Mathematics and Its Applications

J. de Vries

Elements of Topological Dynamics

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by

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God thundereth marvellously with his voice; great things doeth he, which we cannot comprehend. JOB 37:5

Continuous thunder: the symbol of the shock. The Superior Man in fear and trembling seeks to improve himself. I CHING, HEXAGRAM 51

Preface

This book is designed as an introduction into what I call 'abstract' Topological Dynamics (TD): the study of topological transformation groups with respect to problems that can be traced back to the qualitative theory of differential equations. So this book is in the tradition of the books [GH] and [El])*. The title ('Elements . . .' rather than 'Introduction . . .') does not mean that this book should be compared, either in scope or in (intended) impact, with the 'Elements' of Euclid or Bourbaki. Instead, it reflects the choice and organisation of the material in this book: elementary and basic (but sufficient to understand recent research papers in this field). There are still many challenging problems waiting for a solution, and especially among general topologists there is a growing interest in this direction. However, the technical inaccessability of many research papers makes it almost impossible for an outsider to understand what is going on. To a large extent, this inaccessability is caused by the lack of a good and systematic exposition of the fundamental methods and techniques of abstract TD. This book is an attempt to fill this gap.

The guiding principle for the organization of the material in this book has been the exposition of methods and techniques rather than a discussion of the leading problems and their solutions, though the latter are certainly not neglected: they are used as a motivation wherever possible. As a rule, clarity of the exposition has had a higher priority than the completeness of the included material (though is was hard to resist my natural inclination to be encyclopedic). In addition, I have included an abundance of examples, as illustration of results, as 'test-cases' for techniques, and often also for their own interest.

The book can be divided in two parts: Chapters I–III (actions of $\mathbb{R}$ and $\mathbb{Z}$ only, and not exclusively concerned with minimality) form Part One, and Chapters IV–VI (actions of arbitrary topological groups, with the accent on minimal flows and their extensions) form Part Two. The theory in the second part is independent of the first part, but for many examples in Part Two we refer to Part One. A description of the contents of the various chapters can be found in the introductory remarks in each chapter. However, at this place I want to make a remark about Chapter I. This chapter is not needed for the understanding of the rest of the book in a technical sense. But mathematics is more than just formal techniques and results. Though I am not enough of a philosopher to be able to explain briefly what exactly there is more to it, it is my opinion that a good mathematician should know at least how his speciality came to

)* Most references to the literature have the form W.A. VEECH [1976]. For a few often-cited books they are of the form [GH].

being: what are its roots, what are its connections with other fields of mathematics? Chapter I gives a concise answer to these questions: it describes in a nutshell a large part of the field of Dynamical Systems and contains historically orientated motivations for various problems and notions studied in this field.

In every chapter (except Chapter I) the material is organized at three levels: first there is a systematic and essentially self-contained exposition of the theory, then there is a collection of 'Illustrations', containing miscellaneous results, applications and examples (presented as exercises with hints), and finally there is a set of 'Notes', containing references to the sources, additional results (usually without proofs) and references to related material. A little warning is due here: results from references are cited as if they were written in the framework of this book. E.g., I have 'translated' results from [El] about right actions and right ideals into the language of left actions and left ideals. (If no reference is given for a particular result this doesn't mean that I claim any originality: for many results it would have been too time-consuming to trace back the original. A reference to a source only shows where I found the particular result, not who first published it.) In this way I hope the book is also of value for specialists. To increase its value as a reference work there are many cross-references in the book; in addition, I have included an extensive subject index.

Prerequisites for reading the book are a working knowledge of general topology (e.g., as set forth in [Wi], [Du] or [Ke]; I use [Wi] as the main reference) and familiarity with the elements of (Lebesgue) integration theory; also some functional analysis will be useful. For easy reference I have included in four appendices some material from these fields; in a fifth appendix I present the elements of the theory of topological transformation groups as needed in this book.

This book was conceived and written at the CWI (the Centre for Mathematics and Computer Science) in Amsterdam, at the end of the paradisiac period in which justification of mathematical research was not yet restricted to industrial and computational applicability. The idea to write this book was born in the early eighties, when Jaap van der Woude and I organized a seminar at the Mathematical Centre (as the CWI was then called). When Jaap was writing his thesis [Wo] the idea became more concrete: we need a book that contains 'everything that is needed to understand [Wo]'. Our intention was that Jaap would select the material and would write first versions, and that I would write the final version. After I had written the Appendices and Chapter I it became clear that Jaap could not participate in the project: in the Netherlands there was (and still is) no interest in a mathematician with his background, and he had to earn his livings as a computer scientist – which he does as enthusiastically as he did TD. After a period of hesitation I decided to continue the project alone. This not only explains the long time needed to complete the book, it is also responsible for the fact that not all results promised in [Wo] (under the reference [VW?] appear in these 'Elements'.

In this book one will find no revolutionary new results, but for many details the presentation is new. In particular, my presentation of the structure theorems for distal and point-distal extensions of compact minimal flows is entirely based on relatively invariant measures and avoids the use of τ-topologies. However, I could not always trace back whose ideas I used: those of my own, those of Jaap van der Woude, or those of the participants of our seminar at the MC (in particular, Jan Aarts and Ietje Paalman-de Miranda). But of course, the full responsibility for all mistakes is mine.

Josi (M.H. Foe) transformed my handwritten manuscript into a file which caused our printer to produce the output which you now can see before you. She is not responsible for typographical errors, like tildes and bars over symbols that are too high: those are due to the obsolete implementation of the typesetting system (troff). The illustrations were made by me, using MacDraw II on a Macintosh Plus ED.

Jan de Vries Hierden, November, 1992

Notation

Here some non-specific notation is explained. Special symbols and notation introduced in this book are listed in the 'Index of symbols'.

Most notation in this book is standard or self-evident. For example, if in a discussion an index set Λ is fixed then in expressions like $\Sigma_\lambda x_\lambda$, $\bigcap_\lambda X_\lambda$, etc. it will be understood that λ runs through the full set Λ.

Braces are used to indicate alternative reading. For instance: "if $\{P\}\{Q\}$ then $\{R\}\{S\}$" means: "if P then R and if Q then S". Square brackets $[\![\cdots]\!]$ are used to provide a hint or a proof in telegram-style. The symbol $\square$ means "end of proof" (often: " the rest of the proof should be clear now").

The following symbols from symbolic logic will be used: $\Rightarrow$ (if $\cdots$ then), $\Leftrightarrow$ (iff, that is, if and only if), & (and), $\forall$ (for every), $\exists$ (there exists) and $\exists!$ (there exists a unique). In order to reduce the number of parenthesis we often write $\forall x,y \in X$ instead of $\forall (x,y) \in X \times X$, and $\forall x: \Phi(x)$ instead of $\forall x[\Phi(x)]$. The sign $:$ means "such that"; if it immediately precedes a quantifier ($\forall$ or $\exists$) it is omitted. The sign $:=$ means "is by definition equal to"; thus, $P := Q$ and $Q =: P$ both mean that Q defines P.

In what follows, let A, X, Y and Z denote arbitrary sets.

$A \subset X$	means:	A is a *proper* subset of X;
$A \subseteq X$	,,	: $A \subset X$ or $A = X$;
$X \sim A$	,,	: $\{x \in X : x \notin A\}$;
id_X	,,	: the identity mapping of X;
1_A	,,	: the characteristic (or: indicator) function of A;
$\|X\|$	,,	: the cardinality of X;
Δ_X	,,	: $\{(x,x) : x \in X\}$, the diagonal in $X \times X$.

The words "function", "map(ping)" and "transformation" are used, essentially, as synonyms; in particular, a mapping is not automatically assumed to be continuous. Functions are usually denoted like $f: X \to Y$ or $x \mapsto f(x): X \to Y$; sometimes parenthesis are omitted from $f(x)$. If $f: X \to Y$ is a function or, more generally, if $R \subseteq X \times Y$ is a relation, then

$f[A]$	$:= \{f(x) : x \in A\}$, the *image* of A under f for $A \subseteq X$;
$R[A]$	$:= \{y \in Y : (x,y) \in R$ for some $x \in A\}$;
$R^{\leftarrow}$	$:= \{(y,x) : (x,y) \in R\}$, the *inverse* of the relation R;
R_f	$:= \{(x_1,x_2) \in X \times X : fx_1 = fx_2\}$;
$f\|_A$	: *restriction* of f to A for $A \subseteq X$;
f^n	$:= f \circ \cdots \circ f$ (n times, $n \in \mathbb{N}$), $f^0 := \mathrm{id}_X$, provided $Y = X$.

Note that $f^{\leftarrow}$ will denote the inverse of the relation f. Thus, if $B \subseteq Y$, then

$f^{\leftarrow}[B] := \{x \in X : fx \in B\}$, the pre-image of B under f. If $y \in Y$, then we write $R[y]$ and $f^{\leftarrow}[y]$ instead of $R[\{y\}]$ and $f^{\leftarrow}[\{y\}]$. The notation $f^{\leftarrow}$ will be used only to indicate pre-images; in the case that $f : X \to Y$ is a bijection, the inverse *mapping* will be denoted by f^{-1}. Recall that the *composition of two relations* $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is defined as

$$S \circ R := \{(x,z) \in X \times Z : \exists y \in Y \text{ with } (x,y) \in R \ \&\ (y,z) \in S\}.$$

In this context, if $(x,y) \in R$ and $(y,z) \in S$ we shall sometimes write

$$(x,z) = (y,z) \circ (x,y) \in S \circ R.$$

If $\phi: X \times Y \to Z$ is a mapping then the following notation is used:

$$\phi^x(y) := \phi(x,y) =: \phi_y(x) \quad \text{for} \quad (x,y) \in X \times Y.$$

This defines mappings $\phi^x: Y \to Z$ $(x \in X)$ and $\phi_y: X \to Z$.)[†]

The *evaluation mapping* $(f,x) \mapsto f(x)$: $Y^X \times X \to Y$ will be denoted by δ (usually it will be clear from the context which evaluation map is considered, so the same δ is used in all cases). Thus, $\delta_x : f \mapsto f(x)$: $Y^X \to Y$ is the evaluation-at-x $(x \in X)$, and $\delta^f = f$.

Concerning topology, our notation and terminology are standard; in most cases we follow [Wi]. In particular, if X and Y are topological spaces, $\mathcal{T}$ the topology of X, $x \in X$, $A \subseteq X$ and ϕ: $X \to Y$ a continuous mapping, then:

$\overline{A}, \mathrm{cl}A, \mathrm{cl}_X A$	means:	the closure of A, in X (with respect to $\mathcal{T}$);
$\mathrm{int}A, \mathrm{int}_X A$	" :	the interior of A, in X (with respect to $\mathcal{T}$);
nbd of A	" :	any subset of X containing A in its interior;
nbd of x	" :	any subset of X containing x in its interior;
$\mathcal{N}_A, \mathcal{N}_A(X)$	" :	the set of all nbds of A in X;
$\mathcal{N}_x, \mathcal{N}_x(X)$	" :	the set of all nbds of x in X;
$\mathcal{G}_X, \mathcal{G}(X)$	" :	the set of all *non-empty* open subsets of X;
$\mathcal{F}_X, \mathcal{F}(X)$	" :	the set of all *non-empty* closed subsets of X;
$C(X,Y)$	" :	the set of all continuous maps $f : X \to Y$;
$H(X,X)$	" :	the set of all homeomorphisms of X onto X;
$C(X)$	" :	$C(X,\mathbb{R})$;
$C^*(X)$	" :	$\{f \in C(X) : f \text{ is bounded }\}$;
$\lVert f \rVert_A$ for $f \in C(X)$	" :	$\sup \{ \lvert f(x) \rvert : x \in A\}$;
$\tilde{\phi}(f)$ for $f \in C(Y)$	" :	$f \circ \phi$.

The topology on function spaces will be denoted by subscripts: p for the pointwise (or: product) topology, c for the compact-open topology, u for the topology of uniform convergence; thus, e.g., $C_p(X)$, $C_c(X)$, $C_u^*(X)$.

The elements of a uniform structure $\mathcal{U}$ on a space X are denoted by lower case Greek letters (except in metric spaces: see below). Thus, if $\alpha \in \mathcal{U}$ and $x \in X$ then $\alpha[x] = \{x' \in X : (x,x') \in \alpha\}$ is a basic nbd of x in the topology generated by

)[†] For $f : \mathbb{Z} \times X \to \mathbb{Z} \times X$ it will be made clear in the context what f^n means $(n \in \mathbb{Z})$: $f^n(m,x) = (f \circ \cdots \circ f)(m,x)$ or $f^n(x) = f(n,x)$.

(compatible with) $\mathfrak{U}$. However, if the uniformity is generated by a metric d on X, then for $\epsilon>0$ we put

$$S_\epsilon := \{(x,x')\in X\times X:\ d(x,x')<\epsilon\};$$

so $S_\epsilon[x]$ is the open ϵ-ball with radius ϵ and center x.

For topological groups our main reference is [HR] (knowledge of its Sections 4-8 will be assumed) or the relevant sections in [Wi]. Notation and terminology follow these references. Moreover, if T is topological group, then:

e, e_T	denote:	the unit element of T;
ω, ω_T	,, :	the multiplication $(s,t)\mapsto st$ in T;
T_d	,, :	the group T with its discrete topology;
$RUC(T)$	,, :	the set of right uniformly continuous real-valued functions;
$LUC(T)$	,, :	the set of left uniformly continuous real-valued functions;

$RUC^*(T) := RUC(T)\cap C^*(T)$;
$LUC^*(T) := LUC(T)\cap C^*(T)$.

To avoid all misunderstanding (many authors do it just the other way round), if $f\in C(T)$ then $f\in RUC(T)$ iff

$$\forall\epsilon>0\ \exists U\in\mathfrak{N}_e\ \forall s,t\in T:\ st^{-1}\in U \Rightarrow |f(s)-f(t)|<\epsilon,$$

and $f\in LUC(T)$ iff

$$\forall\epsilon>0\ \exists U\in\mathfrak{N}_e\ \forall s,t\in T:\ s^{-1}t\in U \Rightarrow |f(s)-f(t)|<\epsilon.$$

The meaning of $RUC_c(T)$ and $LUC_u^*(T)$ should be clear: the spaces $RUC(T)$ and $LUC^*(T)$ endowed with the compact-open topology, respectively, the topology of uniform convergence on T.

If H is a subgroup of T then T/H will denote the space of all *left* cosets tH ($t\in T$) of H in T, endowed with its quotient topology. The quotient map $T\to T/H$ will often be denoted by $[\,-\,]_H$; so for $t\in T$ and $A\subseteq T$:

$$[t]_H := tH,\quad [A]_H := \{sH:\ s\in A\}.$$

Standard notation:

$\mathbb{C}$	:	the set of all complex numbers;
$\mathbb{R}$	:	the set of all real numbers;
$\mathbb{Q}$	:	the set of all rational numbers;
$\mathbb{Z}$	:	the set of all integers;
$\mathbb{N}$	:	$=\{1,2,3,\cdots\}$;
$\mathbb{S}^1, \mathbb{T}$	:	unit circle in $\mathbb{R}^2$, resp., $\mathbb{C}$;
$\mathbb{T}^2$	:	2-torus $\mathbb{T}\times\mathbb{T}$.

If $A,B\subseteq\mathbb{R}$ then

$A^+ := \{a\in A:\ a\geqslant 0\}$, $A^- := \{a\in A:\ a\leqslant 0\}$;
$-A := \{-a:\ a\in A\}$;
$A^{-1} := \{a^{-1}:\ a\in A\}$ (provided $0\notin A$);
$A+B := \{a+b:\ a\in A\ \&\ b\in B\}$, $A-B := a+(-B)$;

$AB \quad := \{ab \colon a \in A \ \& \ b \in B\}$.

Thus, e.g., $A^+ = A \cap \mathbb{R}^+$; note the difference between $\mathbb{N}$ and $\mathbb{Z}^+$: $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. *Intervals* in $\mathbb{R}$ are denoted as follows ($a,b \in \mathbb{R}$, $a \leqslant b$):

$$[a\,;b] := \{t \in \mathbb{R} \colon a \leqslant t \leqslant b\},\ [a,b) := [a\,;b] \sim \{b\},\ (a\,;b] := [a\,;b] \sim \{a\},$$
$$(a\,;b) := [a\,;b] \sim \{a,b\}.$$

Intervals in $\mathbb{Z}$ are denoted similarly. Thus (when it is clear from the context that an interval in $\mathbb{Z}$ is meant), $[p\,;q] := \{n \in \mathbb{Z} \colon p \leqslant n \leqslant q\}$ for $p,q \in \mathbb{Z}$, $p \leqslant q$, etc. Occasionally, intervals of ordinal numbers are denoted in the same way.

Contents

Chapter I

Various Aspects of the Theory of Dynamical Systems

In this introductory chapter a superficial sketch is given of the theory of dynamical systems: its genesis and its contents. This is done mainly as an attempt to explain the relationship of *Topological Dynamics* with other parts of dynamical systems theory, notably *Ergodic Theory* and *Differentiable Dynamics* (also called the Theory of Smooth Dynamical Systems). Although in this chapter almost no 'abstract' Topological Dynamics will be treated, we hope in this way to contribute to an understanding of its roots.

A current definition of Topological Dynamics says that it is 'the study of transformation groups with respect to those topological properties whose prototype occurred in classical dynamics' (cf. W.H. GOTTSCHALK & G.A. HEDLUND [1955], p. iii). If in this definition the adjective 'topological' is replaced by 'measure-theoretic' then one obtains a description of Ergodic Theory (cf. P. WALTERS [1982], p. 1). Similarly, 'differentiable' (or 'smooth') instead of 'topological' gives a description of Differentiable Dynamics. Thus, in each of these three fields of mathematical research one studies groups (and also semigroups) of transformations of a space that preserve the structure of the space (either a topological, or a measurable, or a differentiable structure). And although the three fields have the same origin, namely, the study of (classical) dynamical systems, the difference in methods and techniques has given them completely different flavours, and they have developed in different directions. Nevertheless, it seems useful to pay attention to what they have in common: the type of properties that occur in classical dynamics (in this context, the phrase 'qualitative properties' is often used).

Thus, in Section 1 we sketch how problems from the study of differential equations can give rise to the notion of a continuous flow, and also how the study of discrete flows can be useful for this. In Section 2 one may find a brief discussion of the fundamentals of Ergodic Theory and its impact on the topological theory of dynamical systems. Section 3 contains some basic ideas from Differential Dynamics. All discussions are quite sketchy: Section 1 because otherwise there would be too much overlap with the remainder of the book, Sections 2 and 3 because of the fact that their contents are outside the scope of this book (in addition, there exist excellent expositions of these topics). The only purpose of this chapter is to give the reader a (maybe, very faint) idea of the theory of Dynamical Systems in its broadest sense. Moreover, we feel that every student in Topological Dynamics should at least know what Ergodic Theory and Differentiable Dynamics are about. The contents of this chapter will not be used in the remainder of the book. In particular, one may start reading in Chapter 2 without any difficulty. (Only occasionally there will be a cross-reference to some item in Chapter 1, for motivation.)

1. DYNAMICAL SYSTEMS

(1.1) A *dynamical system* may be defined as follows. It consists of a space M, the so-called *phase space*, which is to be interpreted as the set of all possible states of some fictitious physical system; in addition, there is a 'rule of evolution' which describes how any state assumed by our fictitious physical system changes with time. To be more precise, consider the case of continuous time†,

† The adjective 'continuous' refers to the nature of the time-variable, which is taken from the real numbers (often $\mathbb{R}$ is called 'continuous' because it has no gaps). Some authors prefer the term *real* time (and, correspondingly, *real* dynamical system instead of continuous dynamical system).

and suppose that our physical system is *stationary*, that is, if at a certain moment the system is in state x then the state which is reached after a time interval of length t depends only on x and t and *not* on the particular moment that this time interval begins (i.e., the moment that state x is assumed). Let $\pi(t,x)$ denote the state of the system reached after a time interval of length t when it starts in state x ($t \geqslant 0$ and $x \in M$). Then $\pi(s, \pi(t,x))$ is the state reached after a time interval $s+t$ when starting in x, i.e.,

$$\pi(s, \pi(t,x)) = \pi(s+t,x). \tag{1}$$

By the definition of π we also have

$$\pi(0,x) = x. \tag{2}$$

In the above we allowed only $s \geqslant 0$ and $t \geqslant 0$. For many systems one can 'reverse time', i.e., $\pi(t,x)$ is defined and (1) holds also for negative values of s and t. In that case we have a mapping $\pi\colon \mathbb{R} \times M \to M$ such that (1) and (2) hold for all $s,t \in \mathbb{R}$ and $x \in M$. Such a mapping is called *an action of (the additive group)* $\mathbb{R}$ *on* M.

The idea that $\mathbb{R}$ 'acts' on M is clarified by the following notational convention: define for every $t \in \mathbb{R}$ the transformation $\pi^t\colon M \to M$ by

$$\pi^t(x) := \pi(t,x) \quad (x \in M). \tag{3}$$

So π^t sends every state to the state that will be reached after a time interval of length t. Using this notation, (1) and (2) can be rewritten as

$$\pi^s \circ \pi^t = \pi^{s+t}; \quad \pi^0 = \mathrm{id}_M \tag{4}$$

($s,t \in \mathbb{R}$). It follows that, for every $t \in \mathbb{R}$,

$$\pi^t \circ \pi^{-t} = \pi^{-t} \circ \pi^t = \pi^0 = \mathrm{id}_M.$$

Consequently, $\pi^t\colon M \to M$ *is a bijection with inverse* π^{-t}. So (4) can be reformulated as follows: the mapping $t \mapsto \pi^t$ is a homomorphism of the additive group $\mathbb{R}$ into the group of all bijections of M (with composition of mappings as group operation).

(1.2) The mathematical theory of dynamical systems originated from classical mechanics[1]: originally, the phrase 'dynamical system' referred to a physical system as studied in classical dynamics. Typically, such a system is described by its *state* and its *law of motion.* The state is usually composed of the *configuration* (= spatial position of the constituents) of the system, and the *velocity* with which this configuration changes in time. In the most simple cases, the state can be represented by a point in $\mathbb{R}^n$ for some $n \in \mathbb{N}$, say with $n = 2k$, where k coordinates are used for the configuration, and the remaining k coordinates for the velocity of the system. The law of motion describes how the state of the system will evolve in time. If this law itself doesn't change in time (i.e., the physical system is stationary) then it has the form of a 'system' of autonomous ordinary differential equations†:

† In order to distinguish the various meanings of the word *system* we use quotation marks when it means 'interdependent family' (of equations).

$$\dot{x}_i(t) = F_i(x_1(t), \ldots, x_n(t)), \quad i=1, \ldots, n. \tag{5}$$

Here $x(t) := (x_1(t), \ldots, x_n(t))$ denotes the state of the physical system at time t and the dot represents differentiation with respect to t. In a more or less symbolic fashion, equations (5) are often denoted as follows: $\dot{x}_i = F_i(x_1, \ldots, x_n)$, $i=1, \ldots, n$, and we shall follow this usage. The set of all possible states in $\mathbb{R}^n$ is called the *phase space* of the system (formerly, the word 'phase' was used instead of 'state'). If the state of the system changes with time then the point representing the state moves along some curve in the phase space: the *trajectory* or *orbit*. It should be clear that this curve is a solution curve of (5).

In many cases, it is not possible to establish a suitable 1,1-correspondence between all states of a physical system and all points of some Euclidean space, but often it is possible to establish such a correspondence locally: in that case the phase space is a (desirably smooth) manifold. Let us mention two examples:

1. The configuration space of a 'planar' double pendulum (see Figure 1.2.1) is the direct product of two circles, i.e., the 2-torus $\mathbb{T}^2$. The corresponding phase space is the tangent bundle of $\mathbb{T}^2$.
2. To describe the motion in $\mathbb{R}^3$ of a rigid body with a fixed point P, introduce two orthogonal coordinate systems S and S', both with the origin in P. Let S be fixed and let S' be rigidly connected with the body. The position of the rigid body is completely described by the position of S' with respect to S, i.e., by an orthogonal 3×3-matrix with determinant $+1$. So for this system the configuration space is $SO(3)$, the special orthogonal group of order 3.

An additional reason for considering more general phase spaces is the following. In classical mechanics much attention is given to so-called 'conserved quantities' or 'constants of motion', also called *first integrals* (of the differential equation describing the mechanical system). A first integral is a real-valued function f, defined on the phase space M which is constant on every trajectory. It follows that for every value c the set $M_c := \{x \in M : f(x) = c\}$ is an *invariant subset* (even a submanifold if f is smooth), i.e., M_c has the property that every trajectory which passes through one of its points lies entirely in M_c. The study of such invariant subsets and of the restriction of the motion to such subsets

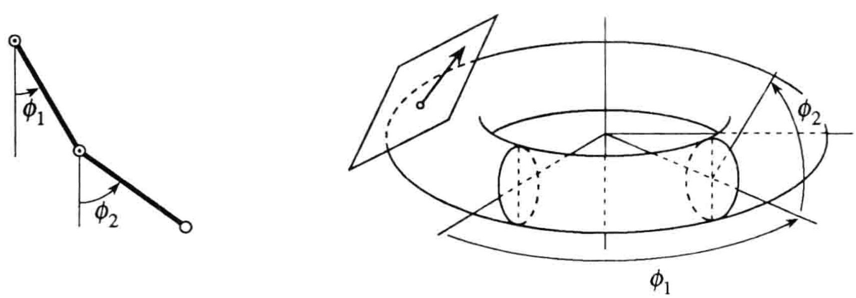

FIGURE 1.2.1. Configuration space of the planar double pendulum.

can reveal much of the behaviour of the system. But if one restricts him/herself to M_c then in general one is working on some manifold (even if the original phase space is Euclidean).

If the phase space of a dynamical system is a manifold M, then *locally* (in each chart) the law of motion of the mechanical system can still be expressed by (5). But *globally* the description is as follows: in every point x of M a tangent vector $F(x)$ is given (so we have a *vector field* on M, or more precisely, a tangent vector field) and if $x(t)$ denotes the state of the mechanical system at time t, then the velocity $\dot{x}(t)$ with which the state changes is required to satisfy the equality

$$\dot{x}(t) = F(x(t)). \tag{6}$$

Consequently, trajectories are curves in M such that in each point of such a curve the vector field is tangent to the curve. The task of finding such curves is often formulated as: solve (or: integrate) the equation $\dot{x} = F(x)$.

Now consider (6) on some manifold M. Using the theory of ordinary differential equations it can be shown that, under quite weak conditions on F (usually satisfied for classical dynamical systems)[2], for every $x \in M$ there exists a *unique* differentiable mapping $\pi_x: \mathbb{R} \to M$ such that

$$\pi_x(0) = x, \quad \dot{\pi}_x(t) = F(\pi_x(t)) \quad (t \in \mathbb{R}), \tag{7}$$

i.e., a solution of (6) with initial value x. Also, the solutions in most cases depend continuously on the initial conditions, i.e.,

$$\pi: (t,x) \mapsto \pi_x(t): \mathbb{R} \times M \to M \tag{8}$$

is a continuous mapping ($\mathbb{R} \times M$ endowed with the product topology). *This mapping π is an action of $\mathbb{R}$ on M*: equality (1) is a straightforward consequence of the unicity of solutions of (6) ⟦both $s \mapsto \pi_y(s)$ with $y := \pi_x(t)$ and $s \mapsto \pi_x(s+t)$ are solutions of the differential equation with initial value y for $s = 0$⟧, and equality (2) is part of (7). The mapping π defined in (8) is called *the action of $\mathbb{R}$ on M defined by the vector field F* (or: by the differential equation $\dot{x} = F(x)$). (The proof that π is an action essentially uses that equation (6) is autonomous. As the fact that the equation is autonomous is an expression of the fact that the corresponding physical system is stationary, this is in accordance with (1.1).)

(1.3) The notions described above enable one to use a 'kinematic' description of the solutions of a 'system' of equations like (5)—or of an equation on a manifold like (6): one considers the phase space as a set of points flowing along their trajectories. Actually, the word *flow* is often used for such a description. Of course, this way of looking at the family of solution curves is applicable to all 'systems' of autonomous differential equations, whether they arise from classical mechanics or not (provided existence and unicity of solutions is guaranteed). In fact, this point of view can even be adopted when one is dealing with an arbitrary action π of $\mathbb{R}$ on a space M: the point $\pi(t,x)$