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Edited by A. Dold and B. Eckmann

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Burnside Groups

Proceedings, Bielefeld, Germany, 1977

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University of Bielefeld, Germany
June–July 1977

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PREFACE

The present notes arose out of a workshop which was held at the University of Bielefeld in summer 1977. The main purpose of the workshop was to survey the present knowledge on Burnside groups, in particular the work of Novikov-Adian. The technical difficulties of this work are such that communication becomes a serious problem. The editors hope that the notes of Professor Adian's lectures will help a prospective reader to find access to the work which is now available in book form. The original Russian book was translated into English by Wiegold and Lennox and has appeared in the Ergebnisse series.

A first attack on the finiteness problem for $B(2, 8)$ is a second part of these notes. The authors are well aware of the incompleteness of the present results. They hope, however, that some of the methods and techniques may be helpful for future progress.

The workers in the field seem to agree that the structure of $B(2, 2^k)$ should become stable for some k .. However, it is not even clear whether one should expect that for large k these groups are finite or infinite. It seems clear, however, that $B(2, 8)$ should be an important test case.

M.F. Newman has compiled a list of problems which we hope will stimulate interest.

It is a pleasure to acknowledge financial support from Deutsche Forschungsgemeinschaft, Heinrich-Hertz-Stiftung, and University of Bielefeld. Our thanks go to the participants, in particular to Professor Sergej I. Adian who did the bulk of the lecturing. Our thanks also go to the technical staff for their valuable help: to the secretaries of the Department of Mathematics, and to the staff of ZiF in Bielefeld, and in particular to Mrs B.M. Geary who did a splendid job in typing the notes.

My personal thanks go to all the cooperators for their patience and efficiency. All the incompleteness of the present volume is my responsibility.

Bielefeld, January 1980 J. Mennicke

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CLASSIFICATIONS OF PERIODIC WORDS AND THEIR APPLICATION IN GROUP THEORY

S.I. ADIAN

1. Introduction

Recently Springer-Verlag published the English translation of my book *The Burnside Problem and Identities in Groups*. The original Russian version was published in Moscow in 1975. In this book we outlined a method for studying periodic groups, which was first introduced in the joint work of P.S. Novikov and the author, "Infinite periodic groups I, II, III", *Izv. Akad. Nauk SSSR Ser. Mat.* **32** No. 1, 2, 3 (1968). The book contains a solution of the well-known Burnside problem for large odd exponent as well as other applications of the method and its modifications to construct groups with some interesting properties.

In 1977 Professor J. Mennicke invited the author to give a series of lectures at Bielefeld University. These lectures were meant as a help for a prospective reader to find his way through the book. The main difficulty which the reader faces is a collection of some 50 notions and more than 100 basic properties of these notions. The proof of the main result then consists of a simultaneous induction step for all the notions and their properties. Because there are too many cross-references to the inductive hypothesis, for nonformal understanding of any of these definitions or properties, depending on the inductive parameter α , the reader must be familiar with the meaning of all considered notions and with many of their properties for lower values of the inductive parameter α .

In order to avoid these difficulties, a rigid exposition requires an axiomatic approach. For $\alpha = 0$, all the relevant properties are more or less obvious. But in the inductive step from α to $\alpha + 1$, we ask the reader to consider all occurring notions for rank $\leq \alpha$ in the beginning as notions defined axiomatically, satisfying some properties named axioms. So in the beginning of this step, starting from the inductive hypothesis we have to give formal deductions of all relevant properties of rank $\alpha + 1$. Then as the reader becomes more and more familiar with the meaning of the notions our exposition will become less formal and we can omit some details, which by then will be clear. After the inductive step is completed (see Adian (1979), V. 2.5, p. 236), all our notions will become meaningful for the reader and all the properties of these notions will be verified for all values of α .

The aim of the introductory lectures given by the author in 1977 at Bielefeld University was to help the nonformal understanding of a prospective reader of the book by introducing the basic ideas of our theory. In the lectures there was given a precise definition of all notions with some necessary comments and demonstrations for rank 1 and 2 . Sometimes the lectures led to informal discussions of questions raised by the audience.

The present article presents more or less the subject of these lectures. We have left out only long precise formal definitions of all notions for rank α , taking into account that in the meantime the book has become available in English, hence the reader may look up the definitions in §4 of Chapter 1. Instead, we introduce an approximative version of the most basic notions of our theory. This version is free from many technical details and demonstrates the basic ideas of our method. At the end, we give a summary of results obtained by our method.

I would like to thank the Mathematische Fakultät der Universität Bielefeld and in particular Professor J. Mennicke for his invitation and for his help during my visit to Bielefeld and the preparation of these notes.

I would like also to thank the Deutsche Forschungsgemeinschaft for their financial support.

2. Free periodic groups

As is well-known, any group can be presented by generators and defining relations. In particular, the free group $F_m = \langle a_1, a_2, \dots, a_m \rangle$ of rank m can be introduced in the following way. Consider an alphabet

(1) $$a_1, a_2, \dots, a_m, a_1^{-1}, a_2^{-1}, \dots, a_m^{-1} .$$

The defining relations of F_m are

(2) $$a_i a_i^{-1} = 1 , \quad a_i^{-1} a_i = 1 \quad (i = 1, 2, \dots, m) ,$$

where 1 denotes the empty word. From (2) we obtain the following consequences

(3) $$P a_i a_i^{-1} Q = PQ , \quad P a_i^{-1} a_i Q = PQ ,$$

where P, Q are arbitrary words in the alphabet (1). Two words X and Y are called equal in F_m (we write $X = Y$ in F_m or $X \equiv Y$) if there exists a finite sequence of words $X_1, X_2, \dots, X_\lambda$ such that $X = X_1$, $X_i = X_{i+1}$ for $1 \le i < \lambda$

and $X_\lambda = Y$ are of the form (3), possibly after permuting the left and right hand sides. This is an equivalence relation in the set of all words in the alphabet (1). Let $\{A\}$ denote the equivalence class containing A. The set of all classes with multiplication $\{X\}\{Y\} = \{XY\}$ is the free group of rank m, that is F_m; the unit element of F_m is $\{1\}$, and the inverse element of $\{X\}$ is the class $\{X^{-1}\}$, where X^{-1} is the result of writing the word X in reverse order and replacing all letters a_i (or a_i^{-1}) by their inverses a_i^{-1} (or a_i).

The presentation of a group

$$G = \langle a_1, a_2, \ldots, a_m; A_j = 1\rangle$$

with defining relations $A_j = 1$, where j runs through some index set J, is obtained from the presentation of F_m by adding the new equations

$$PA_jQ = PQ \quad \text{for all } j \in J . \tag{4}$$

In particular the free periodic group of exponent n,

$$B(m, n) = \left\langle a_1, a_2, \ldots, a_m; x^n = 1\right\rangle$$

is obtained from F_m by adding the identical relation $x^n = 1$, or, what is the same, by adding defining relations $A^n = 1$ for all words A in the alphabet (1).

Let a group G be presented by defining relations. If two words A and B are equal in this group, then by scanning all finite sequences of equations (3) and (4), beginning with the word A, we can find a sequence ending with B. But such a presentation of a group G does not give a method for proving that two words are not equal in G. Moreover, the unsolvability of the word problem for finitely presented groups in the general case means that there does not exist an algorithm which enumerates all pairs of words (x, y) which are not equal in G. To prove that A and B are not equal one has to find a property, which holds for all words equal to A and does not hold for B. To find such properties for the free periodic groups $B(m, n)$ turned out a difficult problem. For this reason the Burnside problem raised in 1902 remained open for a long time (see Burnside (1902)). In the work (Novikov and Adian (1968)) it was first shown that for all $m > 1$ and all odd $n \geq 4381$, the groups $B(m, n)$ are infinite. For the proof of this result the authors introduced a new way of describing the groups $B(m, n)$, based on a classification of periodic words and their transformations depending on a natural parameter, called rank. Below we shall describe the basic ideas of the method for studying periodic groups first introduced in Novikov and Adian (1968) and later improved and refined in Adian (1979).

Unless otherwise stated in what follows n is odd and $n \geq 665$, $q = 90$.

3. Words and Occurrences

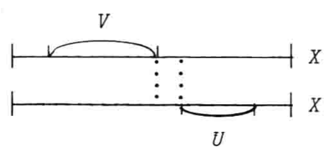
We shall consider words in the alphabet (1). A word X is called *uncancellable* if it has not the form $Pa_i a_i^{-1} Q$ or $Pa_i^{-1} a_i Q$. Letter-for-letter equality of two words X and Y is denoted by $X \overline{\circ} Y$. We denote the length of the word A by $\partial(A)$, that is $\partial(A)$ is the number of letters comprising A .

3.1. If $E \overline{\circ} PQ$, then the word P is called a *start* of E and Q is an *end* of E . If $E \overline{\circ} PQ$, then QP is called a *cyclic shift* of E .

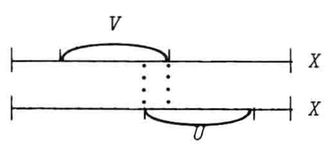
We say that the word E *occurs* in the word X if there are words P and Q such that $X \overline{\circ} PEQ$. One and the same word can occur in a given word X in different places. In order to distinguish between two different occurrences of E in X we use an extra symbol $*$. If $X \overline{\circ} PEQ$ then we call $P*E*Q$ an *occurrence of* E *in* X . Thus an occurrence of E in X is a triple (P, E, Q) of subwords of X . The word E is called the *base* of the occurrence $P*E*Q$, and is denoted by $\mathrm{Bas}(P*E*Q)$. We shall consider only occurrences with non-empty bases. The length of an occurrence is the length of its base. If $W = P*E*Q$, then we denote by W^{-1} the occurrence $Q^{-1}*E^{-1}*P^{-1}$.

One can think of an occurrence in a given word as a segment of this word. Let $X \overline{\circ} PEQ \overline{\circ} RDS$, $V \rightleftharpoons P*E*Q$, $U \rightleftharpoons R*D*S$. Two occurrences V and U in a given word X can be situated in the following ways.

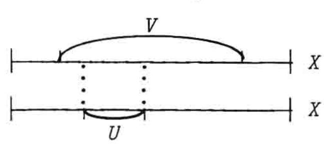
1. $\partial(PE) \leq \partial(R)$



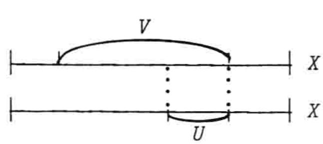
2. $\partial(P) < \partial(R)$, $\partial(S) < \partial(Q)$



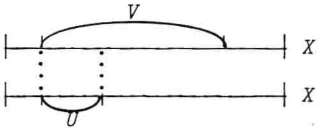
3. $\partial(P) < \partial(R)$, $\partial(Q) < \partial(S)$



4. $\partial(P) < \partial(R)$, $\partial(Q) = \partial(S)$



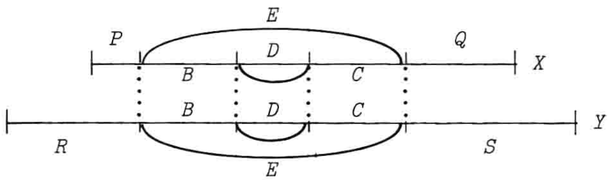
5. $\partial(P) = \partial(R)$, $\partial(Q) < \partial(S)$



In case 1) we say U and V do not intersect, in the other cases they intersect. In cases 1) and 2) we say that V lies to the left of U and write $V < U$. In case 1) we also say that V lies strictly to the left of U and write $V \ll U$. In cases 3), 4) and 5), U occurs in V , thus U is an intersection of U and V . In case 4) we say that U is an end of V , in case 5), U is a start of V .

3.2. The *union* of two occurrences is the occurrence containing both of them that has base of shortest length. In cases 3), 4) and 5), V is the union of V and U . In cases 1) and 2) the union of V and U has the form $P{*}H{*}S$ for some H .

3.3. Let $P{*}E{*}Q$ and $R{*}E{*}S$ be occurrences in words X and Y and V an occurrence in X contained in $P{*}E{*}Q$. Then there exist words B, D and C such that $E \underline{\overline{\circ}} BDC$ and $V = PB{*}D{*}CQ$. We can show this situation geometrically in the following way.



In such a case we denote the occurrence $RB{*}D{*}CS$ in Y by

$$\phi(V;\ P{*}E{*}Q,\ R{*}E{*}S)\ .$$

For any two occurrences W and W_1 with the same base, the function $V_1 = \phi(V;\ W,\ W_1)$ sets up a one-to-one mapping of the set of occurrences in W onto the set of all occurrences in W_1 .

Clearly, the function $V_1 = \phi(V;\ W,\ W_1)$ preserves the relations $<$, $\ll$ and carries the common part (union) of two occurrences contained in W to the common part (union) of their images in W_1 .

If $V_1 = \phi(V;\ W,\ W_1)$, then $V = \phi(V_1;\ W_1,\ W)$.

Let W_1, W_2, W_3 be occurrences with the same base and suppose that V_1 is contained in W_1 . If $V_2 = \phi(V_1;\ W_1,\ W_2)$ and $V_3 = \phi(V_2;\ W_2,\ W_3)$ then $V_3 = \phi(V_1;\ W_1,\ W_3)$.

4. Periodic Words

For any integer $t > 0$ we denote by A^t the word $AA \dots A$, with A repeated t times. For any word A we set

$$A^{-t} \rightleftharpoons (A^{-1})^t \quad \text{and} \quad A^0 \rightleftharpoons 1 .$$

4.1. We call an uncancellable word of the form $A_1 A^t A_2$, where A_1 is an end of A, A_2 is a start of A and $\partial(A_1 A^t A_2) > 2\partial(A)$, a *periodic word* of rank 1 with *period* A. For empty A_1 (or A_2) we call A the *left* (or *right*) *period* of $A_1 A^t A_2$. The set of all periodic words of rank 1 with period A is denoted by $\mathrm{Per}(A)$ or $\mathrm{Per}(1, A)$.

Clearly, if B is a cyclic shift of A, then $\mathrm{Per}(A) = \mathrm{Per}(B)$. If $X \in \mathrm{Per}(A)$, B occurs in X and $\partial(B) = \partial(A)$, then B is a cyclic shift of A.

Clearly, if $A \,\overline{\underline{\circ}}\, B^r$ for $r > 0$, then $\mathrm{Per}(A) \subset \mathrm{Per}(B)$.

4.2. The word A is called *simple*, if it is not of the form D^r for any $r > 1$. Clearly, for every non-empty word A there exists a simple word B such that $A = B^t$ for some $t \geq 1$. If AB is simple, then also BA is simple.

The following property of periodic words is well-known.

4.3. If $A^t A' \,\overline{\underline{\circ}}\, B^r B'$, where A' is a start of A, B' is a start of B and $\partial(A^t A') \geq \partial(AB)$, there exists a word D such that $A \,\overline{\underline{\circ}}\, D^k$ and $B \,\overline{\underline{\circ}}\, D^s$ for some k and s. In particular if A is simple, then $D \,\overline{\underline{\circ}}\, A$.

If $X \in \mathrm{Per}(A)$ for some A, then X has a unique left period, which is also a left period of minimal length of the word X. Similarly for right periods.

5. Basic Notions for Rank 0

We denote by $\mathcal{R}_0$ the set of all uncancellable words in the alphabet (1). These words also will be called *reduced* words of rank 0.

We call an occurrence $P{*}E{*}Q$ in the word $X \in \mathcal{R}_0$ a *kernel of rank* 0 of the word X, if $\partial(E) = 1$, that is if E is a letter. We denote by $\mathrm{Ker}(0, X)$ the set of all kernels of rank 0 of the word X. By $\mathrm{Reg}(0, X)$ we denote the set of all occurrences in the word X with a non-empty base. It is clear that any occurrence $W \in \mathrm{Reg}(0, X)$ begins (and ends) with some kernel of rank 0. By $\partial_0(W)$ we denote the number of kernels of rank 0 contained in W. Clearly, we have $\partial_0(W) = \partial(\mathrm{Bas}(W))$.

Two words $X, Y \in R_0$ are called *equivalent in rank* 0, if $X \,\overline{\underline{\circ}}\, Y$. For this we also write $X \underset{\sim}{0} Y$. If $X \underset{\sim}{0} Y$ and $V \in \mathrm{Reg}(0, X)$, then we set by definition $V = f_0(V; X, Y)$.

We call two occurrences V and W in the words $X \in R_0$ and $Y \in R_0$ *mutually normalized* if $\mathrm{Bas}(V) \,\overline{\underline{\circ}}\, \mathrm{Bas}(W)$.

If $X, Y \in R_0$, then there exists a word T such that $X \,\overline{\underline{\circ}}\, X_1T$, $Y \,\overline{\underline{\circ}}\, T^{-1}Y_1$ and $X_1Y_1 \in R_0$. We say that this word X_1Y_1 is the result of *coupling* of the words X and Y in rank 0, and we write

$$[X, Y]_0 = X_1Y_1 .$$

This operation is clearly unique and associative.

The set R_0 with the operation $[X, Y]_0$ is the simplest form to describe the group F_m with m generators $a_1, a_2, \ldots, a_m$. Our aim will be to find a similar description for the free periodic groups $B(m, n)$.

6. Elementary words of rank 1

The cyclically uncancellable simple words (no cyclic shift is cancellable) are precisely the *minimal periods of rank* 1. From these periods we select elementary periods of rank 1.

6.1. A minimal period A of rank 1 is called an *elementary period of rank* 1 if no periodic word E with period B of rank 1 and $\partial(E) > 8\partial(B)$ occurs in A^8. For instance the words a_1a_2 and $a_2a_3(a_2a_1)^7$ are elementary periods of rank 1. The words $a_1^{10}a_2$ and $a_2a_1a_2a_3(a_2a_1)^7$ are minimal periods of rank 1 but they are not elementary periods of rank 1, because in the second case the word $(a_2a_1)^8a_2$ with the period a_2a_1 occurs in the word $(a_2a_1a_2a_3(a_2a_1)^7)^8$. Notice that if A is a minimal period, then A^{-1} also is. If A is an elementary period of rank 1, then A^{-1} also is.

6.2. If A is an elementary period of rank 1 and E is a periodic word with period A, then E will be called an *elementary word of rank* 1. By 4.1 and 4.3, for any elementary word E of rank 1 one can find a corresponding elementary period A that is unique up to a cyclic shift.

6.3. For an arbitrary elementary word E of rank 1 with period A we call the number

$$\left\lceil\frac{\partial(E)}{\partial(A)}\right\rceil = \text{the smallest integer } r \text{ such that } \frac{\partial(E)}{\partial(A)} \le r \ ,$$

the number of segments of rank 1 of the word E and denote it by $l_1(E)$. We call $l_1(E)$ also the number of segments of an arbitrary occurrence $P*E*Q$ and we write $l_1(P*E*Q) = l_1(E)$.

It follows easily from our definition of $l_1(E)$, that for an arbitrary elemen elementary word E_1E_2 we have the inequalities

$$l_1(E_1) + l_1(E_2) - 1 \le l_1(E_1E_2) \le l_1(E_1) + l_1(E_2) \ .$$

If $l_1(E) \ge r$, then we say that E is an elementary r-power of rank 1. Any occurrence of the elementary word E of rank 1 will be called a *normalized* occurrence of E. For a given word $X \in \mathcal{R}_0$ and given number $r \ge 2$ we denote by

$$\text{Norm}(1, X, r)$$

the set of all occurrences of elementary r-powers of rank 1 in the word X.

6.4. Let an occurrence

$$P*A^tA_1*Q \tag{5}$$

in some word $X \in \mathcal{R}_0$ be given, where A^tA_1 is an elementary word of rank 1 with period $A \overline{\underline{\circ}} A_1A_2$. We say that the occurrence (5) can be continued to the left if P is nonempty and if after adding the last letter of P to the word A^tA_1 we obtain also a periodic word with period A. In other words, the occurrence (5) can be continued to the left if there exists a nonempty word H such that $P \overline{\underline{\circ}} P_1H$ and $HA^tA_1 \in \text{Per}(A)$. In this case we call the occurrence $P_1*HA^tA_1*Q$ a *continuation* of (5) *to the left.* Similarly we define *continuations to the right.* If (5) can be continued to the left or to the right then the corresponding occurrence $P_1*HA^tA_1F*Q_1$ is called a *continuation* of (5). It is convenient also to call (5) a continuation of itself.

6.5. Obviously the occurrence (5) can be continued to the left if and only if P and A have the same last letter. Similarly for a continuation to the right, A_2A_1 and Q must have the same first letter.

We call (5) a *maximal occurrence* if it has no proper continuation to the left or to the right. A continuation of (5) is called a *maximal continuation* of (5) if it is a maximal occurrence. Similarly, we introduce the maximal continuation to the left (or to the right) of (5).

We denote by

$$\mathrm{Max\ Norm}(1, X, r)$$

the set of all maximal occurrences of r-powers of rank 1 in X .

EXAMPLE. The occurrence $a_1^{-1}a_2*\left(a_1^3a_2\right)^{10}*a_2^5$ can be continued to the left and cannot be continued to the right. Its maximal continuation is $a_1^{-1}*a_2\left(a_1^3a_2\right)^{10}*a_2^5$.

6.6. Two occurrences in the same word $X \in R_0$,

$$P*E*Q \quad \text{and} \quad R*D*S \tag{6}$$

of the elementary words E and D of rank 1 we call *compatible* if there exists an occurrence which contains both occurrences (6) and is a continuation of both of them.

It is easy to see that the occurrences (6) are compatible if and only if their union is a continuation of both of them (see 4.3), and consequently E and D have the same periods. If the occurrences (6) are compatible then we write

$$\mathrm{Comp}(P*E*Q,\ R*D*S) \ . \tag{7}$$

Of course in (7) we assume that $RDS \mathrel{\overline{\underline{\circ}}} PEQ$. Obviously (7) is a symmetric and transitive relation.

EXAMPLES. The occurrences

$$a_1*\left(a_2a_1\right)^{15}a_2*a_1a_2\left(a_1a_2\right)^7a_1^3a_2 \tag{8}$$

and

$$a_1\left(a_2a_1\right)^{15}a_2a_1a_2*\left(a_1a_2\right)^7*a_1^3a_2 \tag{9}$$

are compatible. The occurrences

$$a_1*\left(a_2a_1\right)^{15}a_2*a_1a_2^{-1}\left(a_1a_2\right)^7a_1^3a_2$$

and

$$a_1\left(a_2a_1\right)^{15}a_2a_1a_2^{-1}*\left(a_1a_2\right)^7*a_1^3a_2$$

are not compatible, because the base of their union is not a periodic word with the same period.

Notice that any occurrence has a unique maximal continuation. If two occurrences are compatible then they have the same maximal continuation.

The occurrences (8) and (9) have the maximal continuation

$$*(a_1a_2)^{24}a_1*a_1^2a_2 \; .$$

6.7. Two elementary words E and D of rank 1 are called *related* if their periods coincide. In this case any two occurrences $P*E*Q$ and $R*D*S$ are also called related. In this case we shall write

$$\mathrm{Rel}(E, D) \text{ and } \mathrm{Rel}(P*E*Q, R*D*S) \; .$$

6.8 LEMMA. *Suppose* $V \in \mathrm{Norm}(1, Z, 9)$ *and* $W \in \mathrm{Norm}(1, Z, r)$. *If* V *is contained in* W , *then* W *is a continuation of* V *and therefore* $\mathrm{Comp}(V, W)$. *If* $\mathrm{Rel}(V, W)$ *or* $\mathrm{Rel}(V, W^{-1})$, *then the same is true already when* $l_1(V) > 2$.

Proof. Let A be an elementary period of $\mathrm{Bas}(V)$ and C be an elementary period of $\mathrm{Bas}(W)$. By the assumption of our lemma, we have

$$\mathrm{Bas}(V) = A^tA' \,\bar{\underline{\circ}}\, D^rD' \; , \tag{10}$$

where A' is a start of A , D is a cyclic shift of C , D' is a start of D , and $\partial(A^tA') > 8\partial(A)$. By 6.1 we have $\partial(D^rD') > 8\partial(D)$, because D is an elementary period of rank 1 . Then by 4.3 we have $A \,\bar{\underline{\circ}}\, D$. Therefore W is a continuation of V .

The second part of the lemma is easier since by the condition of $\mathrm{Rel}(V, W)$ we have $\partial(A) = \partial(C)$. Then it follows from $l_1(V) \geq 2$ and (10) that $\partial(A^tA') > \partial(A)$ and $A \,\bar{\underline{\circ}}\, D$. Consequently we have $\mathrm{Comp}(V, W)$.

6.9. It follows from 6.8 that if two occurrences $V \in \mathrm{Norm}(1, X, 9)$ and $W \in \mathrm{Norm}(1, Z, 9)$ are not compatible, then the common part of them contains less than 9 segments of each.

7. Reversals of Rank 1

7.1. Suppose that $r \geq 9$, $PA^tA_1Q \in R_0$ and $P*A^tA_1*Q$ is a maximal occurrence of an elementary r-power A^tA_1 with the period $A \,\bar{\underline{\circ}}\, A_1A_2$. Then the transition

$$PA^tA_1Q \to P(A^{-1})^{n-t-1}A_2^{-1}Q \tag{11}$$

is said to be an *r-reversal of rank* 1 of the occurrence $P*A^tA_1*Q$ if $(A^{-1})^{n-t-1}A_2^{-1}$ is also r-power with the elementary period A^{-1} of rank 1 .

7.2. Notice, that by 6.4 it follows easily from $PA^tA_1Q \in R_0$ that $P*(A^{-1})^{n-t-1}A_2^{-1}*Q$ is a maximal occurrence. Then we have $P(A^{-1})^{n-t-1}A_2^{-1}Q \in R_0$

because $P{*}A^{t}A_1{*}Q$ is a maximal occurrence.

Therefore, if (11) is an r-reversal of $P{*}A^{t}A_1{*}Q$, then the transition

$$P\left(A^{-1}\right)^{n-t-1}A_2^{-1}Q \to PA^{t}A_1Q$$

is an r-reversal of $P{*}\left(A^{-1}\right)^{n-t-1}A_2^{-1}{*}Q$.

EXAMPLE. The transition

$$a_1^4\left(a_2a_1^{-2}\right)^q a_2a_1^{-1}a_2^3 \to a_1^4\left(a_1^2a_2^{-1}\right)^{n-q-1} a_1a_2^3$$

is a q-reversal of rank 1 of the maximal occurrence $a_1^4{*}\left(a_2a_1^{-2}\right)^q a_2a_1^{-1}{*}a_2^3$ (see 2.4). Here we have $A = a_2a_1^{-2}$, $A_1 = a_2a_1^{-1}$ and $A_2 = a_1^{-1}$. It is impossible to carry out even a 9-reserval of rank 1 of the occurrence

$$a_1^4{*}\left(a_2a_1^{-2}\right)^{n-7} a_2a_1^{-1}{*}a_2^3 \ .$$

7.3. Any reversal (11) of the occurrence $P{*}A^{t}A_1{*}Q$ will also be called a reversal of an arbitrary occurrence W , compatible with $P{*}A^{t}A_1{*}Q$.

7.4. It follows easily from 6.3 that for an arbitrary elementary period $A \mathrel{\overline{\underline{\circ}}} A_1A_2$ we have

$$n \le l_1\left(A^{t}A_1\right) + l_1\left(\left(A^{-1}\right)^{n-t-1}A_2^{-1}\right) \le n + 1 \ .$$

Then for an arbitrary maximal occurrence $P{*}A^{t}A_1{*}Q$ of the elementary word $A^{t}A_1$ with the period $A \mathrel{\overline{\underline{\circ}}} A_1A_2$ in $X \in R_0$ we can carry out an r-reversal (11) of rank 1 if the inequalities

$$r \le l_1\left(A^{t}A_1\right) \le n - r - 1$$

hold.

7.5. Let P_1 be the set of all words $X \in R_0$ which contain no occurrences of elementary $(n-q)$-powers of rank 1 . Then for any $X \in P_1$ and any maximal occurrence $P{*}A^{t}A_1{*}Q$ in X one can carry out the q-reversal (11) of rank 1 if $l_1\left(A^{t}A_1\right) \ge q$.

7.6. Consider a 9-reversal (11) of the occurrence $P{*}A^tA_1{*}Q \in$ Max Norm(1, X, 9) . An occurrence $F{*}E{*}G$ in the word PA^tA_1Q is said to be ***stable*** in the reversal (11) if either $P \overline{\underline{\circ}}\ FEP_1$ or $Q \overline{\underline{\circ}}\ Q_1EG$. In such a case the occurrence

$$F{*}E{*}P_1(A^{-1})^{n-t-1}A_2^{-1}Q \quad \text{or} \quad P(A^{-1})^{n-t-1}A_2^{-1}Q_1{*}E{*}G$$

will be called the *image* of the occurrence $F{*}E{*}G$ in (11) and will be denoted by

$$f_{X\to Y}(F{*}E{*}G) \ .$$

7.7. An occurrence V in the word $X \in R_0$ is stable in a given reversal of an occurrence $W \in$ Norm(1, X, 9) if and only if V does not intersect the occurrence W .

This follows from 7.6.

7.8. We shall consider q-reversals of rank 1 (see 2.4). Some of the q-reversals of rank 1 we call ***real reversals.*** The precise definition of real reversals of rank 1 is based on the notion of cascades of rank 1 (see I.4.22 and I.4.23), and essentially has been used in Adian (1979) in proofs of some important properties of our notions in rank 2 . The following properties of real reversals are important.

(a) Any real reversal of rank 1 is a q-reversal of rank 1 .

(b) If there exists a real reversal of an occurrence $W \in$ Norm(1, X, 9) , then there exists an occurrence $V \in$ Norm(1, X, q) such that Comp(W, V) and V is stable in the real reversal of an arbitrary occurrence of rank 1 that is not compatible with V (see I.4.23).

(c) If $W \in$ Max Norm(1, X, q+34) and $l_1(W) < n - q - 52$, then one can carry out some real reversal $X \to Y$ of the occurrence W (see III.3.24).

(d) If $X \to Y$ is a real reversal of rank 1 , then $Y \to X$ also is a real reversal of rank 1 (see III.3.24).

If one is interested only in rank 1 , then one can call any q-reversal of rank 1 a real reversal of rank 1 .

We call an occurrence V an *active occurrence of rank* 1 if $V \in$ Norm(1, X, 9) and if it is possible to find some real reversal of it. We denote by

$$\text{Act}(1, X)$$

the set of all active occurrences of rank 1 in a given word $X \in P_1$.

Obviously, if $V, U \in$ Norm(1, X, 9) , Comp(V, U) and $V \in$ Act(1, X) , then $U \in$ Act(1, X) .