Matrix Computer Methods of Vibration Analysis

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A HALSTED PRESS BOOK

LONDON BUTTERWORTHS

THE BUTTERWORTH GROUP

ENGLAND
Butterworth & Co (Publishers) Ltd
London: 88 Kingsway, WC2B 6AB

AUSTRALIA
Butterworths Pty Ltd
Sydney: 586 Pacific Highway, NSW 2067
Melbourne: 343 Little Collins Street, 3000
Brisbane: 240 Queen Street, 4000

CANADA
Butterworth & Co (Canada) Ltd
Toronto: 14 Curity Avenue, 374

NEW ZEALAND
Butterworths of New Zealand Ltd
Wellington: 26-28 Waring Taylor Street, 1

SOUTH AFRICA
Butterworth & Co (South Africa) (Pty) Ltd
Durban: 152-154 Gale Street

First published 1973

ISBN 0 408 70521 3

Text set in 10/12 pt. IBM Press Roman, printed by photolithography, and bound in Great Britain at The Pitman Press, Bath

PREFACE

The purpose of this book is to fill a gap which exists between conventional treatments of vibration analysis and advanced specialist works which presuppose the reader having a considerable level of background knowledge.

The book is an introduction to a particular technique which combines vibration analysis, matrix algebra and computational methods, and has evolved in its present form from a series of lectures presented by the author at the North-East London Polytechnic. The material is intended for use by final-year undergraduate and postgraduate students, and practising engineers and programmers.

A knowledge of Fortran or some other high-level programming language is a necessity for the programming aspect of the work, but the vibration analysis and matrix theory can be assimilated without any such knowledge, and anyone to whom many of the concepts are new will also be able to gain some insight into programming methods from the study of this book.

The programs listed in the appendices have been written in IBM 1130 Fortran, but will require little or no modification for use on any machine with a Fortran compiler.

The author is greatly indebted to the staff of the North-East London Polytechnic Computer Centre for their advice on the program content of the book, and to Mrs. Teresa Taylor who so competently typed the manuscript. Acknowledgement is also made with thanks to the International Business Machines Corporation for their kind permission to reproduce material in Chapters 3 and 5.

Inevitably some errors can be expected in an exposition of this kind. The author will be grateful for indications of them.

David J Hatter

CONTENTS

Chapter One

MATRICES AND THEIR MANIPULATION

1.1 INTRODUCTION

A matrix is an array of numbers or mathematical symbols. It can occur in various forms, and the arrangement and manipulation of matrices allow the carrying out of algebraic operations in a way which is particularly applicable to digital computation. The purpose of this chapter is to set out the rules of matrix operations so that their application to vibration analysis can be developed subsequently.

1.2 NOTATION

The individual building blocks making up the array are called the elements of the matrix. They are arranged in columns and rows and each element may be a constant, a variable or an algebraic expression. The identification of any particular element is achieved by designating its row and column. Thus, if an element is situated at the intersection of the fourth row and the second column of a matrix $[A]$, it will be called A_{42}. That is, the element has two subscripts, the first of which identifies the row and the second the column. The size of the matrix is given as the number of rows by the number of columns. Thus three rows and four columns is termed 3 by 4 and is written (3×4).

1.3 COMMON FORMS OF THE MATRIX

It is found that certain forms occur frequently, and each is given a specific description and notation. Some of the most common ones are listed below.

1.3.1 Rectangular Matrix

The rectangular matrix has m rows and n columns and is set out between square brackets as shown below. The element notation is as given in Section 1.2.

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & --- & A_{1n} \\ A_{21} & A_{22} & A_{23} & A_{24} & --- & A_{2n} \\ A_{31} & A_{32} & A_{33} & A_{34} & --- & A_{3n} \\ | & | & | & | & & | \\ | & | & | & | & & | \\ A_{m1} & A_{m2} & A_{m3} & A_{m4} & & A_{mn} \end{bmatrix}$$

In writing equations using matrices, it is found convenient to use an abbreviation to avoid setting out the matrix in full each time reference to it is made. The abbreviated notation for the above is simply $[A]$.

1.3.2 Column Matrix

The column matrix is the particular case of the rectangular matrix having only a single column:

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ | \\ | \\ A_m \end{bmatrix}$$

Abbreviated notation is $\{A\}$. The curved brackets designate a column matrix (also known as a vector or vector matrix).

1.3.3 Row Matrix

This is similar to the array in Section 1.3.2, but has only a single row.

$$[A_1 \quad A_2 \quad A_3 \; - - - \; A_n]$$

Abbreviated notation is (A). The round brackets designate a row matrix (also known as a transposed vector – for reasons which will be explained subsequently).

1.3.4 Square Matrix

This is, again, a particular case of the rectangular matrix in which the numbers of rows and columns are equal, that is, $m = n$. There are various particular cases of the square matrix which occur in vibration analysis, these being as follows.

Diagonal Matrix

In a square matrix the diagonal joining the top left-hand and bottom right-hand elements is called the leading diagonal. In the diagonal matrix all elements other than those on the leading diagonal are zero:

$$\begin{bmatrix} A_{11} & 0 & 0 & & 0 \\ 0 & A_{22} & 0 & & 0 \\ 0 & & A_{33} & & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{nn} \end{bmatrix}$$

Band Matrix

This form has zero for all elements except those on a band centred on the leading diagonal:

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & - - - 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 & - - - 0 & 0 \\ 0 & A_{32} & A_{33} & A_{34} & 0 & - - - 0 & 0 \\ 0 & 0 & A_{43} & A_{44} & A_{45} & - - - 0 & 0 \\ | & | & | & | & | & | & | \\ 0 & 0 & 0 & 0 & 0 & - - - A_{nn-1} & A_{nn} \end{bmatrix}$$

This matrix is common in vibration analysis, and because it has a band-width of three elements it is known as a *tri-diagonal matrix.*

Symmetric Matrix

The square matrix whose elements are such that those symmetrically placed about the leading diagonal are equal is known as a symmetric matrix:

$$A_{34} = A_{43}, \quad A_{21} = A_{12} \text{ and generally } A_{ij} = A_{ji}$$

For example:

$$\begin{bmatrix} 1 & 3 & -2 & 1 \\ 3 & 2 & 4 & -6 \\ -2 & 4 & 9 & 3 \\ 1 & -6 & 3 & 17 \end{bmatrix}$$

Unit Matrix and Null Matrix

There are two particular numerical forms of the square matrix. The unit matrix has values of unity along the leading diagonal and zeros elsewhere. The null matrix has all zeros. It will be shown later how both of these forms have particular application in program organisation. The notation for the unit matrix is $[I]$, and that for the null matrix is $[0]$. The unit matrix is also known as the identity matrix.

1.4 MATRIX MANIPULATION

A matrix, being simply an array of elements, has no specific numeric or algebraic value. In its symbolic form $[A]$, it must be thought of as a shorthand notation for a large number of numerical values, and manipulation of the matrix involves operations on these individual values. The various matrix operations will now be considered.

1.4.1 Addition

In order to add two matrices they must be of the same dimensions, and the addition is carried out between corresponding elements. For example

$$\begin{bmatrix} 1.6 & 2 \\ 3.7 & -8 \end{bmatrix} + \begin{bmatrix} 4.1 & 3 \\ -2.1 & 6 \end{bmatrix} = \begin{bmatrix} 5.7 & 5 \\ 1.6 & -2 \end{bmatrix}$$

1.4.2 Subtraction

This is identical with the addition process. For example:

$$\begin{bmatrix} 5 & 6 \\ 9 & 10 \end{bmatrix} - \begin{bmatrix} 3.5 & 7.8 \\ -8.3 & 1.2 \end{bmatrix} = \begin{bmatrix} 1.5 & -1.8 \\ 17.3 & 8.8 \end{bmatrix}$$

1.4.3 Transposition

In some cases when using matrix methods for vibration analysis, it is found necessary to interchange the rows and columns of a matrix. This process is known as transposition. Note that the interchanging does not alter the position of the leading (top left-hand) element. For example

$$[X] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad [X]^{\mathrm{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$[X]^{\mathrm{T}}$ is the notation for a transposed matrix, although $[X]^{\mathrm{t}}$ and $[\tilde{X}]$ are sometimes used.

1.4.4 Multiplication

There are three basic rules for this process:

(1) The two matrices must be compatible inasmuch as the number of *columns* in the first matrix must equal the number of *rows* in the second.

(2) In multiplying square matrices the order of multiplication is significant. Multiplication is non-commutative, that is if $[A]$ and $[B]$ are square matrices, then *generally*

$$[A] \times [B] \neq [B] \times [A]$$

(3) The product matrix will have the same number of *rows* as the first matrix (the pre-multiplier), and the same number of *columns* as the second (the post-multiplier). This makes it possible to check on the multiplying process thus: If a (3×4) is post-multiplied by a (4×3), the product is (3×3), that is $(3 \times 4) \times (4 \times 3) = (3 \times 3)$. If the multiplication is reversed $(4 \times 3) \times (3 \times 4) = (4 \times 4)$.

The multiplication procedure is as follows: Consider the product

$$\begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 7 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 13 & 11 \\ 6 & 9 & 10 & 16 \\ 13 & 17 & 25 & 33 \end{bmatrix}$$

Any element, A_{ij}, in the product matrix is found by adding the products of the elements from row i in the pre-multiplier and column j in the post-multiplier starting from the left and the top. This can be illustrated for the above equation by setting out the matrices as shown below:

$$\begin{array}{cc} & \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 6 & 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 7 & 3 \end{bmatrix} & \begin{bmatrix} 1 \times 1 + 2 \times 2 = 5 & 1 \times 2 + 2 \times 1 = 4 & 1 \times 1 + 2 \times 6 = 13 & 1 \times 3 + 2 \times 4 = 11 \\ 4 \times 1 + 1 \times 2 = 6 & 4 \times 2 + 1 \times 1 = 9 & 4 \times 1 + 1 \times 6 = 10 & 4 \times 3 + 1 \times 4 = 16 \\ 7 \times 1 + 3 \times 2 = 13 & 7 \times 2 + 3 \times 1 = 17 & 7 \times 1 + 3 \times 6 = 25 & 7 \times 3 + 3 \times 4 = 33 \end{bmatrix} \end{array}$$

It will be found that setting out matrices for multiplication in this manner simplifies the overall view of the process.

Considering two matrices compatible for multiplication in either order emphasises the importance of the order of multiplication:

If $$\{A\} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad (B) = [3 \quad 4]$$

then

$$\{A\} \times (B) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} [3 \quad 4] = \begin{bmatrix} 6 & 8 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad (B) \times \{A\} = [3 \quad 4] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = [10]$$

Multiplication by a scalar simply multiplies each element in the matrix by the scalar, thus

$$5 \times \begin{bmatrix} 1 & 4 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 20 \\ 45 & 15 \end{bmatrix}$$

1.4.5 Division

Division of matrices cannot be accomplished directly, that is, the expression $[A]/[B]$ has no meaning in matrix algebra. To carry out division a process termed inversion is used.

1.5 INVERSION

Consider the matrix equation

$$[A] \times [B] = [C] \tag{1.1}$$

If $[B]$ is a matrix of unknowns and $[A]$ and $[C]$ are known, it will be seen that $[B]$ is required as a function of $[A]$ and $[C]$. In order to do this, another matrix $[A]^{-1}$ is postulated such that

$$[A]^{-1} \times [A] = [I] \tag{1.2}$$

where $[I]$ is the unit matrix.

$[A]^{-1}$ is termed the inverse of $[A]$, or the reciprocal of $[A]$, and now premultiplying both sides of equation 1.1 by $[A]^{-1}$ gives

$$[A]^{-1} \times [A] \times [B] = [A]^{-1} \times [C]$$

or

$$[I] \times [B] = [A]^{-1} \times [C] \tag{1.3}$$

Premultiplying [B] by [I] leaves [B] unaltered (this is the matrix equivalent of multiplying by unity), so that equation 1.3 gives

$$[B] = [A]^{-1} \times [C] \tag{1.4}$$

Note

(1) An inverse exists for only a square matrix. If equations 1.1 and 1.4 are considered, it is seen that only if [A] and $[A]^{-1}$ are both square can the two equations be meaningful.

(2) Multiplication of a matrix with its inverse yields the product [I] for either order of multiplication. This is one of the exceptions to the non-commutative rule of multiplication.

1.5.1 Expression of Simultaneous Equations in Matrix Form

In order to examine some of the methods of producing the inverse, it is necessary to consider the procedure for solving a number of simultaneous equations. For example:

$$x_1 + 2x_2 + 2x_3 = 2$$

$$2x_1 + 5x_2 + 2x_3 = 4$$

$$x_1 + 2x_2 + 4x_3 = 6$$

From the rules of Section 1.4.4, these equations can be written in matrix form thus:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}. \tag{1.5}$$

Carrying out the matrix multiplication of equation 1.5 reproduces the original equations above.

Now if equation 1.5 is rewritten as

$$[A]\ \{x\} = \{B\} \quad \text{where} \quad [A] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix}, \tag{1.6}$$

$$\{x\} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \{B\} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

it is seen that

$$\{x\} = [A]^{-1}\{B\} \tag{1.7}$$

That is the inverse can be used to solve for x_1, x_2 and x_3. The process of finding the inverse can now be carried out, basically by following procedures used for the solution of simultaneous equations.

1.5.2 Gaussian Elimination

In order to solve equation 1.5, rewrite the equation thus:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Stage 1

Find the largest element in the first column (in this case 2). Place the *whole row* containing this element at the top:

$$\begin{matrix} 2 & 5 & 2 & & 4 \\ 1 & 2 & 2 & & 2 \\ 1 & 2 & 4 & & 6 \end{matrix}$$

Note that it is the element with the largest absolute value which is chosen, so that if there were a value of, say, −3 in the first column, that row would be used.

Stage 2

Take the first value in the *new* second row, and divide by the *new* leading element. This produces a value of 0.5 (called the first coefficient).

Stage 3

Take each element in the first row, multiply by the first coefficient, and subtract each product from the corresponding term in the second row; this gives

2	5	2	4
(1 − 0.5 x 2)	(2 − 0.5 x 5)	(2 − 0.5 x 2)	(2 − 0.5 x 4)
1	2	4	6

or

2	5	2	4
0	−0.5	1	0
1	2	4	6

Stage 4

Repeat stages 2 and 3, this time for the third row: that is, take the first value in the third row and divide by the leading element. This gives the value 0.5 (second coefficient). (Note. The fact that this value is equal to the first coefficient is mere coincidence of values. It has no significance.) Multiplying out gives

2	5	2	4
0	−0.5	1	0
(1 − 0.5 x 2)	(2 − 0.5 x 5)	(4 − 0.5 x 2)	(6 − 0.5 x 4)

or

2	5	2	4
0	−0.5	1	0
0	−0.5	3	4

Stage 5

Consider the numbers enclosed by the box. There is now effectively a (2 x 2) matrix upon which operations can be made starting from stage 1. In this case the first coefficient = 1, giving

−0.5	1	0
0	2	4

The process above is known as triangulation of a matrix. It has modified the original equation 1.5 to

$$\begin{bmatrix} 2 & 5 & 2 \\ 0 & -0.5 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$$

It is thus seen that the solution for x_3, x_2 and x_1 (in this order) may easily be obtained as follows:

$$\begin{aligned} 2x_3 &= 4, \quad \text{hence} \quad x_3 = 2 \\ -0.5x_2 + x_3 &= 0, \quad \text{hence} \quad x_2 = 4 \\ 2x_1 + 5x_2 + 2x_3 &= 4, \quad \text{hence} \quad x_1 = -10 \end{aligned}$$

This has solved three simultaneous equations by eliminating successive terms on the left-hand side. If the equation

$$[A]\ [A]^{-1} = [I]$$

is now considered, where

$$[A] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

this can be written

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} \\ \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (1.8)$$

The terms $\bar{A}_{12}, \bar{A}_{22}$ etc. are the elements of the inverse. Now taking the first columns of both the inverse and the identity matrix:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} \bar{A}_{11} \\ \bar{A}_{21} \\ \bar{A}_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad (1.9)$$