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PREFACE

Apart from being a vehicle for communicating my joy in the subject, this book is intended to serve as a text for an introductory course on special relativity, which is rather more conceptually and mathematically than experimentally oriented. In this context it should be suitable from the upper undergraduate level onwards. But the book might well be used autodidactively by a somewhat more advanced reader. It assumes no prior knowledge of relativity. Thus it elaborates the underlying logic, dwells on the subtleties and apparent paradoxes, and also contains a large collection of problems which should just about cover all the basic modes of thinking and calculating in special relativity. Much emphasis has been laid on developing the student's intuition for space-time geometry and four-tensor calculus. But the approach is not so dogmatically four-dimensional that three-dimensional methods are rejected out of hand when they yield a result more directly. Such methods, too, belong to the basic arsenal even of experts.

In fact, the viewpoint in the first three chapters is purely three-dimensional. Here the reader will find a simple introduction to such topics as the relativity of simultaneity, length contraction, time dilation, the twin paradox, and the appearance of moving objects. But beginning with Chapter 4 (on spacetime) the strongest possible use is made of four-dimensional techniques. Pure tensor theory as such is relegated to an appendix, in the belief that it should really be part of a physicist's general education. Still, this appendix will serve as Chapter '$3\frac{1}{2}$' for readers unfamiliar with that theory. In Chapters 5 and 6—on mechanics and electromagnetism—a purely synthetic four-tensor approach is adopted. Not only is this simpler and more transparent than the historical approach, and a good example of four-dimensional reasoning, but it also brings the student face to face with the 'man-made' aspect of physical laws. In the last chapter (on the mechanics of continua), the synthetic approach is somewhat softened by a heuristic three-dimensional lead-in.

In the discussion of electromagnetism I have reluctantly adopted the SI units now so widely used in spite of their awkwardness for the theoretician. But I have indicated how the equations can easily be translated into their Gaussian (c.g.s.) forms in terms of which most relativists think. A commitment to follow a consistent notation (capital letters for four-dimensional and lowercase for three-dimensional tensors) resulted in some other awkwardnesses, such as **e** and **b** for the electric and magnetic field vectors and **w** for the vector potential (since **a** was already used for the acceleration). I can only hope that the reader will give these symbols a try and not automatically transcribe them.

I should perhaps say a word on the genesis of this book. It has a predecessor after which it is loosely structured, namely my *Special Relativity* (Oliver & Boyd, 1960), which went out of print in 1975. That little book seems to have won some faithful friends and there have been frequent requests for a new edition. But when I finally attempted such an edition I realized how much my ideas—and perhaps the subject itself—had changed and how impossible it was simply to revise the old text. So I found myself much more pleasantly engaged in writing a new book, this book, though a few of the old arguments and problems have been taken over and, I hope, some of the old spirit as well. There are also ties to my *Essential Relativity* (Second Edition, Springer-Verlag, 1977). In a number of contexts I became uneasily aware that I could neither improve upon, nor omit, nor usefully paraphrase what I had already written there. So eventually (with the publisher's kind permission) I decided simply to borrow the relevant passages *verbatim*; these may account for a total of about ten pages of the present book. My conscience was somewhat eased by the fact that, in its time, *Essential Relativity* had similarly borrowed from the older *Special Relativity*.

I clearly owe much to many authors, some by now forgotten. But I would like to acknowledge the special influence on this book of W. G. Dixon, A. Papapetrou, R. Penrose, I. Robinson, D. W. Sciama, R. Sexl, J. L. Synge, H. Weyl, and N. Woodhouse. I also owe a considerable debt to my students. As just one example I like to recall the innocent class question "but what if . . . " which, many years ago, precipitated the 'length contraction paradox'—herein included.

Dallas, November 1981 *Wolfgang Rindler*

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I
THE FOUNDATIONS OF SPECIAL RELATIVITY

1. Introduction

One of the greatest triumphs of Maxwell's electromagnetic theory (*c.* 1864) was the explanation of light as an electromagnetic wave phenomenon. But waves in what? In conformity with the mechanistic view of nature then prevailing, it seemed imperative to postulate the existence of a medium—the *ether*—which would serve as a carrier for these waves (and for electromagnetic 'stress' in general). This led to the most urgent physical problem of the time: the detection of the earth's motion through the ether.

Of the many experiments devised for this purpose, we shall mention just three. Michelson and Morley (1887, see Sec. 2), looked for a directional variation in the velocity of light on earth. Fizeau (1860), Mascart (1872), and later Lord Rayleigh (1902), looked for an expected effect of the earth's motion on the refractive index of certain dielectrics. And Trouton and Noble (1903) tried to detect an expected tendency of a charged plate condenser to face the 'ether drift'. All failed. The facile explanation that the earth might drag the ether along with it only led to other difficulties with the observed aberration of starlight, and could not resolve the problem.

In order to explain nature's apparent conspiracy to hide the ether drift, Lorentz between 1892 and 1909 developed a theory of the ether that was eventually based on two *ad hoc* hypotheses: the longitudinal contraction of rigid bodies[1] and the slowing down of clocks ('time-dilation')[2] when moving through the ether at a speed v, both by a factor $(1 - v^2/c^2)^{1/2}$, where c is the speed of light. This would so affect every apparatus designed to measure the ether drift as to neutralize all expected effects.

In 1905, in the middle of this development, Einstein proposed the *principle of relativity* which is now justly associated with his name. Actually Poincaré had discussed essentially the same principle during the previous year, but it was Einstein who first recognized its full

[1] Notes throughout the book are placed at the end of the relevant section.

significance and put it to brilliant use. In it, he elevated the complete equivalence of all inertial reference frames to the status of an axiom or principle, for which no proof or explanation is to be sought. On the contrary, *it* explains the failure of all the ether-drift expriments, much as the principle of energy conservation explains *a priori* (i.e. without the need for a detailed examination of the mechanism) the failure of all attempts to construct perpetual motion machines.

At first sight Einstein's relativity principle seems to be no more than a whole-hearted acceptance of the null results of all the ether-drift experiments. But by ceasing to look for special explanations of those results, and using them rather as the empirical evidence for a new principle of nature, Einstein had turned the tables: predictions could be made. The situation can be compared to that obtaining in astronomy at the time when Ptolemy's intricate geocentric system (corresponding to Lorentz's 'aetherocentric' theory) gave way to the ideas of Copernicus, Galileo, and Newton. In both cases the liberation from a venerable but inconvenient reference frame ushered in a revolutionary clarification of physical thought, and consequently led to the discovery of a host of new and unexpected results.

Soon a whole theory based on Einstein's principle (and on a 'second axiom' asserting the invariance of the speed of light) was in existence, and this theory is called *special relativity*. Its programme was to modify all the laws of physics, where necessary, so as to make them equally valid in all inertial frames. For Einstein's principle is really a *metaprinciple*: it puts constraints on *all* the laws of physics. The modifications suggested by the theory (especially in mechanics), though highly significant in many modern applications, have negligible effect in most classical problems, which is of course why they were not discovered earlier. However, they were not exactly needed empirically in 1905 either. This is a beautiful example of the power of pure thought to leap ahead of the empirical frontier—a feature of all good physical theories, though rarely on such a heroic scale.

Today, over seventy years later, the enormous success of special relativity theory has made it impossible to doubt the wide validity of its basic premises. It has led, among other things, to a new theory of space and time, and in particular to the relativity of simultaneity and the existence of a maximum speed for all particles and signals, to a new mechanics in which mass increases with speed, to the formula $E = mc^2$, and to de Broglie's association of waves with particles. One of the ironies of these developments is that Newton's theory, which had always been known to satisfy a relativity principle in the classical theory of space and time, now turned out to be in need of modification, whereas Maxwell's theory, with its apparent conceptual dependence on a preferred ether frame, came through with its formalism intact—in itself a powerful recommendation for special relativity.

Apart from leading to new laws, special relativity leads to a useful technique of problem-solving, namely the possibility of switching reference frames. This often simplifies a problem. For although the totality of laws is the same, the configuration of the problem may be simpler, its symmetry enhanced, its unknowns fewer, and the relevant subset of physical laws more convenient, in a judiciously chosen inertial frame.

Our main concern in this chapter will be to set Einstein's principle in its proper perspective and to derive from it the so-called Lorentz transformation equations, which are the mathematical core of the special theory of relativity. With their help we can subject the various branches of classical physics to the test of Einstein's principle, and with their help, too, find the necessary modifications where the principle is not satisfied.

[1] Proposed independently by Fitzgerald as early as 1889.
[2] Based directly on a feature of Einstein's special relativity of 1905.

2. Schematic account of the Michelson–Morley experiment

Certainly the most famous of all the experiments designed to measure the ether drift was that due to Michelson and Morley, first performed

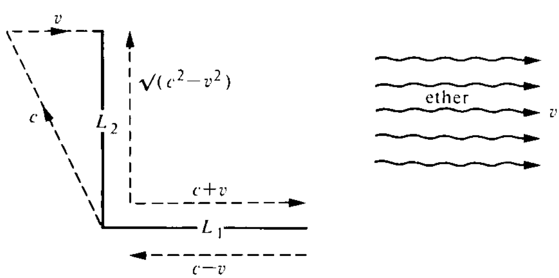

FIG. 1

in 1887 and repeated many times thereafter. Its essential principle was to split a beam of light and then to send the two half-beams along orthogonal arms of equal length, at whose ends mirrors reflected the beams back to the starting point where they were made to interfere. Then the entire apparatus was rotated in the plane of the arms. If this causes a differential change in the to-and-fro light travel times along the two arms, the interference pattern should change. Suppose originally one of the arms, marked L_1 in Fig. 1, lies in the direction of an ether drift of velocity v. Figure 1 should make it clear that the respective to-and-fro light travel times along the two arms would then be expected to be

$$T_1 = \frac{L_1}{c+v} + \frac{L_1}{c-v} = \frac{2L_1}{c(1-v^2/c^2)},$$

$$T_2 = \frac{2L_2}{(c^2-v^2)^{1/2}} = \frac{2L_2}{c(1-v^2/c^2)^{1/2}},$$

where L_1 and L_2 are the purportedly equal lengths of the two arms. Since $T_1 \neq T_2$, a rotation of the experiment through 90° should produce a shift in the interference fringes. None was ever observed, which seems to imply $v = 0$. Yet at some point in its orbit the earth must move through the ether with a speed of at least 18 miles per second (its orbital velocity) and this should have been easily detected by the apparatus. Of course, in Einstein's theory, this null result is expected *a priori*.

In the Lorentz theory the null result of the Michelson–Morley experiment was explained by the contraction of the arm that moves longitudinally through the ether, so that the *actual* lengths of the arms are related by $L_1 = L_2(1-v^2/c^2)^{1/2}$, which yields the observed equality $T_1 = T_2$. (It can be shown that the contraction hypothesis ensures $T_1 = T_2$ for *all* positions of the arms.) That there is also need of a second hypothesis—time dilation—in the Lorentz theory can be appreciated by considering a simple thought experiment. Suppose we could measure the original to-and-fro time T_2 directly with a clock, and suppose we could then move the arm L_2 along with the ether so that v becomes zero. Then the to-and-fro time should be $T_3 = 2L_2/c = T_2(1-v^2/c^2)^{1/2}$. But if nature's conspiracy to hide the ether is complete, we would instead measure $T_3 = T_2$. This could be accounted for by the hypothesis that a clock moving with speed v through the ether goes slow by a factor $(1-v^2/c^2)^{1/2}$. For then the *measured* time in the original position is less by that factor than the actual time T_2, and is thus equal to T_3.

As has been stressed by Sexl, modern equivalents of the Michelson—Morley experiment are being performed daily. For example, the synchrony of the seven atomic clocks around the globe that serve to define 'International Atomic Time' is continually tested by an exchange of radio signals. Any interference with these signals by an ether drift of the expected magnitude could be detected by the clocks. Needless to say, none has been detected: day or night, summer or winter, the signals always arrive with the same time delay. Again, the incredible accuracy of some modern radio navigational systems depends crucially on the directional invariance of the speed of light.

3. Inertial frames in special relativity

A frame of reference is a conventional standard of rest relative to which measurements can be made and experiments described. For example, if we choose a frame rigidly attached to the earth, the various points of the earth remain at rest in this frame while the 'fixed' stars all trace out vast circles in the course of each day; if, on the other hand, we choose a frame attached to the fixed stars then these remain at rest while points on the earth, other than those on its axis, trace out approximate circles in the course of each day, and the earth itself traces out an ellipse in the course of each year; and so on. Among all possible frames there is one class which plays a special role in classical mechanics, namely the class of *inertial frames*. These frames play an even more fundamental role in the special theory of relativity and we shall therefore define and discuss them carefully. *An inertial frame is one in which spatial relations, as determined by rigid scales at rest in the frame, are Euclidean and in which there exists a universal time in terms of which free particles remain at rest or continue to move with constant speed along straight lines* (i.e. in terms of which free particles obey Newton's first law).

Free particles placed without velocity at fixed points in an inertial frame will remain at those points, by definition. We can therefore picture an inertial frame as an aggregate of actual or virtual free test-particles mutually at rest, as determined by rigid scales. The distances between these 'defining' particles satisfy the Euclidean axioms—an important stipulation in view of later developments. Straight lines in such a frame can be defined as geodesics (lines of minimum length) and free particles not belonging to the defining aggregate move along such lines. We can further picture the defining particles as carrying clocks that indicate the universal time throughout the frame.

Now let us see the relevance of this to special relativity. We shall adopt the modern view (largely due to Einstein) that a physical theory is an abstract mathematical model (much like Euclidean geometry) whose applications to the real world consist of correspondences between a subset of it and a subset of the real world. In line with this view, *special relativity is the theory of an ideal physics referred to an ideal set of infinitely extended gravity-free inertial frames*, such as we described above. Why 'gravity-free'? Classically, gravity was regarded as an overlay which did not affect the rest of physics. So it was logical for Newton to treat the frame of the fixed stars as inertial, in the sense that *but for gravity* free particles would move uniformly relative to it. But Einstein, in his *general relativity* (the details of which are beyond the scope of this book) taught us that gravity[1] is *curvature* (of space and time) and so affects *all* the rest of physics, which has no choice but to play on a stage of space and time. In E. T. Whittaker's phrase, gravity ceased to be one of the players and became part of the stage. Thus extended inertial frames cannot be realized in nature, because gravity destroys Euclidicity. But this does not affect in any way the logic of special relativity as an abstract theory (just as it does not invalidate Euclidean geometry). It merely puts limitations on its correspondence with the real world. These are spelled out by Einstein's *equivalence principle* of 1907 (on which he eventually based his general theory of relativity): *the sets of inertial frames in the real world that correspond to (portions of) the ideal set of inertial frames discussed in special relativity consist of freely falling local frames.* At any given place and time in the real world there is one such set, each member of which can be realized by an aggregate of test-particles momentarily at rest relative to each other and falling freely under gravity. Certainly in Newton's theory such a set is locally equivalent to a set of inertial frames from which gravity has been eliminated, for in a gravitational field all particles suffer the same acceleration. Most of us have at least vicariously experienced such frames: we need only recall the televized pictures of space capsules in which astronauts are weightless and, if unrestrained, move according to Newton's first law. Such capsules, then, are the primary reference frames in which the laws of special relativistic physics would be expected to apply most accurately.

In this book *all* reference frames used (unless otherwise stated) will be ideal infinitely extended gravity-free inertial frames, and all observers will be considered to use such frames (‘*inertial observers*’). Sometimes the term ‘inertial’ may be omitted, but it will always be understood.

For our ideal frames we shall assume certain axioms. The first is that any frame in uniform (translatory) motion relative to a given inertial frame is itself inertial. This is certainly the case in Newton's theory. Conversely, a frame *not* moving uniformly relative to an inertial frame cannot be inertial, for Newton's first law would fail in it. So the set of ideal inertial frames consists of infinitely many members all moving uniformly relative to each other.

Our next axiom is that all inertial frames are spatially homogeneous and isotropic, not only in their assumed Euclidean geometry but for the performance of all physical experiments. By this we mean that the outcome of an experiment is the same whenever its initial conditions differ only by a translation (homogeneity) and rotation (isotropy) in some inertial frame. *This is a very strong assumption*, which we are encouraged to make only in view of Einstein's relativity principle. It already eliminates the possible existence of an ether drift in any inertial frame.

It may be noted that, whereas our definition of inertial frame determines the *rate* of time (as that in which free particles move uniformly), the isotropy axiom determines the clock *settings*. For suppose isotropy holds in an inertial frame referred to Cartesian coordinates x, y, z and we define a new time $t' = t + kx$ (k = constant > 0). Then Newton's first law will still hold. But any given rifle will now shoot bullets faster in the negative x-direction than in the positive x-direction (i.e. with greater *coordinate* velocity).

As a final axiom we assume that inertial frames are temporally homogeneous, i.e. that identical experiments (relative to an inertial frame) performed at different times yield identical results. In particular, this implies that all methods of time keeping based on repetitive processes are equivalent, and it denies such possibilities (envisaged by E. A. Milne) as that inertial time—relative to which free particles move uniformly—falls out of step over the centuries with atomic time, e.g. that indicated by a caesium clock.

[1] Actually, the rate of *change* of gravity.

4. Einstein's two axioms for special relativity

As we have seen, Einstein's reaction to the failure of all attempts to detect the ether frame was radical. He advanced the following *principle of relativity: the laws of physics are identical in all inertial frames*, or, equivalently, *the outcome of any physical experiment is the same when performed with identical initial conditions relative to any inertial frame.*

Note that this is a generalization to the whole of physics of a relativity principle long known to be satisfied by Newtonian mechanics. Such a generalization is strongly supported by the essential unity of physics. For it would be very disturbing if, for example, the electromagnetic laws governing the behaviour of matter on the atomic scale possessed different invariance properties from the mechanical laws that govern its macroscopic behaviour. Einstein also cited instances of manifest relativity from electromagnetism. For example, the current induced in a conductor by a magnet is the same whether the conductor is at rest and the magnet moving, or vice versa.

Other *a priori* arguments can be adduced to justify the adoption of Einstein's relativity principle. But in our present development it is in fact already implicit in the homogeneity and isotropy axioms for inertial frames.[1] The demonstration of this depends on the simple lemma[2] that 'between' any two inertial frames S and S′ there exists an inertial frame S″ relative to which S and S′ have equal and opposite velocities. For proof, consider a one-parameter family of inertial frames moving collinearly with S and S′, the parameter being the velocity relative to S. It is then obvious from continuity that there must be one member of this family with the required property (see Fig. 2.) Now imagine two intrinsically identical experiments E and E′ being performed in S and S′, respectively. We can transform E′, by a spatial translation and rotation and a temporal translation, in S′, into a position where it differs from E only by a 180° rotation in S″. Thus,

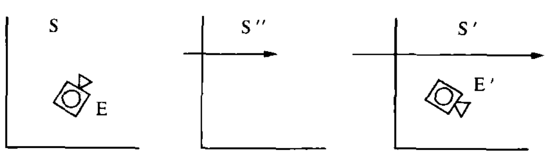

FIG. 2

by the assumed homogeneity and isotropy of S′ and S″, the outcome of E and E′ must be the same, which establishes Einstein's principle (in the form of our second formulation).

The acceptance of this principle—Einstein's *first axiom*—seems harmless enough until we come to his *second axiom*: *There exists an inertial frame in which light signals in vacuum always travel rectilinearly at constant speed c, in all directions, independently of the motion of the source.* (Thc value of c is $2.997\,9245\ldots \times 10^8\ \mathrm{m\,s^{-1}}$, but $c = 3 \times 10^8\ \mathrm{m\,s^{-1}}$ is good enough for many applications.)

By itself, this axiom is also perfectly reasonable. Even Einstein's contemporaries, familiar with Maxwell's electromagnetic theory of light, did not expect the speed of light to depend on the speed of the source, and they had empirical evidence for this axiom in their pseudo-inertial terrestrial frame of reference. In particular, the direction-independence had been very accurately tested by the Michelson–Morley experiment. But when combined with the first axiom, the second leads to the following apparently absurd state of affairs, which we shall call the *law of light propagation*:

Light signals in vacuum are propagated rectilinearly, with the same speed c, at all times, in all directions, in all *inertial frames.*

Thus if a light signal recedes from me and I transfer myself to ever faster-moving inertial frames in pursuit of it, I shall not alter the velocity of that light signal relative to me by one iota. This is totally irreconcilable with our classical concepts of space and time. But it was a mark of Einstein's genius to realize that those concepts were dispensable, and could be replaced by others. The final form of those others is due to the mathematician Minkowski, and consists in a certain blend of space and time into a four-dimensional 'spacetime' (1908), as we shall see in due course.

A first logical consequence of Einstein's two axioms was the elimination of the ether concept from physics. Each inertial frame now has the properties with which the ether was credited, and so it makes no sense to single out one inertial frame arbitrarily and call it the ether frame. It is true that Lorentz's theory—gentler to the classical prejudices than Einstein's, and observationally equivalent to it—kept the ether idea alive a few years longer. But soon Einstein's far more elegant and powerful ideas prevailed, and Lorentz's theory, together with the ether concept, fell into oblivion.

Finally, in spite of its historical and heuristic importance, we must