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Functions of One Complex Variable

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Springer-Verlag Berlin Heidelberg New York

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SPRINGER INTERNATIONAL STUDENT EDITION

Authorised reprint of the original edition published by
Springer-Verlag New York Inc.

ISBN 3-540-78011-4

ISBN 3-540-90062-4 Springer-Verlag Berlin Heidelberg New York

Printed in India by Rekha Printers Private Limited, New Delhi 110020

PREFACE

This book is intended as a textbook for a first course in the theory of functions of one complex variable for students who are mathematically mature enough to understand and execute $\epsilon - \delta$ arguments. The actual prerequisites for reading this book are quite minimal; not much more than a stiff course in basic calculus and a few facts about partial derivatives. The topics from advanced calculus that are used (e.g., Leibniz's rule for differentiating under the integral sign) are proved in detail.

Complex Variables is a subject which has something for all mathematicians. In addition to having applications to other parts of analysis, it can rightly claim to be an ancestor of many areas of mathematics (e.g., homotopy theory, manifolds). This view of Complex Analysis as "An Introduction to Mathematics" has influenced the writing and selection of subject matter for this book. The other guiding principle followed is that all definitions, theorems, etc. should be clearly and precisely stated. Proofs are given with the student in mind. Most are presented in detail and when this is not the case the reader is told precisely what is missing and asked to fill in the gap as an exercise. The exercises are varied in their degree of difficulty. Some are meant to fix the ideas of the section in the reader's mind and some extend the theory or give applications to other parts of mathematics. (Occasionally, terminology is used in an exercise which is not defined—e.g., group, integral domain.)

Chapters I through V and Sections VI.1 and VI.2 are basic. It is possible to cover this material in a single semester only if a number of proofs are omitted. Except for the material at the beginning of Section VI.3 on convex functions, the rest of the book is independent of VI.3 and VI.4.

Chapter VII initiates the student in the consideration of functions as points in a metric space. The results of the first three sections of this chapter are used repeatedly in the remainder of the book. Sections four and five need no defense; moreover, the Weierstrass Factorization Theorem is necessary for Chapter XI. Section six is an application of the factorization theorem. The last two sections of Chapter VII are not needed in the rest of the book although they are a part of classical mathematics which no one should completely disregard.

The remaining chapters are independent topics and may be covered in any order desired.

Runge's Theorem is the inspiration for much of the theory of Function Algebras. The proof presented in section VIII.1 is, however, the classical one involving "pole pushing". Section two applies Runge's Theorem to obtain a more general form of Cauchy's Theorem. The main results of sections three and four should be read by everyone, even if the proofs are not.

Chapter IX studies analytic continuation and introduces the reader to analytic manifolds and covering spaces., Sections one through three can be considered as a unit and will give the reader a knowledge of analytic continuation without necessitating his going through all of Chapter IX.

Chapter X studies harmonic functions including a solution of the Dirichlet Problem and the introduction of Green's Function. If this can be called applied mathematics it is part of applied mathematics that everyone should know.

Although they are independent, the last two chapters could have been combined into one entitled "Entire Functions". However, it is felt that Hadamard's Factorization Theorem and the Great Theorem of Picard are sufficiently different that each merits its own chapter. Also, neither result depends upon the other.

With regard to Picard's Theorem it should be mentioned that another proof is available. The proof presented here uses only elementary arguments while the proof found in most other books uses the modular function.

There are other topics that could have been covered. Some consideration was given to including chapters on some or all of the following: conformal mapping, functions on the disk, elliptic functions, applications of Hilbert space methods to complex functions. But the line had to be drawn somewhere and these topics were the victims. For those readers who would like to explore this material or to further investigate the topics covered in this book, the bibliography contains a number of appropriate entries.

Most of the notation used is standard. The word "iff" is used in place of the phrase "if and only if", and the symbol ■ is used to indicate the end of a proof. When a function (other than a path) is being discussed, Latin letters are used for the domain and Greek letters are used for the range.

This book evolved from classes taught at Indiana University. I would like to thank the Department of Mathematics for making its resources available to me during its preparation. I would especially like to thank the students in my classes; it was actually their reaction to my course in Complex Variables that made me decide to take the plunge and write a book. Particular thanks should go to Marsha Meredith for pointing out several mistakes in an early draft, to Stephen Berman for gathering the material for several exercises on algebra, and to Larry Curnutt for assisting me with the final corrections of the manuscript. I must also thank Ceil Sheehan for typing the final draft of the manuscript under unusual circumstances.

Finally, I must thank my wife to whom this book is dedicated. Her encouragement was the most valuable assistance I received.

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Chapter I

The Complex Number System

§1. *The real numbers*

We denote by $\mathbb{R}$ the set of all real numbers. It is assumed that each reader is acquainted with the real number system as well as all its properties. In particular we assume a knowledge of the ordering of $\mathbb{R}$, the definitions and properties of the supremum and infimum (sup and inf), and the completeness of $\mathbb{R}$ (every set in $\mathbb{R}$ which is bounded above has a supremum). It is also assumed that every reader is familiar with sequential convergence in $\mathbb{R}$ and with infinite series. Finally, no one should undertake a study of Complex Variables unless he has a thorough grounding in functions of one real variable. Although it has been traditional to study functions of several real variables before studying analytic function theory, this is not an essential prerequisite for this book. There will not be any occasion when the deep results of this area are needed.

§2. *The field of complex numbers*

We define $\mathbb{C}$, the complex numbers, to be the set of all ordered pairs (a, b) where a and b are real numbers and where addition and multiplication are defined by:

$$(a, b)+(c, d) = (a+c, b+d)$$

$$(a, b)\,(c, d) = (ac-bd, bc+ad)$$

It is easily checked that with these definitions $\mathbb{C}$ satisfies all the axioms for a field. That is, $\mathbb{C}$ satisfies the associative, commutative and distributive laws for addition and multiplication; $(0, 0)$ and $(1, 0)$ are identities for addition and multiplication respectively, and there are multiplicative inverses for each non zero element in $\mathbb{C}$.

We will write a for the complex number $(a, 0)$. This mapping $a \to (a, 0)$ defines a field isomorphism of $\mathbb{R}$ into $\mathbb{C}$ so we may consider $\mathbb{R}$ as a subset of $\mathbb{C}$. If we put $i = (0, 1)$ then $(a, b) = a+bi$. From this point on we abandon the ordered pair notation for complex numbers.

Note that $i^2 = -1$, so that the equation $z^2+1 = 0$ has a root in $\mathbb{C}$. In fact, for each z in $\mathbb{C}$, $z^2+1 = (z+i)\,(z-i)$. More generally, if z and w are complex numbers we obtain

$$z^2+w^2 = (z+iw)\,(z-iw)$$

By letting z and w be real numbers a and b we can obtain (with both a and $b \neq 0$)

$$\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\left(\frac{b}{a^2+b^2}\right)$$

so that we have a formula for the reciprocal of a complex number.

When we write $z = a+ib$ ($a, b \in \mathbb{R}$) we call a and b the *real* and *imaginary parts* of z and denote this by $a = \operatorname{Re} z$, $b = \operatorname{Im} z$.

We conclude this section by introducing two operations on $\mathbb{C}$ which are not field operations. If $z = x+iy$ ($x, y \in \mathbb{R}$) then we define $|z| = (x^2+y^2)^{\frac{1}{2}}$ to be the *absolute value* of z and $\bar{z} = x-iy$ is the *conjugate* of z. Note that

2.1 $$|z|^2 = z\bar{z}$$

In particular, if $z \neq 0$ then

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

The following are basic properties of absolute values and conjugates whose verifications are left to the reader.

2.2 $$\operatorname{Re} z = \tfrac{1}{2}(z+\bar{z}) \quad \text{and} \quad \operatorname{Im} z = \frac{1}{2i}(z-\bar{z}).$$

2.3 $$(\overline{z+w}) = \bar{z}+\bar{w} \quad \text{and} \quad \overline{zw} = \bar{z}\bar{w}.$$

2.4 $$|zw| = |z|\,|w|.$$

2.5 $$|z/w| = |z|/|w|.$$

2.6 $$|\bar{z}| = |z|.$$

The reader should try to avoid expanding z and w into their real and imaginary parts when he tries to prove these last three. Rather, use (2.1), (2.2), and (2.3).

Exercises

1. Find the real and imaginary parts of each of the following:

$$\frac{1}{z};\ \frac{z-a}{z+a}\ (a \in \mathbb{R});\ z^3;\ \frac{3+5i}{7i+1};\ \left(\frac{-1+i\sqrt{3}}{2}\right)^3;$$

$$\left(\frac{-1-i\sqrt{3}}{2}\right)^6;\ i^n;\ \left(\frac{1+i}{\sqrt{2}}\right)^n \quad \text{for} \quad 2 \leq n \leq 8.$$

2. Find the absolute value and conjugate of each of the following:

$$-2+i;\ -3;\ (2+i)(4+3i);\ \frac{3-i}{\sqrt{2}+3i};\ \frac{i}{i+3};$$

$$(1+i)^6;\ i^{17}.$$

3. Show that z is a real number if and only if $z = \bar{z}$.
4. If z and w are complex numbers, prove the following equations:

$$|z+w|^2 = |z|^2 + 2\text{Re}\, z\bar{w} + |w|^2.$$

$$|z-w|^2 = |z|^2 - 2\text{Re}\, z\bar{w} + |w|^2.$$

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

5. Use induction to prove that for $z = z_1 + \ldots + z_n$; $w = w_1 w_2 \ldots w_n$: $|w| = |w_1| \ldots |w_n|$; $\bar{z} = \bar{z}_1 + \ldots + \bar{z}_n$; $\bar{w} = \bar{w}_1 \ldots \bar{w}_n$.
6. Let $R(z)$ be a rational function of z. Show that $\overline{R(z)} = R(\bar{z})$ if all the coefficients in $R(z)$ are real.

§3. *The complex plane*

From the definition of complex numbers it is clear that each z in $\mathbb{C}$ can be identified with the unique point $(\text{Re}\, z, \text{Im}\, z)$ in the plane $\mathbb{R}^2$. The addition of complex numbers is exactly the addition law of the vector space $\mathbb{R}^2$. If z and w are in $\mathbb{C}$ then draw the straight lines from z and w to 0 $(=(0, 0))$. These form two sides of a parallelogram with 0, z and w as three vertices. The fourth vertex turns out to be $z+w$.

Note also that $|z-w|$ is exactly the distance between z and w. With this in mind the last equation of Exercise 4 in the preceding section states the *parallelogram law*: The sum of the squares of the lengths of the sides of a parallelogram equals the sum of the squares of the lengths of its diagonals.

A fundamental property of a distance function is that it satisfies the triangle inequality (see the next chapter). In this case this inequality becomes

$$|z_1 - z_2| \le |z_1 - z_3| + |z_3 - z_2|$$

for complex numbers z_1, z_2, z_3. By using $z_1 - z_2 = (z_1 - z_3) + (z_3 - z_2)$, it is easy to see that we need only show

3.1 $$|z+w| \le |z| + |w| \quad (z, w \in \mathbb{C}).$$

To show this first observe that for any z in $\mathbb{C}$,

3.2 $$-|z| \le \text{Re}\, z \le |z|$$
$$-|z| \le \text{Im}\, z \le |z|$$

Hence, $\text{Re}\,(z\bar{w}) \le |z\bar{w}| = |z|\,|w|$. Thus,

$$\begin{aligned} |z+w|^2 &= |z|^2 + 2\text{Re}\,(z\bar{w}) + |w|^2 \\ &\le |z|^2 + 2|z|\,|w| + |w|^2 \\ &= (|z| + |w|)^2, \end{aligned}$$

from which (3.1) follows. (This is called the *triangle inequality* because, if we represent z and w in the plane, (3.1) says that the length of one side of the triangle $[0, z, z+w]$ is less than the sum of the lengths of the other two sides. Or, the shortest distance between two points is a straight line.) On encountering an inequality one should always ask for necessary and sufficient conditions that equality obtains. From looking at a triangle and considering the geometrical significance of (3.1) we are led to consider the condition $z = tw$ for some $t \in \mathbb{R}$, $t \geq 0$. (or $w = tz$ if $w = 0$). It is clear that equality will occur when the two points are colinear with the origin. In fact, if we look at the proof of (3.1) we see that a necessary and sufficient condition for $|z+w| = |z|+|w|$ is that $|z\bar{w}| = \text{Re}(z\bar{w})$. Equivalently, this is $z\bar{w} \geq 0$ (i.e., $z\bar{w}$ is a real number and is non negative). Multiplying this by w/w we get $|w|^2(z/w) \geq 0$ if $w \neq 0$. If

$$t = z/w = \left(\frac{1}{|w|^2}\right)|w|^2(z/w)$$

then $t \geq 0$ and $z = tw$.

By induction we also get

3.3 $$|z_1+z_2+\ldots+z_n| \leq |z_1|+|z_2|+\ldots+|z_n|$$

Also useful is the inequality

3.4 $$\big||z|-|w|\big| \leq |z-w|$$

Now that we have given a geometric interpretation of the absolute value let us see what taking a complex conjugate does to a point in the plane. This is also easy; in fact, $\bar{z}$ is the point obtained by reflecting z across the x-axis (i.e., the real axis).

Exercises

1. Prove (3.4) and give necessary and sufficient conditions for equality.
2. Show that equality occurs in (3.2) if and only if $z_k/z_l \geq 0$ for any integers k and l, $1 \leq k, l \leq n$, for which $z_l \neq 0$.
3. Let $a \in \mathbb{R}$ and $c > 0$ be fixed. Describe the set of points z satisfying

$$|z-a|-|z+a| = 2c$$

for every possible choice of a and c. Now let a be any complex number and, using a rotation of the plane, describe the locus of points satisfying the above equation.

§4. *Polar representation and roots of complex numbers*

Consider the point $z = x+iy$ in the complex plane $\mathbb{C}$. This point has polar coordinates (r, θ): $x = r\cos\theta$, $y = r\sin\theta$. Clearly $r = |z|$ and θ is the angle between the positive real axis and the line segment from 0 to z. Notice that θ plus any multiple of 2π can be substituted for θ in the above equations. The angle θ is called the *argument of z* and is denoted by $\theta = \arg z$. Because of the ambiguity of θ, "arg" is not a function. We introduce the notation

4.1 $$\text{cis}\,\theta = \cos\theta + i\sin\theta.$$

Let $z_1 = r_1 \operatorname{cis} \theta_1$, $z_2 = r_2 \operatorname{cis} \theta_2$. Then $z_1 z_2 = r_1 r_2 \operatorname{cis} \theta_1 \operatorname{cis} \theta_2 = r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1)]$. By the formulas for the sine and cosine of the sum of two angles we get

4.2 $$z_1 z_2 = r_1 r_2 \operatorname{cis} (\theta_1 + \theta_2)$$

Alternately, $\arg (z_1 z_2) = \arg z_1 + \arg z_2$. (What function of a real variable takes products into sums?) By induction we get for $z_k = r_k \operatorname{cis} \theta_k$, $1 \leq k \leq n$.

4.3 $$z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n \operatorname{cis} (\theta_1 + \ldots + \theta_n)$$

In particular,

4.4 $$z^n = r^n \operatorname{cis} (n\theta),$$

for every integer $n \geq 0$. Moreover if $z \neq 0$, $z \cdot [r^{-1} \operatorname{cis} (-\theta)] = 1$; so that (4.4) also holds for all integers n, positive, negative, and zero, if $z \neq 0$. As a special case of (4.4) we get *de Moivre's formula*:

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

We are now in a position to consider the following problem: For a given complex number $a \neq 0$ and an integer $n \geq 2$, can you find a number z satisfying $z^n = a$? How many such z can you find? In light of (4.4) the solution is easy. Let $a = |a| \operatorname{cis} \alpha$; by (4.4), $z = |a|^{1/n} \operatorname{cis} (\alpha/n)$ fills the bill. However this is not the only solution because $z' = |a|^{1/n} \operatorname{cis} \frac{1}{n} (\alpha + 2\pi)$ also satisfies $(z')^n = a$. In fact each of the numbers

4.5 $$|a|^{1/n} \operatorname{cis} \frac{1}{n} (\alpha + 2\pi k),\ 0 \leq k \leq n-1,$$

in an nth root of a. By means of (4.4) we arrive at the following: for each non zero number a in $\mathbb{C}$ there are n *distinct* nth roots of a; they are given by formula (4.5).

Example

Calculate the nth roots of unity. Since $1 = \operatorname{cis} 0$, (4.5) gives these roots as

$$1, \operatorname{cis} \frac{2\pi}{n}, \operatorname{cis} \frac{4\pi}{n}, \ldots, \operatorname{cis} \frac{2\pi}{n} (n-1).$$

In particular, the cube roots of unity are

$$1, \frac{1}{\sqrt{2}} (1 + i\sqrt{3}), \frac{1}{\sqrt{2}} (1 - i\sqrt{3}).$$

Exercises

1. Find the sixth roots of unity.

2. Calculate the following:
   (a) the square roots of i
   (b) the cube roots of i
   (c) the square roots of $\sqrt{3}+3i$
3. Show that if a and b are nth and mth roots of unity, respectively, then ab is a kth root of unity for some integer k. What is the smallest possible value of k?
4. Use the binomial equation

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

and compare the real and imaginary parts of each side of de Moivre's formula to obtain the formulas:

$$\cos n\theta = \cos^n \theta - \binom{n}{2} \cos^{n-2} \sin^2 \theta + \binom{n}{4} \cos^{n-4} \theta \sin^4 \theta - \ldots$$

$$\sin n\theta = \binom{n}{1} \cos^{n-1} \theta \sin \theta - \binom{n}{3} \cos^{n-3} \theta \sin^3 \theta + \ldots$$

5. Let $z = \text{cis}\,\frac{2\pi}{n}$ for an integer $n \geq 2$. Show that $1+z+\ldots+z^{n-1} = 0$.
6. Show that $\varphi(t) = \text{cis}\, t$ is a group homomorphism of the additive group $\mathbb{R}$ onto the multiplicative group $T = \{z: |z| = 1\}$.

§5. *Lines and half planes in the complex plane*

Let L denote a straight line in $\mathbb{C}$. From elementary analytic geometry, L is determined by a point in L and a direction vector. Thus if a is any point in L and b is its direction vector then

$$L = \{z = a+tb: -\infty < t < \infty\}.$$

Since $b \neq 0$ this gives, for z in L,

$$\text{Im}\left(\frac{z-a}{b}\right) = 0.$$

In fact if z is such that

$$0 = \text{Im}\left(\frac{z-a}{b}\right)$$

then

$$t = \left(\frac{z-a}{b}\right)$$

implies that $z = a+tb$, $-\infty < t < \infty$. That is

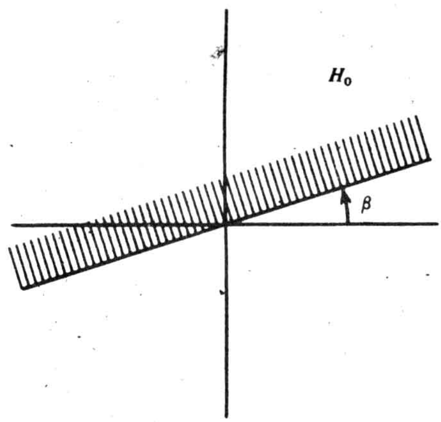
5.1 $$L = \left\{z\colon \operatorname{Im}\left(\frac{z-a}{b}\right) = 0\right\}.$$

What is the locus of each of the sets

$$\left\{z\colon \operatorname{Im}\left(\frac{z-a}{b}\right) > 0\right\},$$

$$\left\{z\colon \operatorname{Im}\left(\frac{z-a}{b}\right) < 0\right\}?$$

As a first step in answering this question, observe that since b is a direction we may assume $|b| = 1$. For the moment, let us consider the case where $a = 0$, and put $H_0 = \{z\colon \operatorname{Im}(z/b) > 0\}$, $b = \operatorname{cis}\beta$. If $z = r\operatorname{cis}\theta$ then $z/b = r\operatorname{cis}(\theta-\beta)$. Thus, z is in H_0 if and only if $\sin(\theta-\beta) > 0$; that is, when $\beta < \theta < \pi+\beta$. Hence H_0 is the half plane lying to the left of the line L if



we are "walking along L in the direction of b." If we put

$$H_a = \left\{z\colon \operatorname{Im}\left(\frac{z-a}{b}\right) > 0\right\}$$

then it is easy to see that $H_a = a+H_0 \equiv \{a+w\colon w \in H_0\}$; that is, H_a is the translation of H_0 by a. Hence, H_a is the half plane lying to the left of L. Similarly,

$$K_a = \left\{z\colon \operatorname{Im}\left(\frac{z-a}{b}\right) < 0\right\}$$

is the half plane on the right of L.

Exercise

1. Let C be the circle $\{z\colon |z-c| = r\}$, $r > 0$; let $a = c+r\operatorname{cis}\alpha$ and put

$$L_\beta = \left\{z\colon \operatorname{Im}\left(\frac{z-a}{b}\right) = 0\right\}$$

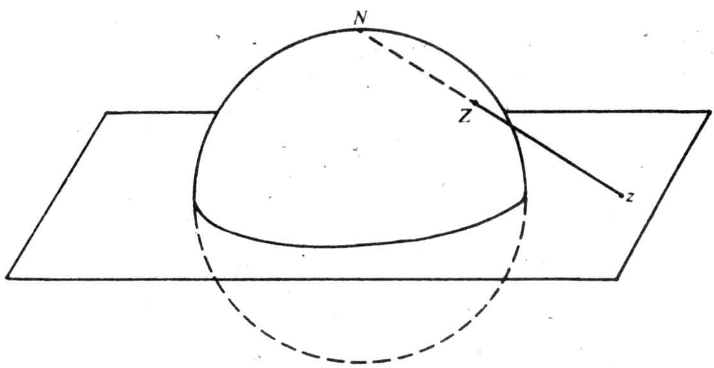
where $b = \operatorname{cis} \beta$. Find necessary and sufficient conditions in terms of β that L_β be tangent to C at a.

§6. *The extended plane and its spherical representation*

Often in complex analysis we will be concerned with functions that become infinite as the variable approaches a given point. To discuss this situation we introduce the *extended plane* which is $\mathbb{C} \cup \{\infty\} \equiv \mathbb{C}_\infty$. We also wish to introduce a distance function on $\mathbb{C}_\infty$ in order to discuss continuity properties of functions assuming the value infinity. To accomplish this and to give a concrete picture of $\mathbb{C}_\infty$ we represent $\mathbb{C}_\infty$ as the unit sphere in $\mathbb{R}^3$,

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon x_1^2+x_2^2+x_3^2 = 1\}.$$

Let $N = (0, 0, 1)$; that is, N is the north pole on S. Also, identify $\mathbb{C}$ with $\{(x_1, x_2, 0)\colon x_1, x_2 \in \mathbb{R}\}$ so that $\mathbb{C}$ cuts S along the equator. Now for each point z in $\mathbb{C}$ consider the straght line in $\mathbb{R}^3$ through z and N. This intersects



the sphere in exactly one point $Z \neq N$. If $|z| > 1$ then Z is in the northern hemisphere and if $|z| < 1$ then Z is in the southern hemisphere; also, for $|z| = 1$, $Z = z$. What happens to Z as $|z| \to \infty$? Clearly Z approaches N; hence, we identify N and the point ∞ in $\mathbb{C}_\infty$. Thus $\mathbb{C}_\infty$ is represented as the sphere S.

Let us explore this representation. Put $z = x+iy$ and let $Z = (x_1, x_2, x_3)$ be the corresponding point on S. We will find equations expressing x_1, x_2, and x_3 in terms of x and y. The line in $\mathbb{R}^3$ through z and N is given by $\{tN+(1-t)z\colon -\infty < t < \infty\}$, or by

6.1 $$\{((1-t)x, (1-t)y, t)\colon -\infty < t < \infty\}.$$

Hence, we can find the coordinates of Z if we can find the value of t at which this line intersects S. If t is this value then

$$1 = (1-t)^2x^2+(1-t)^2y^2+t^2$$
$$= (1-t)^2|z|^2+t^2$$

From which we get

$$1-t^2 = (1-t)^2|z|^2.$$

Since $t \neq 1$ ($z \neq \infty$) we arrive at

$$t = \frac{|z|^2-1}{|z|^2+1}.$$

Thus

6.2 $$x_1 = \frac{2x}{|z|^2+1},\ x_2 = \frac{2y}{|z|^2+1},\ x_3 = \frac{|z|^2-1}{|z|^2+1}$$

But this gives

6.3 $$x_1 = \frac{z+\bar{z}}{|z|^2+1}\quad x_2 = \frac{z-\bar{z}}{(|z|^2+1)}\quad x_3 = \frac{|z|^2-1}{|z|^2+1}.$$

If the point Z is given ($Z \neq N$) and we wish to find z then by setting $t = x_3$ and using (6.1), we arrive at

6.4 $$z = \frac{x_1+ix_2}{1-x_3}$$

Now let us define a distance function between points in the extended plane in the following manner: for z, z' in $\mathbb{C}_\infty$ define the distance from z to z', $d(z, z')$, to be the distance between the corresponding points Z and Z' in $\mathbb{R}^3$. If $Z = (x_1, x_2, x_3)$ and $Z' = (x_1', x_2', x_3')$ then

6.5 $$d(z, z') = [(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2]^{\frac{1}{2}}$$

Using the fact that Z and Z' are on S, (6.5) gives

6.6 $$[d(z, z')]^2 = 2-2(x_1x_1'+x_2x_2'+x_3x_3')$$

By using equation (6.3) we get

6.7 $$d(z, z') = \frac{2|z-z'|}{[(1+|z|^2)(1+|z'|^2)]^{\frac{1}{2}}},\ (z, z' \in \mathbb{C})$$

In a similar manner we get for z in $\mathbb{C}$

6.8 $$d(z, \infty) = \frac{2}{(1+|z|^2)^{\frac{1}{2}}}$$

This correspondence between points of S and $\mathbb{C}_\infty$ is called the *stereographic projection.*