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Ordinary differential equations and stability theory

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Preface

Differential equations play an important role in science, engineering and social sciences. Many phenomena in these branches of knowledge are interpreted in terms of differential equations and their solutions. As a result, the study of differential equations is attaining importance. In particular, during the last two decades many useful and interesting contributions have been made in this field. It is, therefore, necessary to teach the theory and applications to students preparing for advanced training in applied sciences and social sciences.

The aim of this book is to bring together the qualitative theory of differential equations systematically at an introductory level. The contents would familiarise the readers with fundamentals, principles and methods of modern theory of ordinary differential equations. We have introduced here basic concepts and a fairly broad spectrum of qualitative properties of solutions of differential equations. The theory concerning linear differential equations is mainly discussed. However, many chapters deal with results of nonlinear equations. These results have been suitably chosen to illustrate the intricate nature of nonlinear analysis. This choice has remained restrictive due to the introductory level of the book.

The book contains the results of linear equations and systems, solutions by the series method, the existence and uniqueness of nonlinear initial value problems (both local and nonlocal) and the stability theory of linear and nonlinear equations. At an elementary level it also includes the results of oscillations, boundary value problems and elements of control theory. All the results have been presented in easy and lucid language.

Each chapter contains several illustrative examples. The problems given below the articles are simple in nature while those included in the miscellaneous exercises are meant to cover some aspects of the theory not necessarily included in the chapter. Hints are provided at several places. The problems generally can be solved with the help of the theorems and lemmas proved in the text and hence they would lead to a better understanding of the text and help develop the skill and intuition in the theory of differential equations.

Many important qualitative aspects could not be included in the present volume for want of space. The numerical techniques for solving differential equations and the study of complex differential equations are some of the omissions among other topics. Some chapters may need further elaboration to get a complete picture of modern developments. Yet the properties included in the book indicate the take-off points for modern and advanced developments in this field.

The contents of the volume are so organized as to serve as a text to students and teachers. The pre-requisites for an introduction to the book are elementary courses in calculus, mathematical analysis and the theory of matrices. The material of the text can be conveniently covered in one or two semesters depending upon the lecture time available. The book may be also used for shorter courses by omitting some of the chapters without losing the continuity.

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Basic Concepts and Linear Equations of the First Order

1.1 Introduction

A physical system can be studied in various ways. One among them is to create a similar environment in a laboratory and then study the required characters of the system, using calibrated tools. Based on these results a theoretical explanation can be put forth to describe the system. This is but a means of fitting a theory for observed facts. Another method is to propose a theory and to verify it experimentally. Both are well-known procedures adopted in practice. In either case the common feature is "theoretical formulation". It is found that these formulations turn out to be differential equations in many cases. Thus the latent significance of differential equations in studying physical phenomena becomes apparent. This branch of mathematics called "differential equations" is like a bridge linking mathematics and science with its applications. Hence, it is rightly considered as the language of the sciences. Many branches of the sciences have led to some kind of differential equations. The importance of differential equations lies in the abundance of their occurrence and their utility in understanding the sciences.

The name "differential equations" itself suggests that these are equations wherein the unknowns are connected through the concept of derivatives. It is presumed that readers are familiar with the notion of ordinary and partial derivatives studied in elementary calculus. Thus an equation involving ordinary derivatives of an unknown function (for which the search is being made) is called an "ordinary differential equation".

In elementary algebra we have already learnt the meaning of a solution to an algebraic equation

$$ap^2 + bp + c = 0, \quad a \neq 0 \tag{1.1}$$

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z (real or complex) is a solution of (1.1) if z satisfies equation (1.1) or in other words $az^2 + bz + c = 0$. But in differential equations the situation is a bit more complicated in the sense that the solution of a differential equation is neither a real nor a complex number but a function. When a differential equation is written out the immediate query would be to obtain the knowledge of its solution. The unknown quantity, as we had already stressed, is a function. For illustration, consider an equation

$$\frac{dx}{dt} = x. \tag{1.2}$$

Consider whether $x(t) = \mathrm{k}e^t$ (where k is a real constant) is a "solution" of (1.2) or not. The easy test for this is to verify (1.2) by differentiating $x(t) = \mathrm{k}e^t$. For this two things are essential: (i) $x(t)$ needs to be differentiable; (ii) $x(t)$ should satisfy equation (1.2). We observe that

$$\frac{d}{dt}\, x(t) = \frac{d}{dt}\, (\mathrm{k}e^t) = \mathrm{k}e^t = x(t).$$

Thus $x(t)$ is a solution of equation (1.2). In this example, the solution is found almost by inspection. But this is not the case with many other equations. There are a large number of equations whose solutions are not expressible in terms of the usual functions in elementary calculus. In fact, many differential equations have given rise to new sets of functions. Sometimes a solution to an equation is represented by a series. However, it is important to note that many innocent looking equations cannot be solved with the help of well-known functions of elementary calculus. Such equations do occur in many physical problems and their importance needs no emphasis. In Sec. 1.2 we elucidate this point further by citing examples of how differential equations arise.

1.2 How Differential Equations Arise

Differential equations occur quite frequently in our daily life. The motion of an object can always be associated with a differential equation. The change in prices of commodities, the flow of fluids, the concentration of chemicals, etc., often lead to differential equations. Such equations may depend on one or more independent variables. Further, it may include the derivatives of the first or higher order. In order to determine their exact physical significance the unknown function needs to satisfy certain conditions. Some problems which lead to a differential equation are:
(i) growth problem, (ii) electric circuits, (iii) pendulum problem, (iv) the problem of brachistochrone, and (v) family of curves.

Growth Problem

This problem occurs in various fields like economic growth, growth of bacteria in medicine, decay of radioactive elements in physics and so on.

Today the burning problem is the growth of population and the increase in pollution.

First, a general growth problem of some entity x is considered. At this stage, we do not pronounce what x is. Let $x(t)$ denote the quantity of the entity under consideration at time t. An assumption is made that "the rate of growth of $x(t)$ at any time t is proportional to $x(t)$". In mathematical terms this amounts to the differential equation

$$\frac{dx(t)}{dt} = \mathrm{k}x(t) \tag{1.3}$$

where k is a constant of proportionality which is positive if the problem deals with 'growth' and is negative in the case of 'decay'. The justification of the assumption often depends on the careful observation and analysis of experimental results. How far the law is true is not in the domain of mathematics.

When we consider physical problems growth and decay may occur simultaneously. A striking example is that of population in which the death rate acts as a decay. Let us now try to formulate a simple mathematical model for such problems. Let $x(t)$ denote the population at time t. At this stage a similar rule is adopted, namely, the rate of decay of x at a time t is proportional to $x(t)$. Thus for decay alone it is seen that

$$\frac{dx(t)}{dt} = -nx(t) \tag{1.4}$$

where n is positive. Assuming growth and decay simultaneously in view of (1.3) for $\mathrm{k} > 0$ and (1.4) we have the equation,

$$\frac{dx(t)}{dt} = (\mathrm{k} - n)x(t), \quad x(t_0) = x_0 \tag{1.5}$$

where x_0 is the amount of x present initially at the starting time t_0. From equation (1.5) it is interesting to observe that $x(t) \equiv x_0$ when $\mathrm{k} = n$. In other words x remains constant if the rates of growth and decay are the same. Here $x(t_0) = x_0$ denotes the initial condition. Thus finding $x(t)$ in (1.5) is solving the initial value problem. The explicit definition is given in Sec. 1.7.

Electric Circuits

We study an electric circuit which contains in series a capacitor, an inductor and a resistor along with a voltage source E (Fig. 1.1).

Let E be maintained at a constant potential of value E_0. The problem is to find the current $i(t)$ in the system as a function of time when the switch is put on. The physics of the problem states that at any time t the voltage across the inductor is $L\frac{di}{dt}$, across the resistor is iR and across the capacitor is $\int_0^t \frac{i ds}{C}$, assuming that the switch is put on at time $t = 0$.

Thus by Kirchhoff's voltage law, it is seen that at any time t

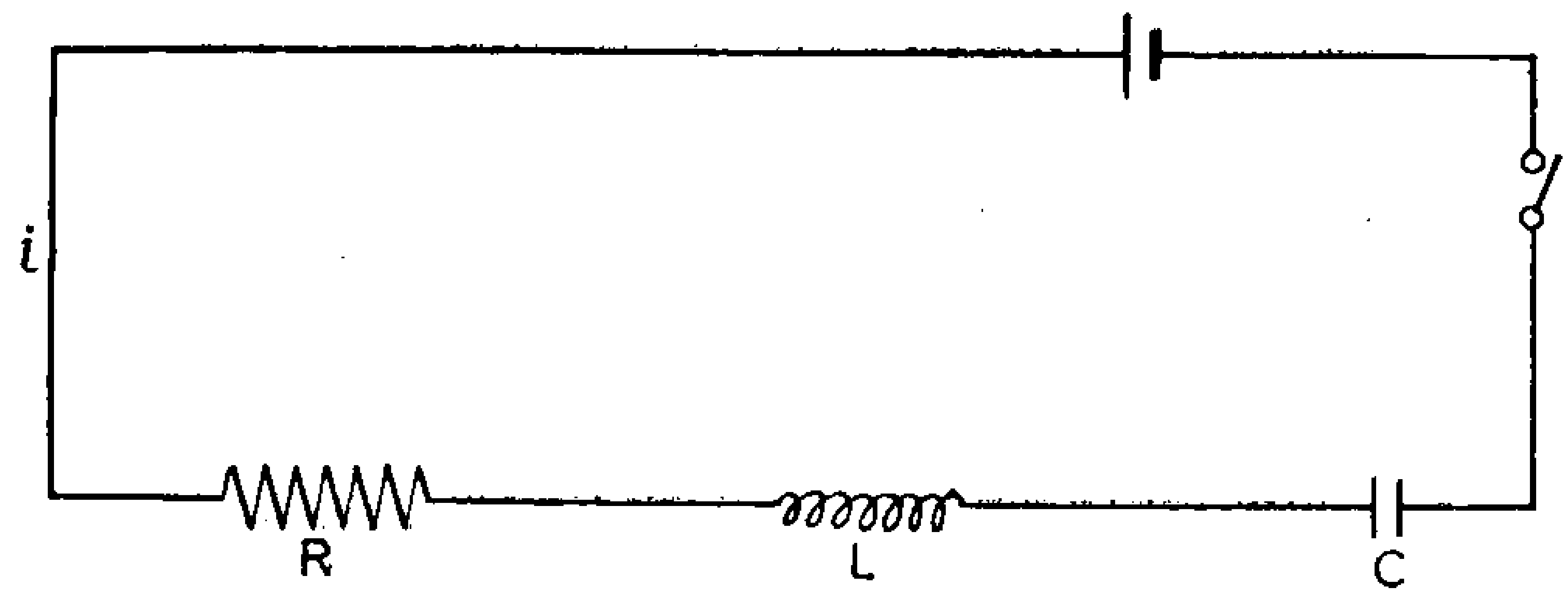

Fig. 1.1

$$L\frac{di}{dt} + Ri + \int_0^t \frac{ids}{C} = E_0. \tag{1.6}$$

First we note that equation (1.6) is not a first order differential equation since $i(t)$ (the unknown to be determined) occurs inside an integral. On differentiating both sides of (1.6) it is seen that

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0.$$

Since the switch is on exactly at $t = 0$, there is no flow of current at $t = 0$. Mathematically, this means $i(0) = 0$.

If $t = 0$ is substituted in equation (1.6), it is seen that

$$L\left.\frac{di}{dt}\right|_{t=0} + Ri(0) = E_0.$$

since $i(0) = 0$, $Li'(0) = E_0$. Thus the current i is determined by the differential equation

$$\left.\begin{aligned} L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} &= 0, \\ i(0) &= 0, \\ Li'(0) &= E_0. \end{aligned}\right\} \tag{1.7}$$

It is noted here that the differential equation involves the derivative of second order and two conditions are given to determine a solution. The nature of such problems is discussed in Sec. 1.7. Note that initially we have $i(0) = 0$ and $Li'(0) = E_0$, and hence these conditions are called the initial conditions. Indeed (1.7) itself is called an initial value problem.

Pendulum Problem

This part of the article is devoted to set up the equation of motion for a simple pendulum. The friction due to the air is neglected. The basic assumption is the conservation of energy.

Let a pendulum bob be suspended from a point O and be at rest. Let OA be the vertical position of the pendulum (see Fig. 1.2). At time t, P denotes the position of the bob and let OP make an angle x with OA.

The maximum displacement of the pendulum is denoted by B; the angle that OB makes with OA is indicated by a. Here a is the maximum

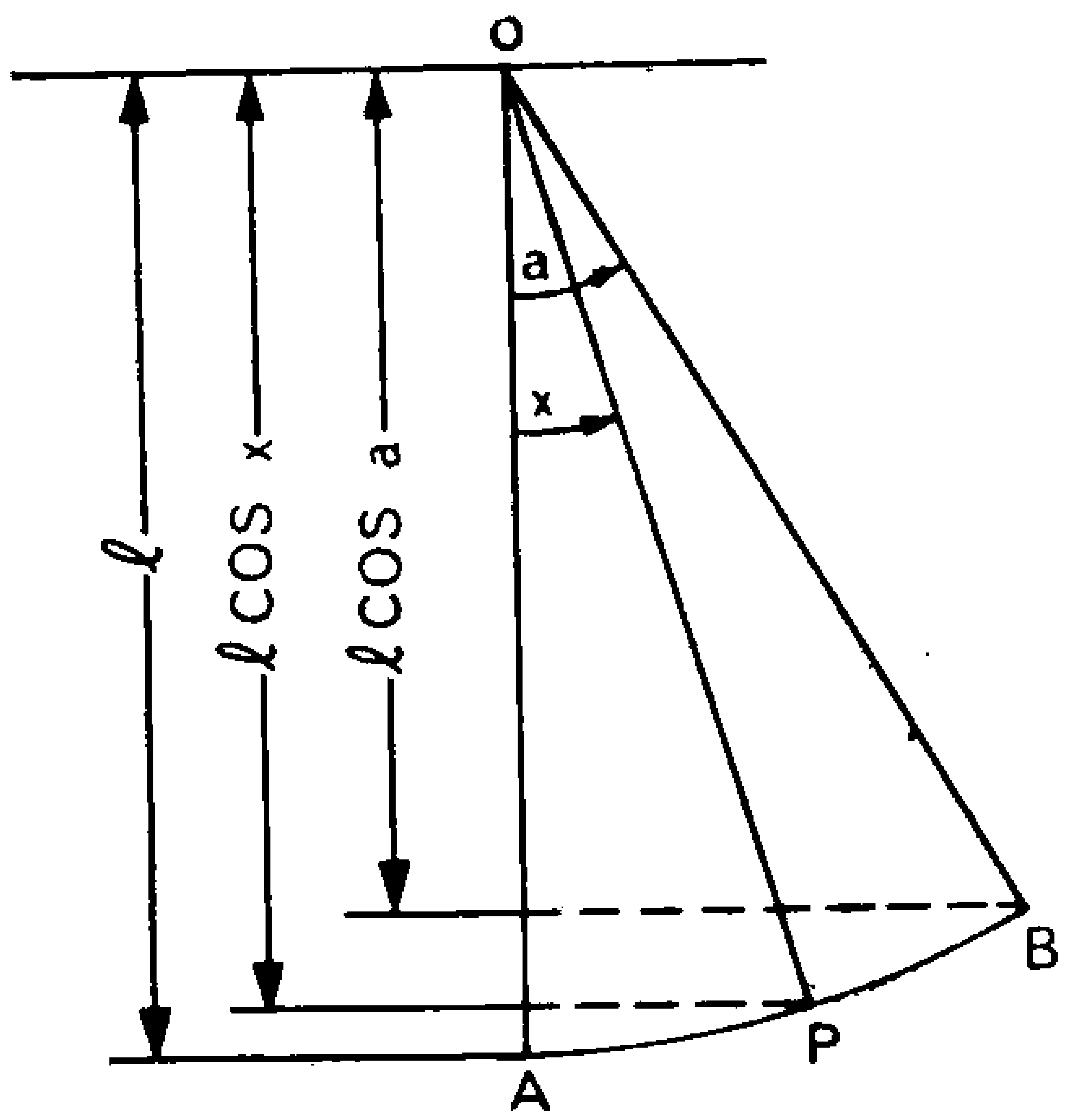

Fig. 1.2

displacement. The work done to change x to the value a is the work required to raise the pendulum bob through a vertical distance $l \cos x - l \cos a$, where l is the length of the pendulum. Since a denotes the angle for maximum displacement, the velocity v (with which the pendulum is swinging) is zero at $x = a$. The conservation of energy implies $\frac{1}{2}mv^2 = \frac{1}{2}ml^2 \left(\frac{dx}{dt}\right)^2 = mgl\,(\cos x - \cos a)$ where m is the mass of the simple pendulum. In other words, since l and m are never zero,

$$\tfrac{1}{2}l\left(\frac{dx}{dt}\right)^2 = g(\cos x - \cos a). \tag{1.8}$$

On differentiating (1.8), it is seen that

$$l\frac{dx}{dt}\frac{d^2x}{dt^2} = -g \sin x \frac{dx}{dt}. \tag{1.9}$$

For a swinging pendulum $\frac{dx}{dt}$ is not identically zero and hence the equation of motion for the pendulum is governed by

$$\frac{d^2x}{dt^2} + \frac{g}{l}\sin x = 0. \tag{1.10}$$

The Problem of Brachistochrone

This problem was first proposed by J. Bernoulli in the year 1696. As the name indicates, the brachistochrone problem is the "problem of quickest descent". This problem deals with finding a curve of quickest descent between two specified points P and Q, the condition being that P and Q do not lie on a vertical line. In other words, a curve joining the two given points P and Q (which are not on the same vertical line) has to be determined so that, an object moving from P to Q under the force of gravity only needs the shortest time. Choosing the point P as the origin for a co-ordinate system, and with respect to this system, let Q have the co-ordinates (m, n). Then the solution $y(x)$ to the brachistochrone problem is the solution of the problem

$$\frac{d}{dx}\left[F - y'\left(\frac{\partial F}{\partial y'}\right)\right] = 0, \quad y(0) = 0, \quad y(m) = n \tag{1.11}$$

where $F = \sqrt{\dfrac{1 + y'^2}{y}}$. Simplifying equation (1.11) with the above value of F we are led to the equation

$$y(1 + y'^2) = M, \quad y(0) = 0, \quad y(m) = n \tag{1.12}$$

where M is a constant to be determined. Thus the unknown function y is a solution of (1.12). Note that the conditions on y are specified at the two end points $x = 0$ and $x = m$. Such problems are called boundary value problems.

Family of Curves

When a differential equation is to be found for a given system of curves, we start with a system of curves and then seek a differential equation which represents them. A simple illustration is to consider a singly infinite system of straight lines represented by

$$y = mx + 2$$

where m is the slope. The equation

$$y = \frac{dy}{dx}\cdot x + 2 \quad \text{or} \quad \frac{dy}{dx} = \frac{y - 2}{x}$$

represents the above family of lines. This is so once we notice the fact that the slope of a line is a constant and is given by $\frac{dy}{dx}$. To elucidate further it can be shown that the doubly infinite system of curves given by

$$y = A \sin x + B \cos x$$

can be represented by a differential equation. To do this, it is observed that

$$y = A \sin x + B \cos x \tag{1.13}$$

$$\frac{dy}{dx} = A \cos x - B \sin x \tag{1.14}$$

$$\frac{d^2y}{dx^2} = -A \sin x - B \cos x \tag{1.15}$$

Eliminating the arbitrary constants A and B by adding the equations (1.13) and (1.15), the following differential equation is arrived at

$$\frac{d^2y}{dx^2} + y = 0.$$

Thus the above differential equation represents the family of curves with which the analysis was started. These two simple illustrations drive home the idea that differential equations may also arise out of a family of curves.

1.3 A Simple Equation

A differential equation which involves only the first order derivative of the unknown function is called a first order equation. In general, a first order differential equation looks like

$$g(x', x, t) = 0.$$

We write the word 'equation' instead of 'differential equation' for convenience. Usually a first order equation is represented in the form

$$x' = f(t, x). \tag{1.16}$$

Here it is to be noted that both g and f are known functions. Surprisingly enough there are no standard formulae to obtain the solution of equation (1.16) even though it looks simple. In fact, the very question of existence of solutions of (1.16) is itself a tough problem. We postpone the discussion of such questions to a later chapter. Section 1.3 presents a simple version of (1.16). An equation of the type

$$x' + c(t)x = 0, \quad t \in I \tag{1.17}$$

is called a "linear homogeneous equation". Here I is a nonempty interval in a real line R, and $c(t)$ is a given continuous function defined on I. The unknown is the function x.

In all of what follows, we restrict ourselves only to equations of the type (1.17). Examples of such equations are

(i) $\dfrac{dx}{dt} = \sin(t^2)x$,

(ii) $\dfrac{dy}{dt} = (1 + t)^{-2}y$.

Since $c(t)$ in (1.17) is assumed to be continuous in t, observe that the function $C(t)$ defined by

$$C(t) = \int_{t_0}^{t} c(s)\, ds, \quad t, t_0 \in I$$

is differentiable for each $t \in I$ and $C(t_0) = 0$. Multiplying both sides of equation (1.17) by $\exp C(t)$, it is seen that

$$\exp C(t) \frac{dx(t)}{dt} + \exp C(t)[c(t)x(t)] = 0.$$

This is precisely

$$\frac{d}{dt}[x(t) \exp C(t)] = 0.$$

Integrating between t_0 and t, it is seen that

$$x(t) \exp C(t) - x(t_0) \exp C(t_0) = 0$$

or, in other words, noting that $C(t_0) = 0$

$$x(t) = \exp\{-C(t)\}x(t_0); \quad t, t_0 \in I. \tag{1.18}$$

Thus the existence of a solution of (1.17) passing through $(t_0, x(t_0))$ has been established. Indeed we have proved the following result.

Theorem 1.1 If $c(t)$ is a continuous function on I then there exists a solution $x(t)$ of (1.17) passing through $(t_0, x(t_0))$, and further $x(t)$ is given by (1.18).

The role played by $\exp C(t)$ is to be observed here. Roughly speaking it has almost led to the integration of (1.17). Here $\exp C(t)$ is called an "integrating factor" for equation (1.17). The solution represented by condition (1.18) is such that it passes through the point $(t_0, x(t_0))$. This is a given point. This pair of numbers is called initial values.

The above procedure is now applied to examples (i) and (ii) given above when $x(0) = p$ and $y(0) = q$. The solution of (i) through the point $(0, p)$ is

$$x(t) = p \exp \int_0^t \sin(s^2)\, ds, \quad t \in [0, \infty) \tag{1.19}$$

and the solution of (ii) through $(0, q)$ is

$$y(t) = q \exp \int_0^t (1 + s)^{-2}\, ds$$

which reduces to

$$y(t) = q \exp\left[1 - \frac{1}{1+t}\right], \quad t \in [0, \infty). \tag{1.20}$$

Here $\int_0^t \sin(s^2)\, ds$ cannot be expressed in terms of elementary functions and hence solutions of the given differential equation may lead to a totally new set of functions. In example (ii) the solution $y(t)$ is bounded on an unbounded interval. The question is whether this boundedness is due to the nature of $c(t) = (1 + t)^{-2}$. The answer is in the affirmative. In fact, if $\int_0^\infty c(s)\, ds$ is finite then solutions of equation (1.17) are bounded but we may not be in a position to express $\int_0^t c(s)\, ds$ in terms of known functions. Thus we have a qualitative property of solutions of (1.17), namely boundedness of solutions, and a sufficient condition for boundedness is that $\int_0^\infty c(s)\, ds$ is finite.

Let us now consider the equation

$$\frac{dx(t)}{dt} + c(t)x(t) = d(t), \quad t \in I, \tag{1.21}$$

where $c(t)$ and $d(t)$ are known continuous functions defined on I. Equation (1.21) is called a non-homogeneous linear equation. Multiplying both sides of (1.21) by exp $C(t)$ it is seen that

$$\exp\{C(t)\}\frac{dx(t)}{dt} + \exp\{C(t)\}\, c(t)x(t) = d(t)\exp\{C(t)\}.$$

The left side of the above equation is $\frac{d}{dt}[\exp\{C(t)\}x(t)]$ and so it is seen that

$$\frac{d}{dt}[\exp\{C(t)\}x(t)] = d(t)\exp\{C(t)\}, \quad t \in I.$$

Integration between t_0 and t now leads to

$$\exp\{C(t)\}x(t) - \exp\{C(t_0)\}x(t_0) = \int_{t_0}^{t} d(s)\exp\{C(s)\}\, ds.$$

Since $C(t_0) = 0$ and the exponential function never vanishes, the above expression reduces to

$$x(t) = \exp\{-C(t)\}x(t_0) + \int_{t_0}^{t} d(s)\exp[C(s) - C(t)]\, ds. \qquad (1.22)$$

The right side of (1.22) is a known function, since it can be computed once $C(t)$ and $d(t)$ are known. Hence, the existence of a solution of non-homogeneous linear equation (1.21) passing through a point $(t_0, x(t_0))$ has been established. Hence the following theorem is proved.

Theorem 1.2 If $c(t)$ and $d(t)$ are continuous functions on an interval I, then there exists a solution of (1.21) on I passing through $(t_0, x(t_0))$ and is given by (1.22).

Remark The non-homogeneous factor or the term $d(t)$ in (1.21) is sometimes called the "forcing term". Thus a linear homogeneous equation is called "unforced".

Once the existence of solutions of equation (1.21) is established it is necessary to investigate if the given equation possesses one or more solutions passing through the given point. We consider the uniqueness of solutions of equation (1.21) passing through the initial point (t_0, x_0). Suppose there are two solutions $x_1(t)$ and $x_2(t)$ which satisfy equation (1.21) and also pass through the point $(t_0, x(t_0))$. Define $z(t) = x_1(t) - x_2(t)$. Clearly $z(t_0) = 0$. Substitute $x_1(t)$ and $x_2(t)$ in (1.21) and subtract the expressions thus obtained to get

$$z'(t) + c(t)z(t) = 0, \quad t \in I.$$

In view of the relation (1.18) we have

$$z(t) = z(t_0)\exp[-C(t)], \quad t, t_0 \in I.$$

Since $z(t_0) = 0$, we have $z(t) = 0$ for all $t \in I$. Hence $x_1(t) = x_2(t)$ on I. Hence the following theorem is proved.

Theorem 1.3 The non-homogeneous linear equation (1.21) possesses a unique solution $x(t)$ passing through (t_0, x_0), where $x(t_0) = x_0$.