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Three Papers on Dynamical Systems

by

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SMOOTH DYNAMICAL SYSTEMS

The literature on smooth dynamical systems is substantial. In selecting material for our lectures we have set ourselves a twofold aim. On the one hand we have tended to give a more or less connected account of a number of contemporary results associated with general problems of the classification of dynamical systems, by describing "rough" and "typical" properties, etc. On the other hand we wish to emphasize that the general constructions arising here are connected with ideas going back to the classics, and they permit us to obtain new information on qualitative properties of some long known problems.

We have not set ourselves the objective of recounting in detail the genesis and evolution of the ideas and notions under consideration, since this would require a substantial increase in size. The bibliographic citations we give do not pretend to be complete; in addition to the few remarks and references of a historical character that are presented by each author, we recommend consultation of the fundamental contribution *Geodesic flows on closed Riemannian manifolds of negative curvature* by D. V. Anosov. As far as we know, this paper gives the most detailed historical survey of everything connected with the "hyperbolic" behavior of the trajectories of dynamical systems–a property playing a basic role in the circle of questions under consideration.

V. M. Alekseev
A. B. Katok
A. G. Kušnirenko

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* The American Mathematical Society scheme for transliteration of Cyrillic may be found at the end of index issues of *Mathematical Reviews.*

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PROBLEMS IN THE GENERAL THEORY OF DYNAMICAL SYSTEMS ON A MANIFOLD*

A. G. KUŠNIRENKO

CONTENTS

§1. Simplest examples of dynamical systems

1. Let M be a C^∞-manifold. By a classical dynamical system on M, we mean either a diffeomorphism $\varphi: M \longrightarrow M$, or a one-parameter group of diffeomorphisms φ_t of M, differentiable in t (i.e. a family of diffeomorphisms φ_t of M, differentiable with respect to t, and satisfying the condition $\varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}$ for any $t_1, t_2 \in \mathbf{R}$). A vector field $X = d\varphi/dt|_{t=0}$ may be associated with every such one-parameter group; and, conversely, in the case where M is compact, every smooth vector field gives rise (by a well-known theorem) to a group of diffeomorphisms of M, depending smoothly upon the parameter t. Such a group of diffeomorphisms (or sometimes the vector field which generates it) is called a *dynamical system with continuous time*, or a *flow*, while the set $\{\varphi^n\}_{n\in \mathbf{Z}}$ of all powers of some diffeomorphism φ (or just this diffeomorphism itself) is often called a *dynamical system with discrete time*. From every dynamical system $\{\varphi_t\}_{t\in\mathbf{R}}$ with continuous time, one can obtain a dynamical system $\{\varphi_{n\cdot t_0}\}_{n\in\mathbf{Z}}$ with discrete time, generated by the powers of the diffeomorphism φ_{t_0}; but not every system with discrete time can be obtained from some flow in this manner.

EXAMPLE 1. *Rotation of a circle.* $M = S^1 = \{x \pmod{2\pi}\}$, $\varphi: S^1 \longrightarrow S^1$, $\varphi x = x + d \pmod{2\pi}$. *Case* a) $d/2\pi$ is rational; *case* b) $d/2\pi$ is irrational.

The set $\{\varphi^n x: n \in \mathbf{Z}\}$ (in the case of a dynamical system with discrete time), or the set $\{\varphi_t x: t \in \mathbf{R}\}$ (in the case of continuous time) is called the *trajectory* or *orbit* of the point $x \in M$.

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In Example 1, case a), every orbit consists of a finite number of points (the trajectories are closed); in case b) all the orbits are infinite sets, and it is easily shown that every orbit is everywhere dense in S^1.

EXAMPLE 2. $M = T^2 = S^1 \times S^1 = \{(x_1, x_2) \pmod 1\}$. A flow on M is given by a system of ordinary differential equations: $\dot{x}_1 = 1, \dot{x}_2 = \lambda$. The corresponding one-parameter group of diffeomorphisms of M may be written down explicitly:

$$\varphi_t\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{matrix} x_1 + t \pmod 1 \\ x_2 + \lambda t \pmod 1 \end{matrix}.$$

Just as in Example 1, two cases are possible: a) λ is rational, and in this case all the trajectories are closed; b) λ is irrational, and in this case it is easily shown that every trajectory is everywhere dense in T^2.

There is a relation between Examples 1 and 2, which will be discussed below.

2. DEFINITION. The point $p \in M$ is said to be a *fixed point* of a dynamical system, provided the orbit of p coincides with p. In the case of continuous time, a fixed point is often called a *singular point*, a *stationary point*, or an *equilibrium position*.

EXAMPLE 3. $M = S^2$ = Riemann sphere = $\widetilde{\mathbf{C}}$. On $\widetilde{\mathbf{C}}\backslash\infty$, a dynamical system is given by the system of two differential equations: $\dot{z} = z$. The substitution $z = 1/V$ shows that φ_t is also smooth in the neighborhood of ∞.

It is easily seen that ∞ is a fixed point and that, for any $t \in \mathbf{R}$ and $z \in \mathbf{C}$, $\varphi_t z = e^t z$. Thus the dynamical system under consideration has two fixed points, 0 and ∞, while for any other point z_0

$$\varphi_t z_0 \longrightarrow \begin{cases} \text{“}\infty\text{''} & \text{as } t \longrightarrow +\infty, \\ 0 & \text{as } t \longrightarrow -\infty. \end{cases}$$

We note that, unlike Examples 1 and 2, φ_t does not preserve any finite regular (i.e., positive on any open set) measure on S^2. In fact, for $t > 0$ we have the strict inclusion $\varphi_t\{z: |z| > 1\} \subset \{z: |z| > 1\}$.

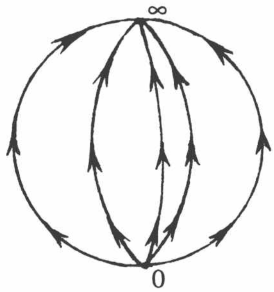

FIGURE 1

EXAMPLE 4. We take the dynamical system (with continuous time) of Example 1, and obtain from it, in the manner described above, the dynamical system (with discrete time) $\{\varphi_t: t \in \mathbf{Z}\}$. This system is generated by the diffeomorphism $\varphi_1: \varphi_1(\infty) = \infty$ and $\varphi_1 z = ez$.

EXAMPLE 5. $M = S^1$, $\dot{x} = \sin 3x$. In this example there are six fixed points, three attractive and three repulsive. All the remaining points form six trajectories. Every nonfixed point approaches one of the attractive fixed points as $t \longrightarrow +\infty$, and one of the repulsive fixed points as $t \longrightarrow -\infty$.

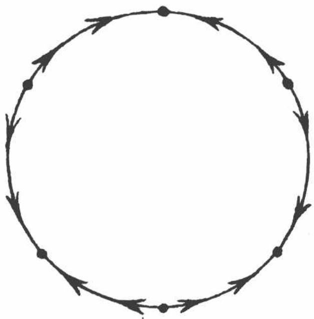

FIGURE 2

EXAMPLE 6. Let $M = S^1$ and let φ_t be the one-parameter group of diffeomorphisms of Example 5.

a) Set $\varphi = \varphi_1$.

b) Define the diffeomorphism ψ by the equality $\psi x = \varphi_1 x + 2\pi/3$.

In Example 6a), just as in Example 5, there are three attractive and three repulsive fixed points, while the closure of any other trajectory contains two fixed points, one attractive and one repulsive. In Example 6b) there are no fixed points. The orbit of the point 0 consists of three points, and likewise the orbit of π consists of three points.

DEFINITION. The point $x \in M$ is said to be a *periodic point* of the diffeomorphism φ if there exists an $n > 0$ such that $\varphi^n x = x$. The smallest such integer n is called the *period* of x.

In the case of continuous time, the trajectory of the point $x \in M$ is said to be *periodic with period* T $(T > 0)$ if $\varphi_T x = x$ and $\varphi_t x \neq x$ for $0 < t < T$. The periodic trajectories of φ_t are also known as *closed* trajectories.

REMARK. Among the fixed points of the diffeomorphism φ^n are included all the periodic points of period n of the diffeomorphism φ; but, in general, not every fixed point of φ^n is periodic with period n.

Thus, in Example 6b), there are two periodic trajectories with period 3 (six periodic points). For any nonperiodic point x, $\psi^n x$ approaches one of the periodic trajectories as $n \longrightarrow +\infty$, and the other as $n \longrightarrow -\infty$.

EXAMPLE 7. As M, we take the sphere S^2, smoothly imbedded in $\mathbf{R}^3$, as shown in Figure 3 for case a), and in Figure 4 for case b). We consider the function $f = x_3|_{S^2}$ (i.e., height) on S^2, and put $-\dot{x} = \operatorname{grad} f$ ($x \in S^2$, and the metric on S^2 is induced by the imbedding in $\mathbf{R}^3$). The trajectories of these dynamical systems are known as *lines of steepest descent.*

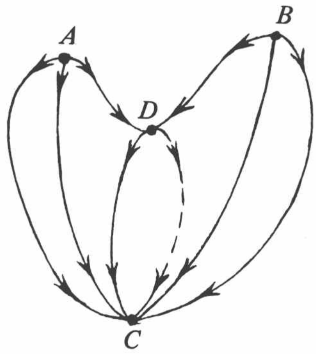

FIGURE 3

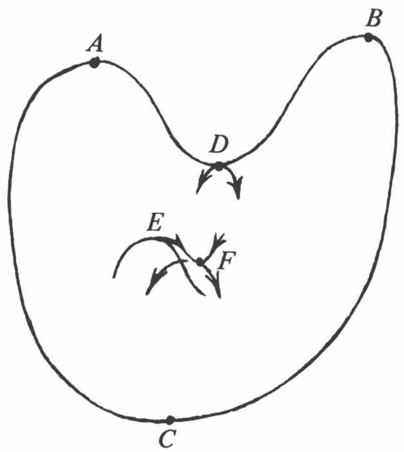

FIGURE 4

Case a). There are four fixed points, A, B, C, and D, corresponding to the critical points of the function f.

In small neighborhoods of A and B, all trajectories recede from the critical point as t increases, and in a small neighborhood of C, all trajectories approach C as t increases. D is a saddle point. There are two trajectories which approach D as $t \longrightarrow +\infty$, and two trajectories which approach D as $t \longrightarrow -\infty$. Any other trajectory bypasses the point D (i.e., leaves any small neighborhood of D as $t \longrightarrow \pm\infty$). The set consisting of those points of M which approach G as $t \longrightarrow +\infty$ is open and everywhere dense in M. This set is homeomorphic to an open disk. The complement of this set is of smaller dimension than M, and consists of the fixed points A, D, B and the two trajectories which approach D as $t \longrightarrow +\infty$. Similarly, the set consisting of those points of M which approach either A or B as $t \longrightarrow -\infty$ is open and everywhere dense in M, while the complement of this set has dimension 1.

Case b). In this example, there are six fixed points: three maximum points of f, one minimum point of f and two saddle points. The behavior of the trajectories on M

essentially depends upon how the separatrices emanating from the point D (that is, the two trajectories which approach D as $t \longrightarrow -\infty$) behave as $t \longrightarrow +\infty$.

EXERCISE. We excise from S^2 a small disk around the point C. The remaining portion of S^2 is also a disk. Sketch the trajectories of system 7b) on this disk, examining the various possible cases with respect to the behavior of the separatrices emanating from the point D.

3. In the next two examples, there naturally appear manifolds more complicated than those in the examples already considered.

EXAMPLE 8. a) We consider the sphere S^2, smoothly imbedded in $\mathbf{R}^3$, and a particle moving on the surface of S^2 without friction and with unit velocity. The phase space of this dynamical system is the manifold of unit vectors tangent to S^2; and, as is easily verified, this manifold is diffeomorphic to three-dimensional real projective space.

b) An analogous construction can be carried out for any surface $M^2 \subset \mathbf{R}^3$. The phase space of such a dynamical system will be the three-dimensional manifold of unit vectors tangent to M^2. The metric on M^2 is induced by the imbedding in $\mathbf{R}^3$, and the trajectories of the dynamical system so obtained correspond to the geodesics on the Riemannian manifold M^2.

EXAMPLE 9. Let A be a real $n \times n$ matrix. We consider the system of linear differential equations $\dot{x} = Ax$ in $\mathbf{R}^n$, and denote by φ_t the corresponding one-parameter group of linear transformations of $\mathbf{R}^n$. We fix an integer k, $1 \leqslant k \leqslant n-1$, and consider the set of all k-dimensional linear subspaces of $\mathbf{R}^n$. This set may naturally be given the structure of a smooth manifold, which is called a *real Grassmann manifold*, and is denoted by G_n^k. Since each diffeomorphism φ_t sends any k-dimensional subspace into some other k-dimensional subspace, φ_t induces a new dynamical system $\hat{\varphi}_t$ on G_n^k.

REMARK. Let the vector field X on the manifold M^n have the fixed point x_0; we wish to find (locally) the k-dimensional invariant manifolds of the flow φ_t passing through x_0. In terms of local coordinates with origin at x_0, the flow φ_t is given by the system of differential equations $\dot{x} = Ax + \cdots$. The tangent plane to any invariant k-dimensional manifold is invariant with respect to the linear system $\dot{x} = Ax$, and therefore constitutes a fixed point of the flow $\hat{\varphi}_t$ on G_n^k. Thus, the determination of the fixed points of $\hat{\varphi}_t$ in G_n^k constitutes a natural step in finding the invariant manifolds passing through x_0. In exactly the same manner, to investigate the behavior of k-dimensional manifolds which are (locally) nearly invariant, it is natural to begin by examining a small neighborhood of a fixed point of the flow $\hat{\varphi}_t$ in G_n^k.

EXERCISE. In the case where $n = 4$, $k = 2$, and A is a diagonal matrix, find the fixed points and invariant submanifolds of $\hat{\varphi}_t$. Describe the behavior of the trajectories* in the two cases of Example 7.

**Translator's note.* Presumably this refers to the trajectories of the flow $\hat{\varphi}_t$ induced by the linearized equations in the neighborhoods of the various fixed points, as explained in the preceding remark.

§2. The relation between diffeomorphisms and flows

Certain formal relations may be established between flows and diffeomorphisms.

1. Succession functions. Let the flow φ_t on an $(n+1)$-dimensional manifold have a closed trajectory γ with period T. We take any point x_0 on γ and any n-dimensional area element P passing through x_0 and transversal to γ.

Let U be some sufficiently small neighborhood of x_0 in P. For any point $x \in U$, the trajectory $\varphi_t x$ of this point remains close to the trajectory of x_0 (that is, to γ), and consequently for some $t_0 > 0$ ($t_0 \approx T$) this trajectory will (for the first time) intersect P.

Denoting this first point of intersection by φx, we obtain a local diffeomorphism $\varphi\colon U \longrightarrow P$, where $\varphi x_0 = x_0$.

DEFINITION. By a *local diffeomorphism* of a (possibly open) manifold P, we mean a pair (U, φ) consisting of an open set $U \subset P$ and a diffeomorphism $\varphi\colon U \longrightarrow \varphi(U) \subset P$.

Let the local diffeomorphism (U, φ) of the manifold P have a fixed point x_0, and let the local diffeomorphism (U', φ') of the manifold P' have the fixed point x'_0. We say that the diffeomorphism φ at x_0 is *differentiably equivalent* to φ' at x'_0 if there exists a diffeomorphism $h\colon V \longrightarrow V'$, where V and V' are open sets, $x_0 \in V \subset U$ and $x'_0 \in V' \subset U'$, such that

$$h \circ \varphi|_{V \cap \varphi^{-1}(V)} = \varphi' \circ h|_{V \cap \varphi^{-1}(V)}.$$

EXERCISE. Let P' be a transversal surface element at some point $x_1 \in \gamma$ and let $\varphi'\colon U' \longrightarrow P'$ be the corresponding local diffeomorphism. Show that φ and φ' are differentiably equivalent.

It is easily seen that the study of the flow φ_t in the vicinity of γ is completely equivalent to the study of the local diffeomorphism φ. For example, the determination of the periodic trajectories of φ_t in the vicinity of γ is equivalent to the determination of the fixed and periodic points of φ, the stability of the periodic solution γ is equivalent to the stability of the fixed point x_0 of φ, and so forth.

The local diffeomorphism φ is often called a *succession function.*

2. Global succession functions. Sometimes a construction similar to the preceding may be effectively carried out in a global context. Let W^{n+1} be a compact, connected $(n+1)$-dimensional manifold, and φ_t a flow on it. We assume that there exists a smooth compact n-dimensional submanifold $M^n \subset W^{n+1}$ having the following properties:

1) no trajectory of φ_t is tangent to M^n;

2) every trajectory has a point of intersection with M^n.

Then,

3) for any point $x \in M^n$, there exists a minimal $t_0 > 0$ such that $\varphi_{t_0} x \in M^n$.

For $x \in M^n$, letting $\varphi(x)$ be the first point of intersection of $\varphi_t x$ with M^n for $t > 0$, we obtain a well-defined global "*succession diffeomorphism.*"

The manifold M^n is called a *global transversal* for the flow φ_t on W^{n+1}.

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It is easily seen that the study of the flow φ_t on W^{n+1} is equivalent to the study of the diffeomorphism φ on M^n. Moreover, it can be shown that every diffeomorphism ψ which is close to φ on M^n is induced in the above-described manner by some flow ψ_t which is close to φ_t on W^{n+1}.

EXERCISE. Let M_1^n be a submanifold of W^{n+1} which is C^1-close to M^n.

a) Show that M_1^n is a global transversal for the flow φ_t.

b) Show that there exists a diffeomorphism $h\colon M^n \longrightarrow M_1^n$ such that the following diagram is commutative:

$$\begin{array}{ccc} M^n & \xrightarrow{\ \varphi\ } & M^n \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ M_1^n & \xrightarrow{\ \varphi_1\ } & M_1^n \end{array}$$

(here φ_1 is the succession diffeomorphism on M_1^n).

3. Suspension. We shall now show that any diffeomorphism φ of an arbitrary manifold M^n is the succession diffeomorphism of some flow.

Consider the manifold-with-boundary $M^n \times [0, 1]$, the boundary of which consists of two copies of M^n (that is, $M^n \times 0$ and $M^n \times 1$), and consider the vector field on this manifold consisting of the unit tangent vectors in the direction of the interval $[0, 1]$. We now identify $M^n \times 0$ and $M^n \times 1$ by means of the diffeomorphism φ; that is, we regard $x \times 0$ and $\varphi x \times 1$ as one and the same point. It is easily seen that we thereby obtain a certain smooth manifold W^{n+1} and a smooth flow φ_t on it; moreover, the locus of the "pasting" operation constitutes a global transversal with succession diffeomorphism φ. The flow φ_t on W^{n+1} is called the *suspension* of the diffeomorphism φ.

EXERCISE. Show that the flow of Example 2 is the suspension of the diffeomorphism of Example 1.

REMARK. In fact, the preceding arguments contained a logical gap. In the set obtained after identifying $M^n \times 0$ and $M^n \times 1$, the structure of a smooth manifold does not just arise by itself, but must be introduced. We shall not do this, but rather mention a construction of the suspension somewhat different in form from the preceding one.

Consider the manifold $\widetilde{W}^{n+1} = M^n \times \mathbf{R}$ and the flow defined by the formula $\widetilde{\varphi}_t(x, \tau) = (x, \tau + t)$, and also the diffeomorphism $\widetilde{\varphi}$ of $\widetilde{W}^{n+1}$ defined by the formula $\widetilde{\varphi}(x, \tau) = (\varphi x, \tau - 1)$. Clearly, $\widetilde{\varphi}_t \circ \widetilde{\varphi} = \widetilde{\varphi} \circ \widetilde{\varphi}_t$ for any t. Now, we shall say that points a and b of the manifold $\widetilde{W}^{n+1}$ are *equivalent* if there exists an integer n such that $a = \widetilde{\varphi}^n b$. We denote by W^{n+1} the set of equivalence classes, and by $\pi\colon \widetilde{W}^{n+1} \longrightarrow W^{n+1}$ the natural projection onto this set. It is easily seen that there exists a unique smooth manifold structure on W^{n+1} such that π becomes a local diffeomorphism. From the fact that $\widetilde{\varphi}$ commutes with $\widetilde{\varphi}_t$, it follows that $\widetilde{\varphi}_t$ preserves the equivalence relation on $\widetilde{W}^{n+1}$ and consequently induces a flow on W^{n+1} (which we denote by φ_t). Finally, it is easily seen that the equivalence classes of points of the form $x \times 0$ $(x \in M^n)$

provide a smooth manifold in W^{n+1}, diffeomorphic to M^n, which constitutes a global transversal for the flow φ_t, and that the succession diffeomorphism on this manifold corresponds to φ.

EXERCISE. Construct the suspension for the diffeomorphism of Example 6b).

DEFINITION. We say that the diffeomorphism $\varphi: M \longrightarrow M$ is *contained in a flow,* if there exists a one-parameter group of homeomorphisms φ_t of M such that $\varphi = \varphi_1$. If φ_t is a group of diffeomorphisms depending differentiably upon t, then we shall say that φ is *contained in a smooth flow.*

PROBLEM. Prove that, for any manifold M, there exists a diffeomorphism φ: $M \longrightarrow M$ which is not contained in a flow.

Hint. Examine the set of periodic points of a diffeomorphism which is included in a flow.

§3. Spaces of dynamical systems. Equivalence relations

Let M be a compact C^∞-manifold. We denote by $\mathrm{Diff}^r(M)$ $(1 \leqslant r \leqslant \infty)$ the set of C^r-diffeomorphisms of M. For any k, $1 \leqslant k \leqslant r \leqslant \infty$, we may convert $\mathrm{Diff}^r(M)$ into a topological (function) space, introducing the topology of C^k-convergence (i.e. uniform convergence together with that of all derivatives of order up to k) in $\mathrm{Diff}^r(M)$. We denote the topological space thus obtained by $\mathrm{Diff}^r_k(M)$. The space Diff^r_k is a complete metric space and possesses the structure of a Banach C^∞-manifold (i.e., one may choose a neighborhood of each point which is homeomorphic to an open ball in a fixed Banach space B such that the transition functions will be C^∞ on B).

Let E be some equivalence relation on Diff^r.

1) For example, let us say that $f, g \in \mathrm{Diff}^r(M)$ are equivalent if there exists a C^l-diffeomorphism $h: M \longrightarrow M$ $(l \leqslant r)$ such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & M \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ M & \xrightarrow{\ g\ } & M \end{array}$$

This equivalence relation is called *C^l-conjugacy.*

2) If we merely require that the mapping h in the previous diagram be a homeomorphism, then we obtain the relation of topological conjugacy.

3) If we impose further conditions on the homeomorphism h, we obtain various other equivalence relations. For example, we may require that h be homotopic to the identity mapping of M.

4) Below, we shall encounter still another interesting equivalence relation, known as *Ω-conjugacy.*

To every equivalence relation E on the set $\mathrm{Diff}^r(M)$, there corresponds a certain notion of stability on $\mathrm{Diff}^r_k(M)$.

DEFINITION. Let E be an equivalence relation on $\mathrm{Diff}^r(M)$. The diffeomorphism $f \in \mathrm{Diff}^r_k(M)$ is said to be *E-stable* if there exists a neighborhood $U = U(f)$ of the point

f in the space $\mathrm{Diff}_k^r(M)$ such that any diffeomorphism $g \in U(f)$ is E-equivalent to f: $g \underset{E}{\sim} f$.

We emphasize that the notion of E-stability in general depends upon the choice of the topology on Diff^r.

In the case where E is the relation of topological equivalence, a stable diffeomorphism is said to be *structurally stable* or *rough.*

The notion of structural stability or roughness, which has turned out to be especially fruitful, first appeared in 1937, in a paper of Andronov and Pontrjagin, in connection with flows [1] (i.e., dynamical systems with continuous time). The definitions of topological equivalence and structural stability for a flow are not entirely parallel to the corresponding definitions for a diffeomorphism, and will be given below.

For a compact manifold M, the space of smooth flows on M may be identified with the linear space $\mathcal{X}^r(M)$ of C^r-smooth vector fields on M. Provided with a C^r-norm, $\mathcal{X}^r$ becomes a Banach space. Given a vector field X on M, for an arbitrary homeomorphism h the objects $X \circ h$ and $h \circ X$ are in general not defined, and we must pass from the field X to the one-parameter group of diffeomorphisms φ_t generated by X. Then, for any fixed t, the homeomorphisms $\varphi_t \circ h$ and $h \circ \varphi_t$ are well-defined. However, the definition of topological conjugacy which suggests itself (that is, $\{\varphi_t\} \sim \{\psi_t\} \iff \exists h \forall t\; h \circ \psi_t = \varphi_t \circ h$) turns out to be unsuccessful, and must be modified. For otherwise, e.g., the periods of closed trajectories turn out to be invariants, and flows having identically disposed trajectories, and differing only in the velocity of motion along these trajectories, would not be equivalent.

DEFINITION. Two smooth flows $\{\varphi_t\}$ and $\{\psi_t\}$ on the manifold M are said to be *topologically equivalent* if there exists a homeomorphism $h: M \longrightarrow M$ which transforms the trajectories of $\{\varphi_t\}$ into the trajectories of $\{\psi_t\}$ with preservation of orientation.

DEFINITION. A flow $\varphi_t \in \mathcal{X}^r$ is said to be *structurally stable* if there exists a neighborhood U of φ_t in $\mathcal{X}^r(M)$ such that any flow $\psi_t \in U$ is topologically equivalent to φ_t.

DEFINITION. Two smooth flows $\{\varphi_t\}$ and $\{\psi_t\}$ in $\mathcal{X}^r(M)$ will be called *C^l-conjugate* if there exists a diffeomorphism $h \in \mathrm{Diff}^l(M)$ such that, for any $t \in \mathbf{R}$, $\psi_t = h^{-1} \circ \varphi_t \circ h$.

EXERCISE. Show that the dynamical systems of Examples 1 and 2 are not rough. Show that the systems of Examples 3–5 lack stability with respect to C^1-conjugacy.

§4. Classification of dynamical systems

First of all, it should be observed that there is no hope of classifying all dynamical systems without exception. There are many reasons for this. For example, the structure of the set of fixed points must certainly be an invariant of the dynamical system for any reasonable equivalence relation; however, any closed subset of a manifold M is the set of fixed points of some C^∞-smooth dynamical system. But we are inclined to regard the problem of describing all closed subsets of a manifold M as unsolvable.

However, it will be shown below that, for all systems belonging to a certain open set which is everywhere dense in $\mathrm{Diff}_r^r(M)$ or in $X^r(M)$, the set of fixed points is finite. Thus, we are led to the necessity of excluding from consideration a closed nowhere dense set in the space of dynamical systems, consisting of systems for which the set of fixed points is infinite (which we regard as an exceptional, nontypical situation). Some other nontypical situations (also encountered only upon a closed, nowhere dense set) will be described below.

The second remark which should be made is that, for the global study of a dynamical system (on the entire manifold M), C^k-conjugacy (even for $k = 1$) is too fine an equivalence relation. One of the reasons for this is that, if a dynamical system has a fixed point, then the eigenvalues of the linear portion of the dynamical system at this point (the rigorous definition will be given below) are C^1-invariants; such invariants, taken with respect to all fixed and periodic points, do not, in general, constitute a complete set of invariants.*

PROBLEM. Consider the flow φ of Example 3. Show that there are flows, arbitrarily close to φ in the C^∞-topology, which coincide with φ on the sets $|z| < 1$ and $|z| > 2$, and are not C^1-equivalent to φ.

We shall now state a theorem, which is basically due (like the formulation of the notion of roughness) to Andronov and Pontrjagin.

THEOREM (ANDRONOV-PONTRJAGIN AND DE BAGGIS-PEIXOTO). *The set of structurally stable flows is everywhere dense* (*and open*) *in* $X^r(M^2)$ *for any compact two-dimensional manifold* M^2.

Andronov and Pontrjagin studied flows on the two-dimensional sphere and disk, and gave a number of simple geometrical conditions, necessary and sufficient for structural stability. Peixoto [2], [3] proved that the same conditions are necessary and sufficient on any compact manifold and that these conditions characterize an open everywhere dense set. The precise formulation of the Andronov-Pontrjagin conditions will be given below.

As has become clear during the last ten years, the analogue of the Andronov-Pontrjagin theorem, unfortunately, does not hold for manifolds of dimension greater than two. On the one hand, this is manifested in the fact that, in attempting to generalize the geometrical conditions for roughness, we obtain only sufficient conditions, since there exist rough systems with very complicated structures (which will be discussed below). On the other hand, we have the following result:

THEOREM OF SMALE. *For any manifold* M^m *with* $m \geqslant 2$, *there exists an open set in* $\mathrm{Diff}_r^r(M)$ $(1 \leqslant r \leqslant \infty)$ *consisting of nonrough diffeomorphisms, and for* $m \geqslant 3$ *there exists an open set of nonrough flows in* $X^r(M)$.

*Translator's note. The purport of this phrase is obscure. Perhaps the intended meaning is "satisfactory" or "meaningful" rather than "complete".

Smale [4] constructed an example of an open set of nonrough diffeomorphisms in $\mathrm{Diff}(T^3)$, where T^3 denotes the three-dimensional torus. Application of the suspension provided an analogous example for flows on a certain four-dimensional manifold.

The construction of the same examples on an arbitrary manifold (the greatest differences arise for diffeomorphisms on two-dimensional (see [5]) and flows on three-dimensional manifolds) was subsequently carried out by a number of authors.

Concerning the formulation of the theorem of Smale, it is worth making one remark. One might attempt to save the situation by introducing a finite number of invariants. In the problem of classifying matrices up to similarity, for example, stability is also lacking: in the neighborhood of any matrix one can find a matrix which is not similar to it. However, on an open, everywhere dense set of matrices there exist a finite number of invariants (the eigenvalues), which completely resolve the classification problem on this set. But a slight modification of the construction of Smale allows one to construct an open set of diffeomorphisms on which the number of invariants of topological conjugacy is infinite.

§5. Fixed points and periodic trajectories

1. Let $f\colon M^n \longrightarrow M^n$, $fx_0 = x_0$. In local coordinates with center at x_0, one may write $fx = Ax + \cdots$. Writing an analogous expansion in another system of local coordinates, we obtain, in place of A, a matrix similar to A. The appropriately defined linear operator $f'|_{x_0}$ on the tangent space $TM^n|_{x_0}$, the matrix of which, in the basis $x_1, \ldots, x_n$ (or, more precisely, in the basis $\partial/\partial x_1|_{x_0}, \ldots, \partial/\partial x_n|_{x_0}$) is A, is called the "linear part" of the diffeomorphism f at the fixed point x_0. The eigenvalues of $f'|_{x_0}$ are called the *eigenvalues of the diffeomorphism f at the fixed point x_0*. If the point x_0 is periodic, with period k, then the preceding construction can be applied to f^k. The eigenvalues of f^k at x_0 are called the *eigenvalues of the diffeomorphism f at the periodic point x_0*.

If the vector field X vanishes at the point x_0, then the corresponding system of differential equations, in the local coordinates $x_1, \ldots, x_n$, has the form $\dot{x} = Ax + \cdots$. Again, the matrix A constitutes the matrix of a properly defined linear operator X' on $TM^m|_{x_0}$, and the eigenvalues of A are known as the *eigenvalues of the vector field at the stationary point x_0*.

Finally, if the vector field X has a periodic trajectory γ, then we apply the preceding definition to the succession function, and we shall refer to the eigenvalues of the succession function as the *eigenvalues of the periodic trajectory γ*.

EXERCISE. Let φ_t be a flow corresponding to the vector field X, let T be the period of γ, let $\lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of γ, and let $x_0 \in \gamma$. Then 1, $\lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of φ_T at the fixed point x_0.

DEFINITION. A nonsingular linear operator A (in a finite-dimensional space) is said to be *hyperbolic* if its spectrum does not intersect the circle $|\lambda| = 1$.

Correspondingly, a fixed or periodic point of a diffeomorphism, or a periodic trajectory of a flow, is said to be hyperbolic if none of its eigenvalues has modulus 1.

A fixed point x_0 of a flow φ_t generated by a vector field X is said to be hyperbolic if x_0 is a hyperbolic fixed point of the diffeomorphism φ_1. This condition is satisfied if and only if the real parts of the eigenvalues of X at x_0 are different from zero.

2. We consider the important special case where all the eigenvalues of the diffeomorphism φ at the fixed point x_0 are of modulus less than 1. In this case, the local structure of φ in the neighborhood of x_0, from the topological point of view, is very simple:

THEOREM 1. *If the abovementioned diffeomorphism φ preserves orientation, then, locally, φ is topologically conjugate to the diffeomorphism $\psi: x \longrightarrow \frac{1}{2}x$, $x \in \mathbf{R}^n$. If φ reverses orientation, then, locally, φ is topologically equivalent to the diffeomorphism $\psi \circ \sigma$, where σ denotes the reflection with respect to the hyperplane $x_1 = 0$.*

We note that the statement of this theorem is only concerned with topological (and definitely not with differentiable) conjugacy.

An equivalent formulation of Theorem 1 is as follows:

THEOREM 1′. *Suppose that the eigenvalues of the diffeomorphisms φ_i: $\mathbf{R}_i^n \longrightarrow \mathbf{R}_i^n$ $(i = 1, 2)$ at the fixed point 0 have moduli less than 1. If φ_1 and φ_2 both preserve (or both reverse) the orientation of $\mathbf{R}^n$, then φ_1 and φ_2 are locally topologically conjugate; that is, there exists a local homeomorphism h such that $\varphi_2 \circ h = h \circ \varphi_1$.*

Still another equivalent formulation is useful:

THEOREM 1″. 1) *If the eigenvalues of the diffeomorphism φ: $\mathbf{R}^n \longrightarrow \mathbf{R}^n$ at the fixed point 0 are less than one in modulus, then φ is locally topologically conjugate to its own linear part.*

2) *Any two linear diffeomorphisms of $\mathbf{R}^n$ with eigenvalues less than one in modulus are topologically conjugate if they lie in the same connected component of the group $GL(n, \mathbf{R})$.*

An analogous theorem is valid for flows:

THEOREM 2. *Suppose that the linear part of a vector field X at the stationary point $0 \in \mathbf{R}^n$ has eigenvalues with negative real parts. Then there exists a homeomorphism h, defined in some neighborhood of the origin, such that the equation $(h \circ e^{-t})x = \varphi_t \circ h$ holds whenever both sides are defined.**

One may likewise formulate Theorems 2′ and 2″, analogous to Theorems 1′ and 1″.

PROOF OF THEOREM 2. By a theorem of Ljapunov, there exists a neighborhood D of 0 such that $\bigcap_{t \geqslant 0} \varphi_t D = 0$. Moreover, one may assume that D is the interior of some ellipsoid, and consequently that there exists a diffeomorphism f of the standard sphere $x_1^2 + \cdots + x_n^2 = 1$ onto $\partial \overline{D}$ (as f one may take, for example, the projection

*Translator's note. φ_t, of course, denotes the flow generated by X.

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