REAL AND COMPLEX ANALYSIS

THIRD EDITION

WALTER RUDIN

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Third Edition

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REAL AND COMPLEX ANALYSIS

ABOUT THE AUTHOR

Walter Rudin is the author of three textbooks, *Principles of Mathematical Analysis*, *Real and Complex Analysis*, and *Functional Analysis*, whose widespread use is illustrated by the fact that they have been translated into a total of 13 languages. He wrote the first of these while he was a C.L.E. Moore Instructor at M.I.T., just two years after receiving his Ph.D. at Duke University in 1949. Later he taught at the University of Rochester, and is now a Vilas Research Professor at the University of Wisconsin–Madison, where he has been since 1959.

He has spent leaves at Yale University, at the University of California in La Jolla, and at the University of Hawaii.

His research has dealt mainly with harmonic analysis and with complex variables. He has written three research monographs on these topics, *Fourier Analysis on Groups*, *Function Theory in Polydiscs*, and *Function Theory in the Unit Ball of* $\mathbf{C}^n$.

PREFACE

This book contains a first-year graduate course in which the basic techniques and theorems of analysis are presented in such a way that the intimate connections between its various branches are strongly emphasized. The traditionally separate subjects of "real analysis" and "complex analysis" are thus united; some of the basic ideas from functional analysis are also included.

Here are some examples of the way in which these connections are demonstrated and exploited. The Riesz representation theorem and the Hahn-Banach theorem allow one to "guess" the Poisson integral formula. They team up in the proof of Runge's theorem. They combine with Blaschke's theorem on the zeros of bounded holomorphic functions to give a proof of the Müntz-Szasz theorem, which concerns approximation on an interval. The fact that L^2 is a Hilbert space is used in the proof of the Radon-Nikodym theorem, which leads to the theorem about differentiation of indefinite integrals, which in turn yields the existence of radial limits of bounded harmonic functions. The theorems of Plancherel and Cauchy combined give a theorem of Paley and Wiener which, in turn, is used in the Denjoy-Carleman theorem about infinitely differentiable functions on the real line. The maximum modulus theorem gives information about linear transformations on L^p-spaces.

Since most of the results presented here are quite classical (the novelty lies in the arrangement, and some of the proofs are new), I have not attempted to document the source of every item. References are gathered at the end, in Notes and Comments. They are not always to the original sources, but more often to more recent works where further references can be found. In no case does the absence of a reference imply any claim to originality on my part.

The prerequisite for this book is a good course in advanced calculus (set-theoretic manipulations, metric spaces, uniform continuity, and uniform convergence). The first seven chapters of my earlier book "Principles of Mathematical Analysis" furnish sufficient preparation.

Experience with the first edition shows that first-year graduate students can study the first 15 chapters in two semesters, plus some topics from 1 or 2 of the remaining 5. These latter are quite independent of each other. The first 15 should be taken up in the order in which they are presented, except for Chapter 9, which can be postponed.

The most important difference between this third edition and the previous ones is the entirely new chapter on differentiation. The basic facts about differentiation are now derived from the existence of Lebesgue points, which in turn is an easy consequence of the so-called "weak type" inequality that is satisfied by the maximal functions of measures on euclidean spaces. This approach yields strong theorems with minimal effort. Even more important is that it familiarizes students with maximal functions, since these have become increasingly useful in several areas of analysis.

One of these is the study of the boundary behavior of Poisson integrals. A related one concerns H^p-spaces. Accordingly, large parts of Chapters 11 and 17 were rewritten and, I hope, simplified in the process.

I have also made several smaller changes in order to improve certain details: For example, parts of Chapter 4 have been simplified; the notions of equicontinuity and weak convergence are presented in more detail; the boundary behavior of conformal maps is studied by means of Lindelöf's theorem about asymptotic values of bounded holomorphic functions in a disc.

Over the last 20 years, numerous students and colleagues have offered comments and criticisms concerning the content of this book. I sincerely appreciated all of these, and have tried to follow some of them. As regards the present edition, my thanks go to Richard Rochberg for some useful last-minute suggestions, and I especially thank Robert Burckel for the meticulous care with which he examined the entire manuscript.

Walter Rudin

CONTENTS

PROLOGUE
THE EXPONENTIAL FUNCTION

This is the most important function in mathematics. It is defined, for every complex number z, by the formula

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{1}$$

The series (1) converges absolutely for every z and converges uniformly on every bounded subset of the complex plane. Thus exp is a continuous function. The absolute convergence of (1) shows that the computation

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{m=0}^{\infty} \frac{b^m}{m!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k} = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!}$$

is correct. It gives the important addition formula

$$\exp(a)\exp(b) = \exp(a+b), \tag{2}$$

valid for all complex numbers a and b.

We define the number e to be $\exp(1)$, and shall usually replace $\exp(z)$ by the customary shorter expression e^z. Note that $e^0 = \exp(0) = 1$, by (1).

Theorem

(*a*) *For every complex z we have $e^z \neq 0$.*
(*b*) exp *is its own derivative:* $\exp'(z) = \exp(z)$.

(*c*) *The restriction of* exp *to the real axis is a monotonically increasing positive function, and*

$$e^x \to \infty \text{ as } x \to \infty, \qquad e^x \to 0 \text{ as } x \to -\infty.$$

(*d*) *There exists a positive number* π *such that* $e^{\pi i/2} = i$ *and such that* $e^z = 1$ *if and only if* $z/(2\pi i)$ *is an integer.*

(*e*) exp *is a periodic function, with period* $2\pi i$.

(*f*) *The mapping* $t \to e^{it}$ *maps the real axis onto the unit circle.*

(*g*) *If* w *is a complex number and* $w \neq 0$, *then* $w = e^z$ *for some* z.

PROOF By (2), $e^z \cdot e^{-z} = e^{z-z} = e^0 = 1$. This implies (*a*). Next,

$$\exp'(z) = \lim_{h \to 0} \frac{\exp(z+h) - \exp(z)}{h} = \exp(z) \lim_{h \to 0} \frac{\exp(h) - 1}{h} = \exp(z).$$

The first of the above equalities is a matter of definition, the second follows from (2), and the third from (1), and (*b*) is proved.

That exp is monotonically increasing on the positive real axis, and that $e^x \to \infty$ as $x \to \infty$, is clear from (1). The other assertions of (*c*) are consequences of $e^x \cdot e^{-x} = 1$.

For any real number t, (1) shows that e^{-it} is the complex conjugate of e^{it}. Thus

$$|e^{it}|^2 = e^{it} \cdot \overline{e^{it}} = e^{it} \cdot e^{-it} = e^{it-it} = e^0 = 1,$$

or

$$|e^{it}| = 1 \qquad (t \text{ real}). \tag{3}$$

In other words, if t is real, e^{it} lies on the unit circle. We define $\cos t$, $\sin t$ to be the real and imaginary parts of e^{it}:

$$\cos t = \operatorname{Re}[e^{it}], \qquad \sin t = \operatorname{Im}[e^{it}] \qquad (t \text{ real}). \tag{4}$$

If we differentiate both sides of Euler's identity

$$e^{it} = \cos t + i \sin t, \tag{5}$$

which is equivalent to (4), and if we apply (*b*), we obtain

$$\cos' t + i \sin' t = ie^{it} = -\sin t + i \cos t,$$

so that

$$\cos' = -\sin, \qquad \sin' = \cos. \tag{6}$$

The power series (1) yields the representation

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots. \tag{7}$$

Take $t = 2$. The terms of the series (7) then decrease in absolute value (except for the first one) and their signs alternate. Hence cos 2 is less than the sum of the first three terms of (7), with $t = 2$; thus $\cos 2 < -\frac{1}{3}$. Since $\cos 0 = 1$ and cos is a continuous real function on the real axis, we conclude that there is a smallest positive number t_0 for which $\cos t_0 = 0$. We define

$$\pi = 2t_0. \tag{8}$$

It follows from (3) and (5) that $\sin t_0 = \pm 1$. Since

$$\sin'(t) = \cos t > 0$$

on the segment $(0, t_0)$ and since $\sin 0 = 0$, we have $\sin t_0 > 0$, hence $\sin t_0 = 1$, and therefore

$$e^{\pi i/2} = i. \tag{9}$$

It follows that $e^{\pi i} = i^2 = -1$, $e^{2\pi i} = (-1)^2 = 1$, and then $e^{2\pi i n} = 1$ for every integer n. Also, (*e*) follows immediately:

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z. \tag{10}$$

If $z = x + iy$, x and y real, then $e^z = e^x e^{iy}$; hence $|e^z| = e^x$. If $e^z = 1$, we therefore must have $e^x = 1$, so that $x = 0$; to prove that $y/2\pi$ must be an integer, it is enough to show that $e^{iy} \neq 1$ if $0 < y < 2\pi$, by (10).

Suppose $0 < y < 2\pi$, and

$$e^{iy/4} = u + iv \qquad (u \text{ and } v \text{ real}). \tag{11}$$

Since $0 < y/4 < \pi/2$, we have $u > 0$ and $v > 0$. Also

$$e^{iy} = (u + iv)^4 = u^4 - 6u^2v^2 + v^4 + 4iuv(u^2 - v^2). \tag{12}$$

The right side of (12) is real only if $u^2 = v^2$; since $u^2 + v^2 = 1$, this happens only when $u^2 = v^2 = \frac{1}{2}$, and then (12) shows that

$$e^{iy} = -1 \neq 1.$$

This completes the proof of (*d*).

We already know that $t \to e^{it}$ maps the real axis *into* the unit circle. To prove (*f*), fix w so that $|w| = 1$; we shall show that $w = e^{it}$ for some real t. Write $w = u + iv$, u and v real, and suppose first that $u \geq 0$ and $v \geq 0$. Since $u \leq 1$, the definition of π shows that there exists a t, $0 \leq t \leq \pi/2$, such that $\cos t = u$; then $\sin^2 t = 1 - u^2 = v^2$, and since $\sin t \geq 0$ if $0 \leq t \leq \pi/2$, we have $\sin t = v$. Thus $w = e^{it}$.

If $u < 0$ and $v \geq 0$, the preceding conditions are satisfied by $-iw$. Hence $-iw = e^{it}$ for some real t, and $w = e^{i(t+\pi/2)}$. Finally, if $v < 0$, the preceding two cases show that $-w = e^{it}$ for some real t, hence $w = e^{i(t+\pi)}$. This completes the proof of (*f*).

If $w \neq 0$, put $\alpha = w/|w|$. Then $w = |w|\alpha$. By (*c*), there is a real x such that $|w| = e^x$. Since $|\alpha| = 1$, (*f*) shows that $\alpha = e^{iy}$ for some real y. Hence $w = e^{x+iy}$. This proves (*g*) and completes the theorem. ////

We shall encounter the integral of $(1 + x^2)^{-1}$ over the real line. To evaluate it, put $\varphi(t) = \sin t/\cos t$ in $(-\pi/2, \pi/2)$. By (6), $\varphi' = 1 + \varphi^2$. Hence φ is a monotonically increasing mapping of $(-\pi/2, \pi/2)$ onto $(-\infty, \infty)$, and we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\pi/2}^{\pi/2} \frac{\varphi'(t)\, dt}{1 + \varphi^2(t)} = \int_{-\pi/2}^{\pi/2} dt = \pi.$$

CHAPTER

ONE

ABSTRACT INTEGRATION

Toward the end of the nineteenth century it became clear to many mathematicians that the Riemann integral (about which one learns in calculus courses) should be replaced by some other type of integral, more general and more flexible, better suited for dealing with limit processes. Among the attempts made in this direction, the most notable ones were due to Jordan, Borel, W. H. Young, and Lebesgue. It was Lebesgue's construction which turned out to be the most successful.

In brief outline, here is the main idea: The Riemann integral of a function f over an interval $[a, b]$ can be approximated by sums of the form

$$\sum_{i=1}^{n} f(t_i)m(E_i)$$

where $E_1, \ldots, E_n$ are disjoint intervals whose union is $[a, b]$, $m(E_i)$ denotes the length of E_i, and $t_i \in E_i$ for $i = 1, \ldots, n$. Lebesgue discovered that a completely satisfactory theory of integration results if the sets E_i in the above sum are allowed to belong to a larger class of subsets of the line, the so-called "measurable sets," and if the class of functions under consideration is enlarged to what he called "measurable functions." The crucial set-theoretic properties involved are the following: The union and the intersection of any countable family of measurable sets are measurable; so is the complement of every measurable set; and, most important, the notion of "length" (now called "measure") can be extended to them in such a way that

$$m(E_1 \cup E_2 \cup E_3 \cup \cdots) = m(E_1) + m(E_2) + m(E_3) + \cdots$$