THE MECHANICS OF VIBRATION

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PREFACE

The understanding of the physical nature of a vibration problem is often both more difficult and more important than the calculation of frequencies and modal shapes. The latter can be tackled readily once a few techniques have been mastered; and modern computing machinery has removed the one difficulty which previously existed—that of excessive complexity. While, however, the study of methods of frequency calculation is often an important aid to understanding the physical nature of a problem, it is not always sufficient to enable the engineer to obtain the grasp that he requires. Further, the ease with which numerical work can now be handled makes a thorough understanding of *physical* problems even more necessary; without it the most important calculations may not be attempted and time may be wasted on others which are of little value.

It was when one of us was working in industry that he came to appreciate the value of the concept of receptance as an aid to understanding as well as to calculation. The concept was, at that time, being used in studying the vibration of aeroplane propellers in conjunction with the torsional vibration of the engine shafting on which they were mounted; it allowed the vibrational characteristics of the propellers and of the shafting to be discussed independently and this was helpful because—amongst other reasons—the propellers and engines were often made by different manufacturers! The receptance concept is used extensively throughout this book and, in particular, the classical theory of the small oscillation of a linear system is developed by means of it. We believe that the approach will be satisfying to the reader and that the familiarity with receptances which he will gain will enable him to apply them when examining new problems.

The notion of receptances is particularly useful when the vibration of one portion of a system, such as a single turbine blade, is being studied independently of the rest of the system. The justification for isolating a part of a system in this way, for the purposes of analysis, is commonly that of intuition and nothing more; the receptance idea should often enable the engineer to reconcile the intuitive results with sound reasoning or alternatively to reject them when they cannot be so reconciled. This revaluation of the results of intuition is one of the responsibilities of the academically trained engineer.

Since the aim of this book is to present ideas, rather than to describe the application of those ideas to particular engineering problems, much of the discussion is in terms of 'academic' systems only, that is to say of spring-connected rigid masses, taut strings and uniform elastic bodies. In Chapter 5, however, there is some discussion of the extent to which real systems may be analysed in terms of these ideal systems.

Some guidance may be needed by those who wish to study the receptance concept without working through the whole book, or by others who have other special needs. Chapter 1 introduces receptance ideas but the discussion is limited to very simple mechanical systems. Chapter 2 deals with dynamical methods for analysing more complex systems and Chapter 3 uses these methods in a general treatment of the vibration of multi-freedom systems. This latter chapter contains most of the essential ideas in receptance theory. After this, with the exception of Chapter 5, which has been mentioned above, the discussion is extended to include elastic bodies and the various chapter headings explain which types of system are treated in each. Systems with damping are then introduced, the general theory for these being developed in a similar way to that of Chapter 3. Finally the excitation of a system by transient forces is treated.

It will be seen that many items which are of interest to the engineer have been omitted. For instance, questions of non-linearity, instability, self-excitation, the vibration of rotating elastic bodies, as well as matters of analytical and experimental technique have not been dealt with. This is admittedly a somewhat arbitrary division; but it was our original intention to write more than one volume. Perhaps a second will be written, although we no longer enjoy the advantage of working in the same laboratory.

Much of the theory that is presented is capable of concise proof by matrix methods. These methods have acquired an important place in vibration analysis, particularly as a result of modern developments in high-speed computing. Nevertheless it is our belief that matrices should not be used by engineers when mastering the fundamentals of mechanical vibration theory, if only because matrices *are* so convenient. It is all too easy to become facile in the mathematical sense without acquiring a true understanding of the physical side of things.

Examples are provided at the ends of most of the sections in this book and we have used the 'absolute' system of units both in these and in the text. That is to say, the units of force, mass, length and time are the poundal, pound, foot and second respectively. In doing this we have made free use of a convenient approximation, namely that $g = 32$ ft./sec.^{2} (and that 32 poundals = 1 lb.wt.).

Finally we should like to mention a number of helpers, other than those whose names are mentioned later in connection with specific items. First we acknowledge our debt to Professor W. J. Duncan for his encouragement and for the benefit which we obtained from his pioneer work in the use of admittance methods for mechanical systems; without this our book would never have been started. Secondly we thank Miss P. L. A. Baker for typing the complete manuscript, and Messrs. B. Wood, S. Hother-Lushington and A. G. Parkinson for reading the proofs.

R. E. D. B.
D. C. J.

December 1957

GENERAL NOTATION

See also the Supplementary Lists of Symbols at the ends of chapters

A	Compound oscillatory system.
${}_kA_{rs}$	Numerator of kth partial fraction in series representation of α_{rs} [see equation (3.4.3)].
a_{rs}	Inertia coefficient [see equation (3.1.2)].
a_r	Inertia coefficient [see equation (3.6.9)].
$B, C, D, \ldots$	Sub-systems which, together, make up A.
b	Viscous damping coefficient of isolated damping element.
b_{rs}	Viscous damping coefficient [see equation (8.4.8)].
c_{rs}	Stability (or 'stiffness') coefficient [see equation (3.1.7)].
c_r	Stability (or 'stiffness') coefficient [see equation (3.6.9)].
D	Dissipation function [see equation (8.4.19)].
d_{rs}	Hysteretic damping coefficient [see equation (9.4.6)].
E	Young's modulus.
e	The exponential constant.
F	Amplitude of harmonic applied force or torque.
F_r	Amplitude of harmonic force or torque applied at x_r. If a letter (as well as a numerical) subscript is carried, this refers to a sub-system.
G	Shear modulus.
g	Gravitational constant (taken as 32 ft./sec. throughout this book).
h	Particular value of x defining a section of a taut string, shaft, bar, beam, distant h from origin ($0 \leqslant h \leqslant l$); hysteretic damping coefficient of isolated damping element.
I	Moment of inertia of rigid disk; second moment of area of cross-section of beam about its neutral axis.
i	Imaginary operator [see equation (1.3.1)].
J	Second polar moment of area of circular shaft.
k	Stiffness.
l	Length of taut string, shaft, bar, beam.
M	Mass.
m	Mass of particle.
N	Magnification factor (see § 8.2).
n	Magnification factor [see equation 9.2.10)].
P_r	Generalized force corresponding to p_r.
p_r	rth principal co-ordinate.
Q_r	Generalized force corresponding to q_r.
${}_kQ_1, {}_kQ_2, \ldots, {}_kQ_n$	Set of generalized forces $Q_1, Q_2, \ldots, Q_n$ which produce distortion in kth principal mode only.

q_r	rth generalized co-ordinate.
$_kq_r$	Displacement at q_r in the kth principal mode.
R	Amplitude of x (a real constant).
S	Dissipation function for hysteretic damping [see equation (9.4.8)].
T	Kinetic energy.
t	Time.
U	$= V+S$.
V	Potential energy.
v	Lateral deflexion of taut string or beam.
v_r	Lateral deflexion of taut string or beam in rth principal mode.
W_r	Work done by forces of a system due to displacement at q_r.
w	Applied lateral load/unit length on beam or taut string.
w_r	Applied lateral load/unit length causing deflexion of a beam or taut string in its rth principal mode only.
X_r	Amplitude of displacement at x_r (may be complex). If a letter (as well as a numerical) subscript is carried, this refers to a sub-system.
x	Distance along taut string, shaft, bar, beam ($0 \leqslant x \leqslant l$).
x_r	rth co-ordinate.
α_{rs}	Receptance of system A.
α_r	Receptance at rth principal co-ordinate.
$\beta_{rs}, \gamma_{rs}, \delta_{rs}, \ldots$	Receptances of sub-systems $B, C, D, \ldots$.
Δ	Denominator of expressions for the receptances of a system having finite freedom [see equation (3.2.6), (8.5.9) or (9.5.5)], being the determinant of the coefficients of equations of motion.
Δ_{rs}	Determinant formed from Δ by omission of the rth column and sth row.
Ξ_r	Amplitude of P_r.
Π_r	Amplitude of p_r.
ρ	Mass density.
Φ_r	Amplitude of Q_r.
$\phi_r(x)$	rth characteristic function of taut string, shaft, bar, beam.
Ψ_r	Amplitude of q_r.
Ω_r	rth anti-resonance frequency (rad./sec.).
ω	Circular frequency of excitation (rad./sec.).
ω_r	rth natural frequency (rad./sec.).

CONTENTS

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CHAPTER 1

INTRODUCTION

The position of the moving parts of a steam-engine indicator clamped to an engine may be specified by giving the displacement of the pencil from some fixed point, or the displacement of the indicator piston, or the inclination of the pencil lever to the horizontal. Here again, the specification will be only a good approximation in actual fact, because, for example, the pencil lever will bend to some extent when rapidly moved.

B. Hopkinson, *Vibrations of Systems having One Degree of Freedom* (1910)

1.1 Preliminary remarks

The subject of this book is the theory of the small oscillations of a dynamical system. This subject is important to the engineer because all materials that are used in the construction of machinery possess mass and the ability to store potential energy through the property of elasticity; the combination of these properties renders vibration possible. Gravity and other effects can also affect the potential energy of a system and so modify the vibration.

For the purpose of the study it will be convenient to classify systems by their 'degrees of freedom'. The number of degrees of freedom that a system possesses is the number of co-ordinates which must be specified in order to define its configuration. Thus a simple mathematical pendulum has one degree of freedom if its motion is confined to a single plane but it has two degrees of freedom if it can swing in more than one plane. Now all real systems have an infinite number of degrees of freedom and this renders impossible a complete vibration analysis. Even if such an analysis were possible the effort of making it would be mostly wasted because it would yield far more information than could be used. It is therefore necessary to make some simplifying assumptions about the motion of a system if an analysis is to be feasible.

Now it usually appears from inspection that some types of motion are likely to be unimportant; these can then be imagined to be suppressed entirely for the purpose of analysis. It will be shown later that the extent to which such restrictions can be justified depends on the frequencies of vibration to which the system is subjected. One of the most common simplifications of this type is to regard each vibrating link of a mechanism as a number of spring-connected rigid masses, the springs being treated as massless and therefore incapable of surging; it is sometimes possible to simplify further by restricting the rigid bodies to zero dimensions so that the links become systems of spring-connected particles. We may consider as an example the system that is formed by two flywheels which are joined by a shaft; the torsional oscillations of the system may be examined under the assumption that the wheels are rigid and the shaft massless. Simplifications of this type reduce the number of degrees of freedom of the system and thereby simplify the mathematical treatment.

We shall be concerned, in this book, with the analysis of systems after they have been simplified. The reader should be warned, however, that this is only part of the engineer's problem; the process of simplification of the system is often difficult and may need considerable knowledge and experience.

The vibrations which we shall discuss will be linear. That is to say, they will be governed by linear differential equations which have constant coefficients. In some problems this will be a result of the form taken by the system under discussion; thus, a linear equation governs the free oscillations of a mass attached to the free end of a massless spring. The form of the equation is due, in this case, to the nature of the spring—that is, it obeys Hooke's law—and to the nature of the oscillation—namely one of motion along the axis of the spring. But in order to arrive at a linear equation it is sometimes necessary, not only to have a suitable system, but also to restrict its motion to *small* oscillations. This is required when the geometry of the system would otherwise vary appreciably during the motion. For instance, a simple pendulum of length l oscillates according to the equation

$$\ddot{x}+\frac{g}{l}\sin x = 0, \tag{1.1.1}$$

where x is the angle made with the vertical. Only by limiting the angle of swing can we arrive at the linear equation

$$\ddot{x}+\frac{g}{l}x = 0. \tag{1.1.2}$$

All real mechanisms are affected by frictional forces, of which there are several causes. It is expedient to neglect these forces entirely during the development of the analysis. Their effects will be discussed in Chapters 8 and 9.

We shall begin our analysis by considering the motion of a system due to a sinusoidally varying force. This motion is of direct importance where the effects of lack of balance of rotating machinery are to be discussed. Such analysis also covers, through the medium of Fourier analysis, the effects of more complex periodic excitation. The motion of a system under transient loading, for instance pulse excitation, will be treated in Chapter 11 where it will be shown that the results for sinusoidal forcing provide useful data for the solution of the transient problem.

EXAMPLE 1.1

1. How many degrees of freedom have the following systems?
   (*a*) A particle in space.
   (*b*) A particle which is constrained to move along a fixed tortuous curve.
   (*c*) A lamina which can move in its own plane only.
   (*d*) A four-bar kinematic chain with one link fixed.
   (*e*) A four-bar kinematic chain with no links fixed.
   (*f*) A rigid gyroscope rotor in gimbals.
   (*g*) A tramcar.
   (*h*) A motor car.
   (*i*) An aeroplane.

For (*g*), (*h*) and (*i*) it is intended that the bodies should be treated as rigid.

1.2 Systems with one degree of freedom

The simplest type of vibrating system has a single degree of freedom. Consider, as an example, that shown in fig. 1.2.1, in which the rigid body of mass M is attached to a fixed abutment through the massless spring of stiffness k. Displacements of the mass from its equilibrium position will be denoted by x. The equation of motion of the system, when it is vibrating freely, is

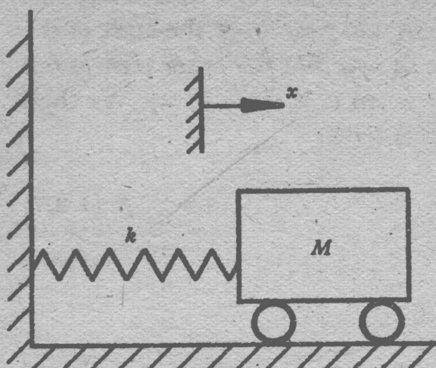


Fig. 1.2.1

$$\ddot{x}+\omega_1^2 x = 0, \tag{1.2.1}$$

where $\omega_1^2 = k/M$. The solution of (1.2.1) may be written in the forms

$$\left.\begin{aligned} x &= A\cos\omega_1 t + B\sin\omega_1 t \\ &= R\cos(\omega_1 t-\phi) \\ &= R\sin(\omega_1 t+\psi), \end{aligned}\right\} \tag{1.2.2}$$

where A, B, R, ϕ and ψ are constants which are determined by the initial conditions. The quantity ω_1 is sometimes called the 'circular frequency'; it will be convenient, when there is no danger of confusion, to refer to it simply as 'the frequency'.

Equation (1.2.1) is easily formed by the application of Newton's laws. With some single-degree-of-freedom systems, however, this method is somewhat laborious and an energy method may be used instead. Then, the kinetic energy T and the potential energy V of the system concerned are written as functions of the single co-ordinate. The equation of free motion can then be found from the relation

$$\frac{d}{dt}(T+V) = 0 \tag{1.2.3}$$

because the total energy of the system remains constant in the absence of external and frictional forces. Alternatively we may assume the motion to be harmonic and can then find the frequency from the relation

$$T_{\text{max.}} = V_{\text{max.}}, \tag{1.2.4}$$

that is, by equating the maximum values of T and V. This latter method requires the value of the potential energy to be taken as zero in the equilibrium position. These two energy methods can be tested on the system of fig. 1.2.1. A third, and much more powerful energy method—that of Lagrange—will be introduced in Chapter 2.

Now let a harmonic force $F\sin\omega t$ be applied to the mass M along the direction of the motion. The equation now becomes

$$\ddot{x}+\omega_1^2 x = \omega_1^2\left(\frac{F}{k}\right)\sin\omega t \tag{1.2.5}$$

and the general solution to this is the sum of the complementary function (1.2.2) and a particular integral. The latter corresponds, as may be expected, to a motion of the mass with the frequency ω of the disturbing force; the general solution is found to be

$$x = [A\cos\omega_1 t + B\sin\omega_1 t] + \frac{F/k}{1-\dfrac{\omega^2}{\omega_1^2}}\sin\omega t. \qquad (1.2.6)$$

The terms within the bracket in this expression allow, by a suitable choice of A and B, for the initial conditions to be satisfied; it will be shown later that in any real system these terms are reduced to zero, after a suitable time from the start of the motion, by the action of damping. It will also be shown that the other term is not greatly affected by small damping forces except when ω is close to ω_1. We shall therefore confine our attention to the forced motion term

$$x = \frac{F/k}{1-\dfrac{\omega^2}{\omega_1^2}}\sin\omega t. \qquad (1.2.7)$$

The solution (1.2.7) may be written in the form

$$x = \left(\frac{F}{k}\right) N\sin(\omega t - \zeta), \qquad (1.2.8)$$

where N is the 'magnification factor' which is defined by the relation

$$N = \left|\frac{1}{1-\dfrac{\omega^2}{\omega_1^2}}\right|, \qquad (1.2.9)$$

and ζ is a phase angle which is zero for $\omega < \omega_1$ and π for $\omega > \omega_1$. These two quantities are functions of ω/ω_1 only and are shown sketched in fig. 1.2.2. It must be emphasized that any point on these curves represents a particular solution to equation

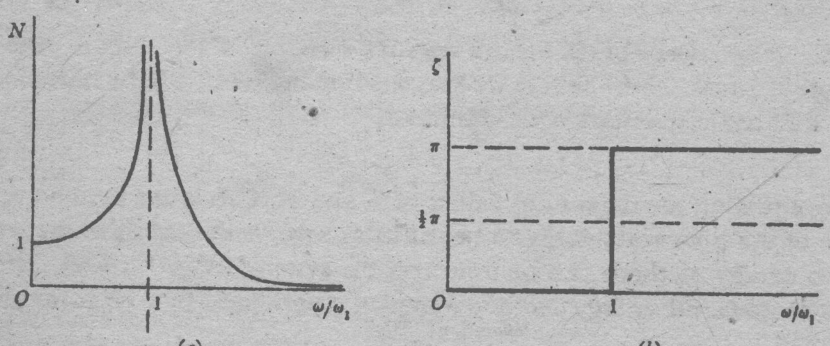


Fig. 1.2.2

(1.2.5) for the appropriate value of ω/ω_1; the curves do not imply that if, for a specified system, ω were varied, then the motion at any moment would be given by them. The forcing of the system at a *varying* frequency would imply transient excitation for which it is not permissible to neglect the complementary function.

Later on, we shall often meet the curves of fig. 1.2.2 in a slightly different form. This is obtained by writing (1.2.7) in the form

$$x = \left[\frac{1}{M(\omega_1^2-\omega^2)}\right] F \sin \omega t \tag{1.2.10}$$

and then plotting the quantity in the square bracket against ω. The curve is shown in fig. 1.2.3.

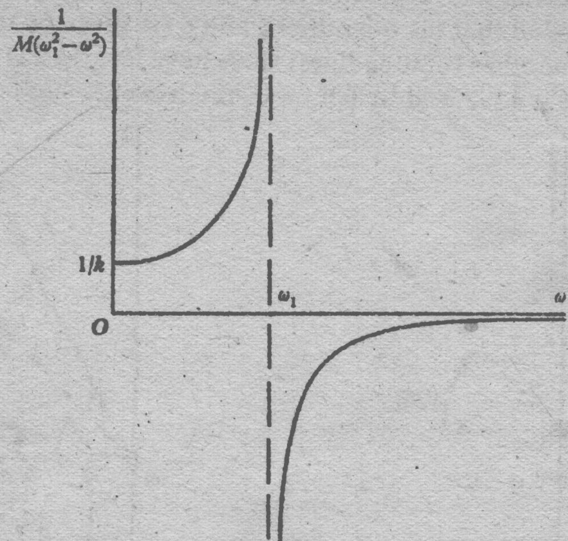


Fig. 1.2.3

As ω is taken closer and closer to ω_1, the amplitude of the motion increases without limit if the force amplitude is kept the same. Alternatively the force amplitude must be diminished indefinitely if the displacement amplitude is not to increase. The latter statement is usually preferable because an infinitely large amplitude is inconsistent with the assumption of small motion. The concept of reducing the force indefinitely as ω tends to ω_1 may be used to provide an alternative definition of natural frequency, namely that it is the frequency at which a finite response is produced by an infinitesimal force. We shall find it more convenient later to adopt this definition rather than to work explicitly with the equations of free vibration.

In using the expression (1.2.7) rather than the general solution to the differential equation the reader may prefer, at this stage, to suppose that the initial conditions were such as to make the constants A and B equal to zero. This will not impair the validity of the mathematics though it will restrict the physical application of it. Such a viewpoint avoids the apparent contradiction in the assumptions that the damping will eliminate the free motion but may be neglected in discussing the forced motion. This contradiction, and the important effect of damping on the forced motion in the immediate neighbourhood of the natural frequency, will be discussed in Chapters 8 and 9.

Both the free and forced motions of (1.2.2) and (1.2.7) are harmonic; we now introduce a method for representing such quantities graphically. A harmonically varying quantity

$$\xi = Z \sin(\Omega t + \psi) \quad (1.2.11)$$

may be represented by a rotating line. Thus if a line of length Z rotates with angular velocity Ω about one end, as shown in fig. 1.2.4, then the value of ξ is given by the projection of the moving line on a fixed line; this is the vertical line in the figure. Further, it will be found that the quantities $\dot{\xi}$ and $\ddot{\xi}$ are given by the projections on the same fixed line of other rotating lines; these have lengths $Z\Omega$ and $Z\Omega^2$ respectively, and each one is advanced by 90° on its predecessor.

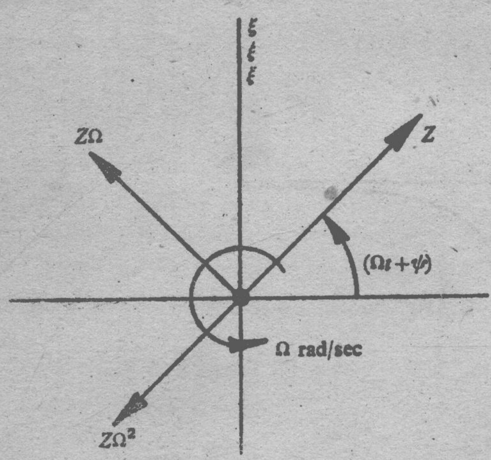


Fig. 1.2.4

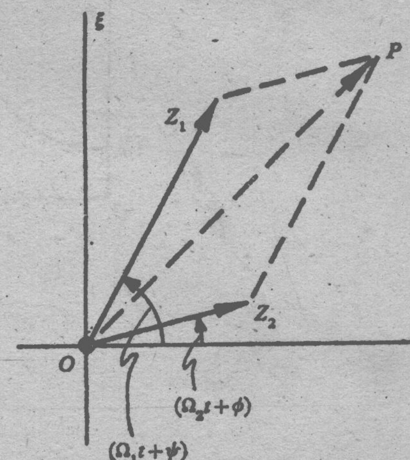


Fig. 1.2.5

An important property of these rotating lines is that they may be added vectorially. Let some quantity ξ be given as the sum of two harmonic components so that we might write

$$\xi = Z_1 \sin(\Omega_1 t + \psi) + Z_2 \sin(\Omega_2 t + \phi). \quad (1.2.12)$$

It may be seen from fig. 1.2.5 that ξ is given by the projection of the line OP, which is the vector sum of the lines whose projections give the two components of ξ. If the frequencies of the two components are not the same then the instantaneous value of ξ is still given by the projection of the vector-sum line; but under these conditions this line will not remain of constant length as it moves. In many problems only equal-frequency components are involved; when this is so the fixed line may be omitted from the figure because it is only the relative positions of the rotating lines, that is to say the shape of the diagram which depends on the relative phases, that concerns us.

When the rotating line representation is applied to the free vibration expression for the oscillator, then R of equation (1.2.2) will be identified with Z, ω_1 with Ω and x with ξ.

The forced vibration equation may also be represented in this way. The equation of motion may be written in the form

$$M\ddot{x} + kx = F \sin \omega t, \quad (1.2.13)$$

so that the right-hand side will correspond to a rotating line of length F. We confine attention to the variation of x with the frequency ω so that ω is identified with Ω. The value of Z can now be deduced from the diagram of fig. 1.2.6. In order that the three lines in the figure should fit together as shown, which they must if the right-hand side of (1.2.13) is to be the vector sum of the left-hand side, it is necessary that

$$Z(k-M\omega^2) = F,$$

or

$$Z = \frac{F}{(k-M\omega^2)} = \frac{F/k}{1-\frac{\omega^2}{\omega_1^2}}, \tag{1.2.14}$$

which agrees with the analytical solution.

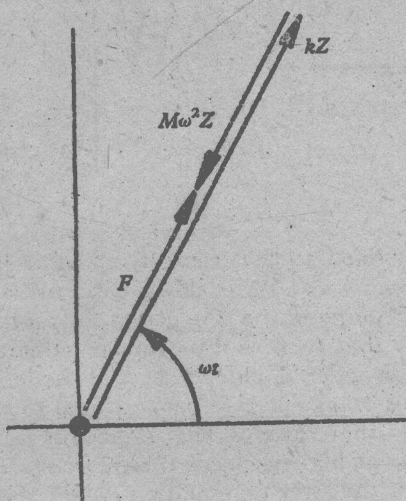
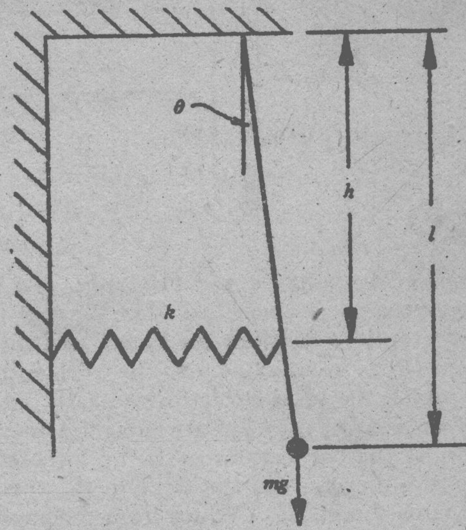


Fig. 1.2.6



Ex. 1.2.1

EXAMPLES 1.2

1. The figure (Ex.1.2.1) shows a simple pendulum, to the arm of which is attached a light spring of stiffness k. The system is in equilibrium when the arm is vertical. Find the frequency of free oscillations by

(*a*) applying Newton's laws to the free system,

(*b*) using an energy method,

(*c*) finding the resonance condition when a harmonic, horizontal disturbing force is applied to the arm at a distance $\frac{1}{2}h$ below the point of suspension.

[Note. While it is convenient to specify the point of application of the force, the choice cannot influence the value of the natural frequency which is found by this method.]

2. Two equal rollers, each of radius r, are placed with their axes at the same level and distant $2{\cdot}4r$ apart, in contact with a concave cylindrical surface of radius $3r$. They are maintained in this position by a third equal roller which is placed in contact with each of them, its axis thus coinciding with the axis of the concave surface. All the rollers are solid and homogeneous.

Show that the periodic time of a small oscillation of the three rollers about the above position of equilibrium, assuming no slipping, will be $5\pi\sqrt{(r/g)}$.

(C.U.M.S.T. Pt I, 1953)