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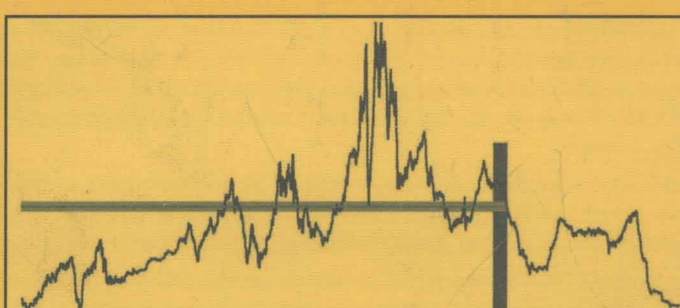

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Preface

This is the second volume of the Paris-Princeton Lectures in Mathematical Finance. The goal of this series is to publish cutting edge research in self-contained articles prepared by well known leaders in the field or promising young researchers invited by the editors. Particular attention is paid to the quality of the exposition, and the aim is at articles that can serve as an introductory reference for research in the field.

The series is a result of frequent exchanges between researchers in finance and financial mathematics in Paris and Princeton. Many of us felt that the field would benefit from timely exposés of topics in which there is important progress. René Carmona, Erhan Cinlar, Ivar Ekeland, Elyes Jouini, José Scheinkman and Nizar Touzi will serve in the first editorial board of the Paris-Princeton Lectures in Financial Mathematics. Although many of the chapters in future volumes will involve lectures given in Paris or Princeton, we will also invite other contributions. Given the current nature of the collaboration between the two poles, we expect to produce a volume per year. Springer Verlag kindly offered to host this enterprise under the umbrella of the Lecture Notes in Mathematics series, and we are thankful to Catriona Byrne for her encouragement and her help in the initial stage of the initiative.

This second volume contains three chapters. The first one is written by Tomasz Bielecki, Monique Jeanblanc and Marek Rutkowski. It reviews recent developments in the *reduced form* approach to credit risk and offers an exhaustive presentation of the hedging issues when contingent claims are subject to counterparty default. The second chapter is contributed by Tomas Bjork and is based on a short course given by him during the Spring of 2003 at Princeton University. It gives a detailed introduction to the geometric approach to mathematical models of fixed income markets. This contribution is a welcome addition to the long list of didactic surveys written by the author. Like the previous ones, it is bound to become a reference for the newcomers to mathematical finance interested in learning how and why the geometric point of view is so natural and so powerful as an analysis tool. The last chapter is due to José Scheinkman and Wei Xiong. It considers dynamic trading by agents with heterogeneous beliefs. Among other things, it uses short sale constraints and overconfidence of groups of agents to show that equilibrium prices can be consistent with speculative bubbles.

It is anticipated that the publication of this volume will coincide with the *Third World Congress* of the Bachelier Finance Society, to be held in Chicago (July 21-24, 2004).

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Hedging of Defaultable Claims

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Summary. The goal of this chapter is to present a survey of recent developments in the practically important and challenging area of hedging credit risk. In a companion work, Bielecki et al. (2004a), we presented techniques and results related to the valuation of defaultable claims. It should be emphasized that in most existing papers on credit risk, the risk-neutral valuation of defaultable claims is not supported by any other argument than the desire to produce an arbitrage-free model of default-free and defaultable assets. Here, we focus on the possibility of a perfect replication of defaultable claims and, if the latter is not feasible, on various approaches to hedging in an incomplete setting.

Key words: Defaultable claims, credit risk, perfect replication, incomplete markets, mean-variance hedging, expected utility maximization, indifference pricing.
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Introduction

The present chapter is naturally divided into three different parts.

Part I is devoted to methods and results related to the replication of defaultable claims within the reduced-form approach (also known as the intensity-based approach). Let us mention that the replication of defaultable claims in the so-called structural approach, which was initiated by Merton (1973) and Black and Cox (1976), is entirely different (and rather standard), since the value of the firm is usually postulated to be a tradeable underlying asset. Since we work within the reduced-form framework, we focus on the possibility of an exact replication of a given defaultable claim through a trading strategy based on default-free and defaultable securities. First, we analyze (following, in particular, Vaillant (2001)) various classes of self-financing trading strategies based on default-free and defaultable primary assets. Subsequently, we present various applications of general results to financial models with default-free and defaultable primary assets are given. We develop a systematic approach to replication of a generic defaultable claim, and we provide closed-form expressions for prices and replicating strategies for several typical defaultable claims. Finally, we present a few examples of replicating strategies for particular credit derivatives. In the last section, we present, by means of an example, the PDE approach to the valuation and hedging of defaultable claims within the framework of a complete model.

In Part II, we formulate a new paradigm for pricing and hedging financial risks in incomplete markets, rooted in the classical Markowitz mean-variance portfolio selection principle and first examined within the context of credit risk by Bielecki and Jeanblanc (2003). We consider an investor who is interested in dynamic selection of her portfolio, so that the expected value of her wealth at the end of the pre-selected planning horizon is no less then some floor value, and so that the associated risk, as measured by the variance of the wealth at the end of the planning horizon, is minimized. If the perfect replication is not possible, then the determination of a price that the investor is willing to pay for the opportunity, will become subject to the investor's overall attitude towards trading. In case of our investor, the bid price and the corresponding hedging strategy is to be determined in accordance with the mean-variance paradigm.

The optimization techniques used in Part II are based on the mean-variance portfolio selection in continuous time. To the best of our knowledge, Zhou and Li (2000) were the first to use the embedding technique and linear-quadratic (LQ) optimal control theory to solve the continuous-time mean-variance problem with assets having deterministic diffusion coefficients. Their approach was subsequently developed in various directions by, among others, Li et al. (2001), Lim and Zhou (2002), Zhou and Yin (2002), and Bielecki et al. (2004b). For an excellent survey of most of these results, the interested reader is referred to Zhou (2003).

In the final part, we present a few alternative ways of pricing defaultable claims in the situation when perfect hedging is not possible. We study the indifference pricing approach, that was initiated by Hodges and Neuberger (1989). This method leads us to solving portfolio optimization problems in an incomplete market model, and we shall use the dynamic programming approach. In particular, we compare the indifference prices obtained using strategies adapted to the reference filtration to the indifference prices obtained using strategies based on the enlarged filtration, which encompasses also the observation of the default time. We also solve portfolio optimization problems for the case of the exponential utility; our method relies here on the ideas of Rouge and El Karoui (2000) and Musiela and Zariphopoulou (2004). Next, we study a particular indifference price based on the quadratic criterion; it will be referred to as the quadratic hedging price. In a default-free setting, a similar study was done by Kohlmann and Zhou (2000). Finally, we present a solution to a specific optimization problem, using the duality approach for exponential utilities.

Part I. Replication of Defaultable Claims

The goal of this part is the present some methods and results related to the replication of defaultable claims within the *reduced-form approach* (also known as the *intensity-based approach*). In contrast to some other related works, in which this issue was addressed by invoking a suitable version of the martingale representation theorem (see, for instance, Bélanger et al. (2001) or Blanchet-Scalliet and Jeanblanc (2004)), we analyze here the possibility of a perfect replication of a given defaultable claim through a trading strategy based on default-free and defaultable securities. Therefore, the important issue of the choice of primary assets that are used to replicate a defaultable claim (e.g., a vulnerable option or a credit derivative) is emphasized. Let us stress that replication of defaultable claims within the structural approach to credit risk is rather standard, since in this approach the default time is, typically, a predictable stopping time with respect to the filtration generated by prices of primary assets.

By contrast, in the intensity-based approach, the default time is not a stopping time with respect to the filtration generated by prices of default-free primary assets, and it is a totally inaccessible stopping time with respect to the enlarged filtration, that is, the filtration generated by the prices of primary assets and the jump process associated with the random moment of default.

Our research in this part was motivated, in particular, by the paper by Vaillant (2001). Other related works include: Wong (1998), Arvanitis and Laurent (1999), Greenfield (2000), Lukas (2001), Collin-Dufresne and Hugonnier (2002) and Jamshidian (2002).

For a more exhaustive presentation of the mathematical theory of credit risk, we refer to the monographs by Cossin and Pirotte (2000), Arvanitis and Gregory (2001), Bielecki and Rutkowski (2002), Duffie and Singleton (2003), or Schönbucher (2003).

This part is organized as follows. Section 1 is devoted to a brief description of the basic concepts that are used in what follows. In Section 2, we formally introduce the definition of a generic defaultable claim (X, Z, C, τ) and we examine the basic features of its ex-dividend price and pre-default value. The well-known valuation results for defaultable claims are also provided. In the next section, we analyze (following, in particular, Vaillant (2001)) various classes of self-financing trading strategies based on default-free and defaultable primary assets.

Section 4 deals with applications of results obtained in the preceding section to financial models with default-free and defaultable primary assets. We develop a systematic approach to replication of a generic defaultable claim, and we provide closed-form expressions for prices and replicating strategies for several typical defaultable claims. A few examples of replicating strategies for particular credit derivatives are presented.

Finally, in the last section, we examine the PDE approach to the valuation and hedging of defaultable claims.

1 Preliminaries

In this section, we introduce the basic notions that will be used in what follows. First, we introduce a default-free market model; for the sake of concreteness we focus on default-free zero-coupon bonds. Subsequently, we shall examine the concept of a random time associated with a prespecified hazard process.

1.1 Default-Free Market

Consider an economy in continuous time, with the time parameter $t \in \mathbb{R}_+$. We are given a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P}^*)$ endowed with a d-dimensional standard Brownian motion W^*. It is convenient to assume that $\mathbb{F}$ is the $\mathbb{P}^*$-augmented and right-continuous version of the natural filtration generated by W^*. As we shall see in what follows, the filtration $\mathbb{F}$ will also play an important role of a *reference filtration* for the intensity of default event. Let us recall that any (local) martingale with respect to a Brownian filtration $\mathbb{F}$ is continuous; this well-known property will be of frequent use in what follows.

In the first step, we introduce an arbitrage-free default-free market. In this market, we have the following primary assets:

- A *money market account* B satisfying

$$dB_t = r_t B_t \, dt, \quad B_0 = 1,$$

  or, equivalently,

$$B_t = \exp\left(\int_0^t r_u \, du\right),$$

where r is an $\mathbb{F}$-progressively measurable stochastic process. Thus, B is an $\mathbb{F}$-adapted, continuous, and strictly positive process of finite variation.

- *Default-free zero-coupon bonds* with prices

$$B(t,T) = B_t \, \mathbb{E}_{\mathbb{P}^*}(B_T^{-1} \,|\, \mathcal{F}_t), \quad \forall \, t \le T,$$

where T is the bond's maturity date. Since the filtration $\mathbb{F}$ is generated by a Brownian motion, for any maturity date $T > 0$ we have

$$dB(t,T) = B(t,T)\big(r_t \, dt + b(t,T) \, dW_t^*\big)$$

for some $\mathbb{F}$-predictable, $\mathbb{R}^d$-valued process $b(t,T)$, referred to as the *bond's volatility*.

For the sake of expositional simplicity, we shall postulate throughout that the default-free term structure model is complete. The probability $\mathbb{P}^*$ is thus the unique martingale measure for the default-free market model. This assumption is not essential, however. Notice that all price processes introduced above are continuous $\mathbb{F}$-semimartingales.

Remarks. The bond was chosen as a convenient and practically important example of a tradeable financial asset. We shall be illustrating our theoretical derivations with examples in which the bond market will play a prominent role. Most results can be easily applied to other classes of financial models, such as: models of equity markets, futures markets, or currency markets, as well as to models of LIBORs and swap rates.

1.2 Random Time

Let τ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, termed a *random time* (it will be later referred to as a *default time*). We introduce the jump process $H_t = \mathbb{1}_{\{\tau \le t\}}$ and we denote by $\mathbb{H}$ the filtration generated by this process.

Hazard process. We now assume that some *reference filtration* $\mathbb{F}$ such that $\mathcal{F}_t \subseteq \mathcal{G}$ is given. We set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{H}_t)$ for every $t \in \mathbb{R}_+$. The filtration $\mathbb{G}$ is referred to as to the *full filtration*: it includes the observations of default event. It is clear that τ is an $\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time (but not necessarily an $\mathbb{F}$-stopping time). The concept of the hazard process of a random time τ is closely related to the process F_t which is defined as follows:

$$F_t = \mathbb{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\}, \quad \forall \, t \in \mathbb{R}_+.$$

Let us denote $G_t = 1 - F_t = \mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\}$ and let us assume that $G_t > 0$ for every $t \in \mathbb{R}_+$ (hence, we exclude the case where τ is an $\mathbb{F}$-stopping time). Then the process $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$, given by the formula

$$\Gamma_t = -\ln(1 - F_t) = -\ln G_t, \quad \forall\, t \in \mathbb{R}_+,$$

is termed the *hazard process* of a random time τ with respect to the reference filtration $\mathbb{F}$, or briefly the $\mathbb{F}$-*hazard process* of τ.

Notice that $\Gamma_\infty = \infty$ and Γ is an $\mathbb{F}$-submartingale, in general. We shall only consider the case when Γ is an increasing process (for a construction of a random time associated with a given hazard process Γ, see Section 1.2). This indeed is not a serious compromise to generality. We refer to Blanchet-Scalliet and Jeanblanc (2004) for a discussion regarding completeness of the underlying financial market and properties of the process Γ. They show that if the underlying financial market is complete then the so-called (H) hypothesis is satisfied and, as a consequence, the process Γ is indeed increasing.

Remarks. The simplifying assumption that $\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\} > 0$ for every $t \in \mathbb{R}_+$ can be relaxed. First, if we fix a maturity date T, it suffices to postulate that $\mathbb{Q}^*\{\tau > T \,|\, \mathcal{F}_T\} > 0$. Second, if we have $\mathbb{Q}^*\{\tau \leq T\} = 1$, so that the default time is bounded by some $U = \operatorname{ess\,sup} \tau \leq T$, then it suffices to postulate that $\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\} > 0$ for every $t \in [0, U)$ and to examine separately the event $\{\tau = U\}$. For a general approach to hazard processes, the interested reader is referred to Bélanger et al. (2001).

Deterministic intensity. The study of a simple case when the reference filtration $\mathbb{F}$ is trivial (or when a random time τ is independent of the filtration $\mathbb{F}$, and thus the hazard process is deterministic) may be instructive. Assume that τ is such that the cumulative distribution function $F(t) = \mathbb{Q}^*\{\tau \leq t\}$ is an absolutely continuous function, that is,

$$F(t) = \int_0^t f(u)\, du$$

for some density function $f : \mathbb{R}_+ \to \mathbb{R}_+$. Then we have

$$F(t) = 1 - e^{-\Gamma(t)} = 1 - e^{-\int_0^t \gamma(u)\, du}, \quad \forall\, t \in \mathbb{R}_+,$$

where (recall that we postulated that $G(t) = 1 - F(t) > 0$)

$$\gamma(t) = \frac{f(t)}{1 - F(t)}, \quad \forall\, t \in \mathbb{R}_+.$$

The function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ is non-negative and satisfies $\int_0^\infty \gamma(u)\, du = \infty$. It is called the *intensity function* of τ (or the *hazard rate*). It can be checked by direct calculations that the process $H_t - \int_0^{t \wedge \tau} \gamma(u)\, du$ is an $\mathbb{H}$-martingale.

Stochastic intensity. Assume that the hazard process Γ is absolutely continuous with respect to the Lebesgue measure (and therefore an increasing process), so that there exists a process γ such that $\Gamma_t = \int_0^t \gamma_u\, du$ for every $t \in \mathbb{R}_+$. Then the $\mathbb{F}$-predictable version of γ is called the *stochastic intensity* of τ with respect to $\mathbb{F}$, or simply the $\mathbb{F}$-intensity of τ. In terms of the stochastic intensity, the conditional probability of the default event $\{t < \tau \leq T\}$, given the full information $\mathcal{G}_t$ available at time t, equals

$$\mathbb{Q}^*\{t < \tau \leq T \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}} \, \mathbb{E}_{\mathbb{Q}^*}\Big(1 - e^{-\int_t^T \gamma_u \, du} \,\Big|\, \mathcal{F}_t\Big).$$

Thus

$$\mathbb{Q}^*\{\tau > T \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}} \, \mathbb{E}_{\mathbb{Q}^*}\Big(e^{-\int_t^T \gamma_u \, du} \,\Big|\, \mathcal{F}_t\Big).$$

It can be shown (see, for instance, Jeanblanc and Rutkowski (2002) or Bielecki and Rutkowski (2004)) that the process

$$H_t - \Gamma_{\tau \wedge t} = H_t - \int_0^{\tau \wedge t} \gamma_u \, du = \int_0^t (1 - H_u)\gamma_u \, du, \quad \forall \, t \in \mathbb{R}_+,$$

is a (purely discontinuous) $\mathbb{G}$-martingale

Construction of a Random Time

We shall now briefly describe the most commonly used construction of a random time associated with a given hazard process Γ. It should be stressed that the random time obtained through this particular method – which will be called the *canonical construction* in what follows – has certain specific features that are not necessarily shared by all random times with a given $\mathbb{F}$-hazard process Γ. We start by assuming that we are given an $\mathbb{F}$-adapted, right-continuous, increasing process Γ defined on a filtered probability space $(\widetilde{\Omega}, \mathbb{F}, \mathbb{P}^*)$. As usual, we assume that $\Gamma_0 = 0$ and $\Gamma_\infty = +\infty$. In many instances, the hazard process Γ is given by the equality

$$\Gamma_t = \int_0^t \gamma_u \, du, \quad \forall \, t \in \mathbb{R}_+,$$

for some non-negative, $\mathbb{F}$-predictable, stochastic intensity γ. To construct a random time τ such that Γ is the $\mathbb{F}$-hazard process of τ, we need to enlarge the underlying probability space $\widetilde{\Omega}$. This also means that Γ is not the $\mathbb{F}$-hazard process of τ under $\mathbb{P}^*$, but rather the $\mathbb{F}$-hazard process of τ under a suitable extension $\mathbb{Q}^*$ of the probability measure $\mathbb{P}^*$. Let ξ be a random variable defined on some probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{Q}})$, uniformly distributed on the interval $[0, 1]$ under $\widehat{\mathbb{Q}}$. We consider the product space $\Omega = \widetilde{\Omega} \times \widehat{\Omega}$, endowed with the product σ-field $\mathcal{G} = \mathcal{F}_\infty \otimes \widehat{\mathcal{F}}$ and the product probability measure $\mathbb{Q}^* = \mathbb{P}^* \otimes \widehat{\mathbb{Q}}$. The latter equality means that for arbitrary events $A \in \mathcal{F}_\infty$ and $B \in \widehat{\mathcal{F}}$ we have $\mathbb{Q}^*\{A \times B\} = \mathbb{P}^*\{A\}\widehat{\mathbb{Q}}\{B\}$. We define the random time $\tau : \Omega \to \mathbb{R}_+$ by setting

$$\tau = \inf \{ \, t \in \mathbb{R}_+ : e^{-\Gamma_t} \leq \xi \, \} = \inf \{ \, t \in \mathbb{R}_+ : \Gamma_t \geq \eta \, \},$$

where the random variable $\eta = -\ln \xi$ has a unit exponential law under $\mathbb{Q}^*$. It is not difficult to find the process $F_t = \mathbb{Q}^*\{\tau \leq t \,|\, \mathcal{F}_t\}$. Indeed, since clearly $\{\tau > t\} = \{\xi < e^{-\Gamma_t}\}$ and the random variable Γ_t is $\mathcal{F}_\infty$-measurable, we obtain

$$\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_\infty\} = \mathbb{Q}^*\{\xi < e^{-\Gamma_t} \,|\, \mathcal{F}_\infty\} = \widehat{\mathbb{Q}}\{\xi < e^{-x}\}_{x=\Gamma_t} = e^{-\Gamma_t}.$$

Consequently, we have

$$1 - F_t = \mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\} = \mathbb{E}_{\mathbb{Q}^*}\left(\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_\infty\} \,|\, \mathcal{F}_t\right) = e^{-\Gamma_t},$$

and so F is an $\mathbb{F}$-adapted, right-continuous, increasing process. It is also clear that Γ is the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}^*$. Finally, it can be checked that any $\mathbb{P}^*$-Brownian motion W^* with respect to $\mathbb{F}$ remains a Brownian motion under $\mathbb{Q}^*$ with respect to the enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$.

2 Defaultable Claims

A generic defaultable claim (X, C, Z, τ) with maturity date T consists of:

- The *default time* τ specifying the random time of default and thus also the default events $\{\tau \le t\}$ for every $t \in [0, T]$. It is always assumed that τ is strictly positive with probability 1.
- The *promised payoff* X, which represents the random payoff received by the owner of the claim at time T, if there was no default prior to or at time T. The actual payoff at time T associated with X thus equals $X \mathbb{1}_{\{\tau > T\}}$.
- The finite variation process C representing the *promised dividends* – that is, the stream of (continuous or discrete) random cash flows received by the owner of the claim prior to default or up to time T, whichever comes first. We assume that $C_T - C_{T-} = 0$.
- The *recovery process* Z, which specifies the recovery payoff Z_τ received by the owner of a claim at time of default, provided that the default occurs prior to or at maturity date T.

It is convenient to introduce the *dividend process* D, which represents all cash flows associated with a defaultable claim (X, C, Z, τ). Formally, the dividend process D is defined through the formula

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T,\infty)}(t) + \int_{(0,t]} (1 - H_u)\, dC_u + \int_{(0,t]} Z_u \, dH_u,$$

where both integrals are (stochastic) Stieltjes integrals.

Definition 1. *The* ex-dividend price process U *of a defaultable claim of the form* (X, C, Z, τ) *which settles at time* T *is given as*

$$U_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{(t,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t\Big), \quad \forall\, t \in [0, T),$$

where $\mathbb{Q}^*$ *is the* spot martingale measure *and* B *is the savings account. In addition, at maturity date we set* $U_T = U_T(X) + U_T(Z) = X \mathbb{1}_{\{\tau > T\}} + Z_T \mathbb{1}_{\{\tau = T\}}$.

Observe that $U_t = U_t(X) + U_t(Z) + U_t(C)$, where the meaning of $U_t(X), U_t(Z)$ and $U_t(C)$ is clear. Recall also that the filtration $\mathbb{G}$ models the full information, that is, the observations of the default-free market and of the default event.

2.1 Default Time

We assume from now on that we are given an $\mathbb{F}$-adapted, right-continuous, increasing process Γ on $(\Omega, \mathbb{F}, \mathbb{P}^*)$ with $\Gamma_\infty = \infty$. The default time τ and the probability measure $\mathbb{Q}^*$ are constructed as in Section 1.2. The probability $\mathbb{Q}^*$ will play the role of the *martingale probability* for the defaultable market. It is essential to observe that:

- The Wiener process W^* is also a Wiener process with respect to $\mathbb{G}$ under the probability measure $\mathbb{Q}^*$.
- We have $\mathbb{Q}^*_{|\mathcal{F}_t} = \mathbb{P}^*_{|\mathcal{F}_t}$ for every $t \in [0, T]$.

If the hazard process Γ admits the integral representation $\Gamma_t = \int_0^t \gamma_u \, du$ then the process γ is called the (stochastic) *intensity of default* with respect to the reference filtration $\mathbb{F}$.

2.2 Risk-Neutral Valuation

We shall now present the well-known valuation formulae for defaultable claims within the reduced-form setup (see, e.g., Lando (1998), Schönbucher (1998), Bielecki and Rutkowski (2004) or Bielecki et al. (2004a)).

Terminal payoff. The valuation of the terminal payoff is based on the following classic result.

Lemma 1. *For any $\mathcal{G}$-measurable, integrable random variable X and any $t \leq T$ we have*

$$\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{\tau>T\}} X \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{\tau>T\}} X \,|\, \mathcal{F}_t)}{\mathbb{Q}^*(\tau > t \,|\, \mathcal{F}_t)}.$$

If, in addition, X is $\mathcal{F}_T$-measurable then

$$\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{\tau>T\}} X \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \, \mathbb{E}_{\mathbb{Q}^*}(e^{\Gamma_t - \Gamma_T} X \,|\, \mathcal{F}_t).$$

Let X be an $\mathcal{F}_T$-measurable random variable representing the promised payoff at maturity date T. We consider a defaultable claim of the form $\mathbb{1}_{\{\tau>T\}} X$ with zero recovery in case of default (i.e., we set $Z = C = 0$). Using the definition of the ex-dividend price of a defaultable claim, we get the following *risk-neutral valuation formula*

$$U_t(X) = B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} \mathbb{1}_{\{\tau>T\}} X \,|\, \mathcal{G}_t)$$

which holds for any $t < T$. The next result is a straightforward consequence of Lemma 1.

Proposition 1. *The price of the promised payoff X satisfies for $t \in [0, T]$*

$$U_t(X) = B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} X \mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \widetilde{U}_t(X), \tag{1}$$

where we define

$$\widetilde{U}_t(X) = B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} e^{\Gamma_t - \Gamma_T} X \,|\, \mathcal{F}_t) = \widehat{B}_t \, \mathbb{E}_{\mathbb{Q}^*}(\widehat{B}_T^{-1} X \,|\, \mathcal{F}_t),$$

where the risk-adjusted savings account $\widehat{B}_t$ equals $\widehat{B}_t = B_t e^{\Gamma_t}$. If, in addition, the default time admits the intensity process γ then

$$\widehat{B}_t = \exp\left(\int_0^t (r_u + \gamma_u)\, du\right).$$

The process $\widetilde{U}_t(X)$ represents the *pre-default value* at time t of the promised payoff X. Notice that $\widetilde{U}_T(X) = X$ and the process $\widetilde{U}_t(X)/\widehat{B}_t$, $t \in [0, T]$, is a continuous $\mathbb{F}$-martingale (thus, the process $\widetilde{U}(X)$ is a continuous $\mathbb{F}$-semimartingale).

Remark. The valuation formula (1), as well as the concept of pre-default value, should be supported by replication arguments. To this end, we need first to construct a suitable model of a defaultable market. In fact, if we wish to use formula (1), we need to know the joint law of all random variables involved, and this appears to be a non-trivial issue, in general.

Recovery payoff. The following result appears to be useful in the valuation of the recovery payoff Z_τ which occurs at time τ. The process $\widetilde{U}(Z)$ introduced below represents the pre-default value of the recovery payoff.

For the proof of Proposition 2 we refer, for instance, to Bielecki and Rutkowski (2004) (see Propositions 5.1.1 and 8.2.1 therein).

Proposition 2. *Let the hazard process Γ be continuous, and let Z be an $\mathbb{F}$-predictable bounded process. Then for every $t \in [0, T]$ we have*

$$\begin{aligned} U_t(Z) &= B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_\tau^{-1} Z_\tau \mathbb{1}_{\{t<\tau\le T\}} \,|\, \mathcal{G}_t) \\ &= \mathbb{1}_{\{\tau>t\}} B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_t^T Z_u B_u^{-1} e^{\Gamma_t - \Gamma_u}\, d\Gamma_u \,\Big|\, \mathcal{F}_t\Big) = \mathbb{1}_{\{\tau>t\}} \widetilde{U}_t(Z). \end{aligned}$$

where we set

$$\widetilde{U}_t(Z) = \widehat{B}_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_t^T Z_u \widehat{B}_u^{-1} d\Gamma_u \,\Big|\, \mathcal{F}_t\Big), \quad \forall\, t \in [0, T].$$

If the default intensity γ with respect to $\mathbb{F}$ exists then we have

$$\widetilde{U}_t(Z) = \mathbb{E}_{\mathbb{Q}^*}\Big(\int_t^T Z_u e^{-\int_t^u (r_v+\gamma_v)\,dv}\gamma_u\,du\,\Big|\,\mathcal{F}_t\Big).$$

Remark. Notice that $\widetilde{U}_T(Z) = 0$ while, in general, $U_T(Z) = Z_T 1\!\!1_{\{\tau=T\}}$ is non-zero. Note, however, that if the hazard process Γ is assumed to be continuous then we have $\mathbb{Q}^*\{\tau = T\} = 0$, and thus $\widetilde{U}_T(Z) = 0 = U_T(Z)$.

Promised dividends. To value the promised dividends C that are paid prior to default time τ we shall make use of the following result. Notice that at any date $t < T$ the process $\widetilde{U}(C)$ gives the pre-default value of future promised dividends.

Proposition 3. *Let the hazard process Γ be continuous, and let C be an $\mathbb{F}$-predictable, bounded process of finite variation. Then for every $t \in [0,T]$*

$$\begin{aligned} U_t(C) &= B_t\,\mathbb{E}_{\mathbb{Q}^*}\Big(\int_{(t,T]} B_u^{-1}(1-H_u)\,dC_u\,\Big|\,\mathcal{G}_t\Big) \\ &= 1\!\!1_{\{\tau>t\}} B_t\,\mathbb{E}_{\mathbb{Q}^*}\Big(\int_{(t,T]} B_u^{-1} e^{\Gamma_t-\Gamma_u}\,dC_u\,\Big|\,\mathcal{F}_t\Big) = 1\!\!1_{\{\tau>t\}}\widetilde{U}_t(C), \end{aligned}$$

where we define

$$\widetilde{U}_t(C) = \widehat{B}_t\,\mathbb{E}_{\mathbb{Q}^*}\Big(\int_{(t,T]} \widehat{B}_u^{-1}\,dC_u\,\Big|\,\mathcal{F}_t\Big), \quad \forall\, t \in [0,T].$$

If, in addition, the default time τ admits the intensity γ with respect to $\mathbb{F}$ then

$$\widetilde{U}_t(C) = \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{(t,T]} e^{-\int_t^u (r_v+\gamma_v)\,dv}\,dC_u\,\Big|\,\mathcal{F}_t\Big).$$

2.3 Defaultable Term Structure

For a defaultable discount bond with zero recovery it is natural to adopt the following definition (the superscript 0 refers to the postulated zero recovery scheme) of the price

$$D^0(t,T) = B_t\,\mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} 1\!\!1_{\{\tau>T\}}\,|\,\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\widetilde{D}^0(t,T),$$

where $\widetilde{D}^0(t,T)$ stands for the *pre-default value* of the bond, which is given by the following equality:

$$\widetilde{D}^0(t,T) = \widehat{B}_t\,\mathbb{E}_{\mathbb{Q}^*}(\widehat{B}_T^{-1}\,|\,\mathcal{F}_t).$$

Since $\mathbb{F}$ is the Brownian filtration, the process $\widetilde{D}^0(t,T)/\widehat{B}_t$ is a continuous, strictly positive, $\mathbb{F}$-martingale. Therefore, the pre-default bond price $\widetilde{D}^0(t,T)$ is a continuous, strictly positive, $\mathbb{F}$-semimartingale. In the special case, when Γ is deterministic, we have $\widetilde{D}^0(t,T) = e^{\Gamma_t-\Gamma_T}B(t,T)$.