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Lectures on FUNCTIONAL EQUATIONS AND THEIR APPLICATIONS

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LECTURES ON FUNCTIONAL EQUATIONS AND THEIR APPLICATIONS was originally published under the title *Vorlesungen über Funktionalgleichungen und ihre Anwendungen.*

Foreword to the German Edition

The solution of functional equations is one of the oldest topics of mathematical analysis. D'Alembert, Euler, Gauss, Cauchy, Abel, Weierstrass, Darboux, and Hilbert are among the great mathematicians who have been concerned with functional equations and methods of solving them. In this field of mathematics, as in others, the literature has grown markedly during the past fifty years. (See the chronological bibliography at the end of this volume.) However, results found in earlier decades have often been presented anew because through the years there has been no systematic presentation of this field, in spite of its age and its importance in application.

In this monograph, an attempt is made to remedy this situation, at least in part. Results are usually presented with proofs, in contrast to S. PINCHERLE's German and French encyclopedia articles published in 1906 and 1912, which, of course, were written for a different purpose. Earlier works (such as those by E. CZUBER 1891, E. PICARD 1928, G. H. HARDY, J. E. LITTLEWOOD, and G. PÓLYA 1934, M. FRÉCHET 1938, and B. HOSTINSKY 1939) (see bibliography) also give some attention to functional equations, but the special functional equations treated are subordinate to their applications. We prefer to arrange the subject matter according to actual types of functional equations. We also cover a different and, as we think, somewhat broader range of problems than does the book of M. GHERMANESCU 1960[b]. A. R. Schweitzer's plan of 1918 to compile a bibliography of the theory of functional equations was, alas, never carried out; therefore the list of references at the end of this book, although incomplete, can partly serve as a bibliography too.

The term *functional equation* is interpreted here in its modern, more restricted sense (cf. exact definition in the introduction, Sect. 0.1), so that the definition does not apply to differential, integral, integro-differential, differential functional, and similar equations. As defined, however, the field of functional equations is still vast, and it was necessary to limit even further the material to be treated. Although our interpretation of this concept includes difference equations, we decided to omit them because many standard treatises on the subject are available.

For consistency, and because ample systematic discussion is partly available elsewhere, those functional equations are also omitted in which all the unknown functions contain at least as many variables as the total number of independent variables in the equation; for example, all iterative equations have been left out. Since functional equations eliminated by this decision have entirely different methods of solution from all the others, this fact may be regrettable to some readers, but the omission was necessary to keep the size of the book within reasonable limits and to preserve systematic unity.

A number of factors were considered in organizing the subject matter: functional equations for functions of one or several variables, for one function, or several functions, simple and composite equations, different elementary methods of solution and reduction to differential and other equations, special applications of the equations, historical considerations, and the like. The classification is not rigid; some investigations may be considered as belonging in a particular chapter but are treated elsewhere because of their specific relationships. Functional equations for vectors and matrices of a finite number of dimensions are treated briefly as a link between equations with one and with several unknown functions of one and of several variables. On the other hand, equations for operators and functionals are not considered, since to do so would entail delving too deeply into functional analysis. As indicated by the title, there is no treatment of functional inequalities; an investigation of this would have to include, for example, the entire theory of convex functions. Within its framework, the book merely touches on the use of functional equations to define functions and their extension from the real to the complex region, their extension to matrices and other forms, their use in constructing functions of several variables by means of functions of fewer variables, and similar questions—all of them outside the scope of this book, which is limited mainly to methods of solution.

Certain restrictions are imposed in regard to the domains and the range of functions, as well as the "regularity" of the functions figuring in the equations; otherwise, for instance, most of the algebra would have to be included in the treatment of functional equations of associativity, transformation, and distributivity. However, algebraic structures with other laws are mentioned, and most of the publications dealing with them are included in the bibliography. Also, the independent fields of mathematics which are largely concerned with the solution and application of functional equations, such as the theory of continuous groups and the theory of geometrical objects, had to be omitted. In this connection, the booklet (J. Aczél and S. Gołąb 1960) on functional equations in the theory of geometrical objects might be mentioned. Nevertheless, we have included in the bibliography works in this theory which utilize principally functional equations. On the other hand, emphasis is placed on the relation of the discipline under discussion to algebra and to many "algebraized" fields of geometry (continuous groups, vector analysis, and the like). The broad fields of application, such as probability theory, non-Euclidean geometry, and mechanics, which have contributed greatly to developing the discipline of functional equations, should also play a significant role. In this book, however, the treatment of applications is subordinated to the equations, and preliminaries and consequences in these fields cannot be discussed in detail. Particulars of applications which are required for clarity but which themselves do not use functional equations are sometimes given in fine print, as are less important examples and more elaborate parts of certain proofs.

The book, aside from the foreword, introduction, concluding remarks, and bibliography, is divided into two parts. Smaller divisions include chapters, sections, and subsections. The chapters are numbered in sequence through the book, the sections are numbered within the chapters, and the subsections within the sections. Theorems and formulas are numbered within the subsections. For example, 1.2.3 (1) indicates formula (1) of subsection 3 in Section 2 of Chapter 1 (which in turn is in Part I). References to the bibliography are made as follows: a work cited in the text or in a footnote, or in this foreword—for example, A. R. Schweitzer 1918 [c]—will be found in the chronologically arranged bibliography under the year (1918); within that year-group, alphabetically under the author's name (Schweitzer, A. R.); and among that author's references, under the letter indicated ([c]).

Unfortunately, very few existence and uniqueness theorems are included, and very little is included about the influence of the form of the equation on the form of the solution, since such information is almost nonexistent. When possible, we investigate the generality of given methods of solution. Hence, *types* of functional equations are investigated with more general methods of solution, in addition to special functional equations with individual methods of solution which have been mainly investigated up to now. As a result, a certain order and correlation are imposed upon this disorganized field, although the lack of a unified theory is still quite apparent.

Another peculiarity of functional equations, compared with differential, integral, difference, and other equations, as observed by Abel, is that *one* functional equation can contain *several* unknown functions in such a way that all unknown functions can be determined from it. This fact is encountered frequently throughout the book and can serve as a unifying principle, since such functional equations often unite many seemingly different equations.

In systematized fields, books are often arranged so that individual parts are independently comprehensible; here, emphasis is placed on relations between otherwise isolated investigations. This arrangement often makes it necessary for the reader to leaf through the book, although nearly all chapters and many sections actually stand as independent entities.

The origins of the investigations are, to the best of our knowledge, always explicitly presented. The expert will also find some new material. As a link between textbook, monograph, and reference work, this book contains theorems with both sketchy and detailed proofs; in a few cases proofs are omitted. The hypotheses and assertions are sometimes formulated in advance, sometimes later, but nearly always explicitly. There is no attempt to prove the strongest assertions under the weakest hypotheses. Some of the numerous and diverse gaps, problems, and conjectures which are still unresolved are explicitly formulated in this book, which, it is hoped, will intensify interest in this field, rich in problems and important for applications.

The expert may consider the elementary character of the book a shortcoming, but this approach was intentional, in order to benefit the student as much as possible. The concepts of function, monotonicity, and continuity suffice for understanding a considerable amount of the material; with knowledge of the concepts of integrability (measurability) of the (partial) derivatives, of the Jacobian functional determinant, and of the basis of real numbers, most of the text should be readily comprehensible, with the possible exception of certain applications. As a matter of fact, the elementary character of the present status of the functional equation theory has both advantages and disadvantages. The object of this book is to win new supporters through an elementary approach, in the hope that they, in turn, will help advance the field.

The book is based partly on lectures delivered from 1953–1960 at Debrecen University, and should be judged with the indulgence due a first attempt to summarize a large, ramified, and unsystematically developed field. The author will welcome comments from colleagues in this and other disciplines concerning the content and methods used in the book.

The author is grateful to Professors S. Gołąb (Kracow), M. Ghermanescu (Bucharest), H. P. Thielman (Iowa), Dr. M. Hosszú (Miskolc), Dr. H. Kiesewetter (Berlin), Dr. M. Kuczma (Katowice), and Dr. E. Vincze (Miskolc) for their critical review of the first draft and the final manuscript of the book and for their valuable comments and suggestions. Warmest thanks go to Dr. J. Merza, who drew the figures, and to the author's wife, who helped to prepare the manuscript and who, with Mr. Hosszu, compiled the author index. Numerous colleagues supplied important literature references.

The manuscript of the book contained the dedication "To my highly esteemed and beloved teacher—Leopold Fejér—in warmest gratitude and friendship." To our great sorrow, L. Fejér succumbed to a serious illness in his 80th year, prior to publication of this work. It is dedicated to his memory.

J. Aczél

February, 1960
Debrecen

Foreword to the English Edition

Although the English edition is not a new book, it is more than a translation of J. Aczél 1961 [h]. Much new material has been added to the text and the bibliography is almost twice its original size. Not all new entries date from later than 1960, the year the German edition was finished, but the many new contributions since 1960 reflect the vigorous development of this theory. I would feel gratified if the English edition helps continue this progress.

The principles governing the topics chosen remain essentially unchanged. Although the presentation is more formal (separating theorems and proofs) and generalizations for more abstract structures are often considered, the elementary and ever-developing character of the material is still evident. Topics such as the domains of functional equations and noncontinuous solutions are more stressed in the new edition.

A new feature in the Bibliography, which the reader might find helpful, is that after each item he finds the page-numbers where that work was quoted.

The revisions in this new edition are partly based on experiences of further lectures in Debrecen (Hungary), Gainesville (Florida), Giessen and Cologne (Germany), and Waterloo (Ontario). Warmest thanks go to students and colleagues at all these places for their valuable comments.

The author is grateful to Dr. M. Kuczma (Katowice), Dr. E. Vincze (Miskolc), and Dr. M. A. McKiernan, who read the galley proofs and the repros, and made many important suggestions; to the translator and referee for the English text; and to Academic Press for readily accepting changes resulting in improvement.

Last but not least I thank those who have made valuable suggestions and remarks on the German edition and hope they and others will continue this activity in connection with the English edition.

J. Aczél

Waterloo, Ontario, Canada
October, 1965

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Introduction

0.1. Definition and Examples

Nearly everyone working in mathematics has encountered functional equations, so that examples need not be given at this point. A definition of the concept "functional equation" is difficult. A somewhat loose paraphrase of what is generally meant by this expression follows:

Functional equations are equations, both sides of which are terms constructed from a finite number of unknown functions (of a finite number of variables) and from a finite number of independent variables. This construction is effected by a finite number of known functions of one or several variables (including the four species) and by finitely many substitutions of terms which contain known and unknown functions into other known and unknown functions. The functional equations determine the unknown functions. We speak of functional equations or systems of functional equations, depending on whether we have one or several equations. (Also, a single functional equation can determine several desired functions occurring within the equation; as a result, the number of equations is not related to the number of functions to be determined.)

Since this description, which can hardly be considered a definition, contains the concept *term*, we shall begin with the definition of the concept itself[1]:

1. Definition of Term. (a) *The independent variables* $x_1, x_2, \ldots, x_k$ *are terms.*

(b) *Given that* $A_1, A_2, \ldots, A_m$ *are terms and that* F *is a function of* m *variables, then* $F(A_1, \ldots, A_m)$ *is also a term.*

(c) *There are no other terms.*

[1] J. Aczél and H. Kiesewetter 1957, D. S. Mitrinović and D. Ž. Đoković 1962[a], M. Kuczma 1964.

A given term thus contains a definite number (k) of variables and a definite number of functions (n). (In functional equations, some of the functions are *known*, others are *unknown*.)

2. Definition of Functional Equation. *A functional equation is an equation*

$$A_1 = A_2$$

between two terms A_1 and A_2, which contains k independent variables $x_1, x_2, \ldots, x_k$ and $n \geqslant 1$ unknown functions $F_1, F_2, \ldots, F_n$ of $j_1, j_2, \ldots, j_n$ variables respectively, as well as a finite number of known functions.

k is the *rank*[2] and n is the *number of functions* of the functional equation, $j = \min(j_1, \ldots, j_n)$ is the *minimal number* of the variables in the functions of the functional equation.

The fact that we are excluding the possibility of infinitely many variables or functions as well as the possibility of known and unknown operators and functionals excludes functional equations in the broader sense from our definition (for example, operator equations, differential, integral, integrodifferential, functional differential equations and the equations of optimization in dynamic programming). It does include, however, difference equations, iteration equations, equations defining implicit functions, etc., so that we apply the limitation $j < k$ in this book, i.e.:

The rank must be larger than the minimal number of variables in the functions of the equation.[3]

3. Definition of System of Functional Equations. *A system of functional equations consists of $p \geqslant 2$ functional equations, which contain $n \geqslant 1$ unknown functions altogether. p is the number of equations, n the number of functions of the system.*

[2] W. Maier 1957. [We translate "Stufe" as "rank".] For other definitions cf. A. R. Schweitzer 1916[e]; H. Steinhaus 1956; M. Hosszú 1962[c]; B. Schweizer and A. Sklar 1962[a]; M. Kuczma 1964.

[3] In any event, to a certain extent these are only formal limitations: It is also possible to write differential equations in the form of functional equations in the above sense. For example, the differential equation

$$f'(x) = f(x)$$

can be written in the form

$$f(x + y) - f(x)(1 + y) = F(x, y)y, \qquad F(x, 0) = 0, \qquad F(x, y) \text{ continuous.}$$

It is assumed here too that *in at least one equation of the system, the rank is greater than the minimal number of variables of the functions appearing in this equation*. It is further assumed that the ranks, numbers of functions, minimal numbers, and the number of equations are *essential*, that is, that none of the variables, functions, and equations can be eliminated in a trivial manner; thus, no identities should occur, no variables are to be included in functions in which they are constant, etc.

Naturally, it is permitted that the "known" functions of the above definition appear in an arbitrary form, for example, implicitly.

The functional equations or systems must be identically satisfied for certain values of the variables $x_1, x_2, \ldots, x_k$ figuring in them. (Nevertheless we use the sign $=$ and not $\equiv$.) This is their *domain*. It usually consists of sets of k-tuples of real or complex numbers, but can also be a domain in a vector space, a set of matrices, or even an abstract algebraic system. *If we do not say otherwise, the domain is supposed to be that of real numbers or k-tuples of real numbers*. Also, the range of the unknown functions can consist of varied quantities (real or complex numbers, vectors, matrices, elements of abstract sets, and the like). An important additional concept is the *class of admissible functions* for the desired functions. This class can be defined by the analytic properties (measurability, invertibility, boundedness, monotonicity, continuity, integrability, differentiability, analyticity, etc.) or by initial and boundary conditions (function values on a subset of the domain). Sometimes conditions are given in the form of additional functional or other equations.

A particular *solution* of a functional equation or of a system of functional equations is a function or a system of functions (when $n \geqslant 2$) which satisfies the equation or equations in the given domain (reduces them to identities). The *general solution* is the totality of all solutions belonging to the class of admissible functions. To *solve* a functional equation, or a system, means to find the general solution. This solution depends naturally on the domain and the class of admissible functions.

The following are examples of functional equations and of systems of functional equations:

$$f(x+y) = f(x)f(y), \tag{1}$$

$$f(x)^2 = f(x+y)f(x-y), \tag{2}$$

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}, \tag{3}$$