressed with particular force by engineers who are especially sensitive to the effects of rapid technological change. The rôle of teaching is to prepare students for change and, on the whole, they are ready to recognise this aim as essential. Such preparation for change is necessary whatever the student's future professional activity will be. Thus there is no contradiction - indeed quite the reverse - between a teaching devoted more to fundamentals and a teaching nearer to practice.

More fundamental, more practical, less technical; it seems to us that these trends should obtain as a general rule for the teaching of mathematics as a service subject.

III <u>What is being done and could be done. With whom? How?</u>

1 We have just written of technologies, of sciences, of

subjects and of curricula. Yet at the meeting in Udine, the main part of the discussion centred on another aspect: teaching and learning methods, pedagogical experiences and problems, the relationships between teacher and students, and the social function of those engaged in service teaching.

We shall consider in order:

- students, from those of the first year to those in continuous education (2,3,4),

- desirable directions for developments (concerning, inter alia, mathematical reasoning, rigour, theory and examples, and modelling) (4-11),

- tools (computers, books, examinations) (12,13,14),

- relationships between teaching colleagues and those within the mathematical community at large (15-21).

2 The entry of students to higher education deserves special attention and Fred Simons' paper is devoted to this topic. Let us abstract from it a few topics which he describes.

It is a remarkable and somewhat paradoxical fact that first-year syllabi should be practically the same throughout the world for all service-teaching to students of engineering and the physical sciences. Yet students come to university with very different levels of attainment. Some of them, ill-trained during their secondary schooling, find themselves in difficulty on courses which their peers find accessible. Two solutions have been mentioned: imposing more strictly a minimum level of attainment on entry (a move which would often run against national traditions and mores) and organising special entry programmes. These last, 'booster', courses have given rise to some interesting experiments, but there appears to have been little done in the way of evaluation. In any case an essential effort is required to spell out the prerequisites to first-year teaching by giving precise indications on what subjects will be needed, when they will be used, and in what context. In some places, such clarification of prerequisites, together with the production of complementary documentation and the establishment of booster programmes has already occurred, and been welcomed by students and colleagues (see Shannon, Selected Papers). This is also a useful way in which to help and influence secondary schools.

An example of another experiment in first-year teaching can be found in Southampton. This is self-paced, individualised instruction (with opportunities for consulting tutors) which is controlled by means of tests taken at the end of each 'unit'.

Numerically - whether in terms of the number of students involved, the number of lecturing hours, or the number of lecturers - the teaching of first-year service courses is of considerable significance. It is at this level that the most crucial factors common to all service-teaching commitments arise: student motivation and that of their lecturers. It is at this level that an ill-adapted course can so easily deprive students of an interest in mathematics and can conceal from them the true flavour of, and creativity inherent in, the subject. It is also at this stage that vocations can reveal themselves. This is then a time when the need to exercise 'pedagogical care' (see Martha Siegel) is uppermost. It is, clearly, a level at which considerable pedagogical research is needed. One can assume that students of mathematics enter university motivated to study the subject (although how long that motivation will last will depend very much upon the courses they are then given); but for those taking mathematics as a service subject it is usually necessary to create/foster motivation. Yet it is at this stage that lecture rooms are at their most packed - a time when the need for small classes and tutorial groups is at its greatest. Where sequential courses are not set out from the start, it is also the stage at which students will have the opportunity to opt for different career directions - and this brings a corresponding need for multivalent types of mathematics teaching. The first-year, too, is often the time when mathematics is used as a sieve to separate out the 'clever' from the 'dull' students. Assessment then becomes over-important with the result that students devote their major intellectual effort to cramming for the end-of-year examinations.

3 At the other end of the time-scale, continuous education is now a fascinating field in which there are already many valuable experiments to report. Yet it is still an insufficiently explored area. The account of the development of continuous education in Bell Laboratories is well worth studying (Pollak). Here, motivation is clear. But the teaching approaches most suitable for adults with considerable professional expertise will differ considerably from what is traditional practice for university academics. Students must be given the opportunity, and encouraged, to proceed at their own pace (books, papers, software) and the teacher should assume (more even than elsewhere) the rôle of expert and adviser. The provision of materials suitable for use on continuous education programmes is an urgent need.

4 The present position so far as motivation to study is concerned is often described in gloomy tones:

(a) users frequently demand a fantastic quantity of techniques, of tricks, while allowing mathematics only a ridiculously small fraction of the students' time;

(b) students bother only with examinations and prefer to learn and apply formulae rather than to develop their reasoning powers;

(c) students couldn't care less about what worried Fourier or what prompted the development of Hilbert spaces - that will not help them to pass the examination!