Series in Analysis Vol. 5

Lecture Notes on Applied Analysis

Roderick Wong

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City University of Hong Kong, Hong Kong

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Lecture Notes on Applied Analysis

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To my wife Edwina,

and to my daughters Priscilla and Letitia.

Preface

This is a set of lecture notes that I have written for a course taught at City University of Hong Kong in the last few years. It is a course for beginning graduate students majoring in Analysis or Applied Mathematics. The book consists of six chapters, and each chapter deals with a specific topic. They are quite independent of each other; and can be used in any order that the instructor wishes. The idea is to give students a broad view of the mathematics that is frequently used in applications. Each chapter has only four sections, and I did not go into depth in any one of the topics in these chapters. But, the mathematics presented in the book is not simple, and requires a good knowledge of advanced calculus, ordinary differential equations and functions of a complex variable at the undergraduate level. This book can be used either as a text book or as a reference book for self-reading. Readers, who wish to learn more about any one of these subjects touched upon in this book, can find plenty of references at the end of each chapter.

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Chapter 1

COMPLEX ANALYSIS

1.1. CAUCHY'S THEORM

One of the most useful applications of complex analysis is the evaluation of definite integrals. For instance, the following examples can be found in nearly all standard books on the subject:

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}, \tag{1.1.1}$$

$$\int_{-\infty}^\infty \frac{x^{\alpha-1}}{1+x} dx = \frac{\pi}{\sin \alpha\pi}, \qquad 0 < \alpha < 1, \tag{1.1.2}$$

$$\int_{-\infty}^\infty \frac{\cos x}{x^2+a^2} dx = \frac{\pi e^{-a}}{a}, \qquad a > 0, \tag{1.1.3}$$

$$\int_{-\infty}^\infty \frac{1}{1+x^4} dx = \frac{\pi}{\sqrt{2}}; \tag{1.1.4}$$

see, e.g., [5, pp. 137 & 139] and [13, pp. 231 & 235]. In this section, we wish to illustrate the method of complex integration by studying a slightly more complicated integral; namely, the integral

$$I(z) = \int_a^b \frac{\log t}{\sqrt{(t-a)(b-t)}} \frac{dt}{t-z}, \tag{1.1.5}$$

where $0 < a < b < \infty$, $z \notin (a, b)$ and $|\arg z| < \pi$, which occurred in a recent study of the asymptotic behavior of the Stieltjes-Wigert polynomials [21]. The main tools in complex integration are the following two results; see [2].

Theorem 1.1.1. (Cauchy's theorem) *Let γ be the oriented piecewise smooth boundary of a compact subset K of an open set Ω, and let $f(z)$ be an analytic function in Ω. Then*

$$\int_\gamma f(z)dz = 0. \tag{1.1.6}$$

Theorem 1.1.2. (Cauchy's integral formula) *Let γ be the positively oriented piecewise smooth boundary of a compact subset K of an open set Ω, $f(z)$ be an analytic function in Ω, and z_0 be an interior point of K. Then,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz = f(z_0). \tag{1.1.7}$$

To evaluate the Cauchy-type integral $I(z)$ in (1.1.5), we first need an auxiliary result on the integral

$$I^*(z) = \int_0^\infty \frac{1}{\sqrt{(a+s)(b+s)}}\frac{ds}{s+z}. \tag{1.1.8}$$

Lemma 1.1. *For any $0 < a < b < \infty, z \notin [a,b]$ and $|\arg z| < \pi$, we have*

$$\begin{aligned} I^*(z) = &\frac{1}{\sqrt{(z-a)(z-b)}} \\ &\times \log\frac{[z+\sqrt{ab}+\sqrt{(z-a)(z-b)}]^2}{(\sqrt{a}+\sqrt{b})^2 z}, \end{aligned} \tag{1.1.9}$$

where the branches of the square root and the logarithm are taken to be positive when $z \in (b,\infty)$.

Proof. Make the change of variable

$$s = \frac{b-a}{4}\left(t+\frac{1}{t}\right) - \frac{a+b}{2}.$$

This transformation takes the s-interval $[0,\infty)$ onto the t-interval $[1/c,\infty)$, where $c = (\sqrt{b}-\sqrt{a})/(\sqrt{b}+\sqrt{a})$. Simple calculation gives

$$I^*(z) = \int_{1/c}^\infty \frac{dt}{\frac{1}{4}(b-a)(t^2+1)+(z-\frac{1}{2}(a+b))t}.$$

Let

$$t_\pm = \frac{2}{b-a}\left[-\left(z-\frac{b+a}{2}\right)\pm\sqrt{(z-a)(z-b)}\right],$$

and note that

$$t_+ - t_- = \frac{4}{b-a}\sqrt{(z-a)(z-b)}.$$

Here, we take the branch of the square roots to be positive when $z \in (b,\infty)$. By partial fractions,

$$I^*(z) = \frac{1}{\sqrt{(z-a)(z-b)}}\int_{1/c}^\infty \left(\frac{1}{t-t_+}-\frac{1}{t-t_-}\right)dt.$$

An integration then yields

$$I^*(z) = \frac{1}{\sqrt{(z-a)(z-b)}} \log \frac{z+\sqrt{ab}+\sqrt{(z-a)(z-b)}}{z+\sqrt{ab}-\sqrt{(z-a)(z-b)}}.$$

Note that $I^*(z)$ is positive when $z \in (b, \infty)$. Thus, we need to take the branch of the logarithm on the right-hand side to be also positive when $z \in (b, \infty)$. The last equation is clearly equivalent to (1.1.9). ∎

To evaluate the integral $I(z)$ in (1.1.5), we consider the contour integral

$$J(z) = \int_C \frac{\log \zeta}{\sqrt{(\zeta-a)(\zeta-b)}} \frac{d\zeta}{\zeta - z}, \qquad z \in \mathbb{C}\backslash(-\infty, 0] \cup [a, b], \quad (1.1.10)$$

where C is a positively oriented contour consisting of a large circle $\Gamma_R = \{z : |z| = R\}$, two straight lines Σ_+ and Σ_-, one above and one below the cut along the negative real-axis, and a closed curve Γ embracing a cut along the interval $[a, b]$; see Figure 1.1. In (1.1.10), the square root and the logarithm take their principal values. By Cauchy's integral formula,

$$J(z) = 2\pi i \frac{\log z}{\sqrt{(z-a)(z-b)}}. \quad (1.1.11)$$

On the large circle Γ_R, it is easily seen that the integrand in (1.1.10) is dominated by $(\log R)/R^2$; thus,

$$\int_{\Gamma_R} \frac{\log \zeta}{\sqrt{(z-a)(z-b)}} \frac{d\zeta}{\zeta - z} = O\left(\frac{\log R}{R}\right) \quad (1.1.12)$$

as $R \to \infty$. We deform the curve Γ into two straight line segments joining a and b. Due to the cut along the interval $[a, b]$, we have $\sqrt{\zeta - b} = \sqrt{b - \zeta}e^{i\pi/2}$ for ζ on the upper edge of the cut and $\sqrt{\zeta - b} = \sqrt{b - \zeta}e^{-i\pi/2}$ for ζ on the lower edge of the cut. Thus,

$$\int_\Gamma \frac{\log \zeta}{\sqrt{(\zeta-a)(\zeta-b)}} \frac{d\zeta}{\zeta - z} = \frac{2}{i} \int_a^b \frac{\log t}{\sqrt{(t-a)(b-t)}} \frac{dt}{t - z}, \quad (1.1.13)$$

where the path of integration on the left-hand side is oriented in the clockwise direction. Also, since $\log \zeta = \log |\zeta| \pm i\pi$ for $\zeta \in \Sigma_\pm$, we have

$$\int_{\Sigma_+ + \Sigma_-} \frac{\log \zeta}{\sqrt{(\zeta-a)(\zeta-b)}} \frac{d\zeta}{\zeta - z} = 2\pi i \int_0^\infty \frac{1}{\sqrt{(a+s)(b+s)}} \frac{ds}{s+z} \quad (1.1.14)$$

when $R \to \infty$. A combination of (1.1.10), (1.1.11), (1.1.12) and (1.1.13) gives the following result.

Lemma 1.2. *For any* $0 < a < b < \infty, z \notin [a, b]$ *and* $|\arg z| < \pi$, *the integral* $I(z)$ *in* (1.1.5) *is given by*

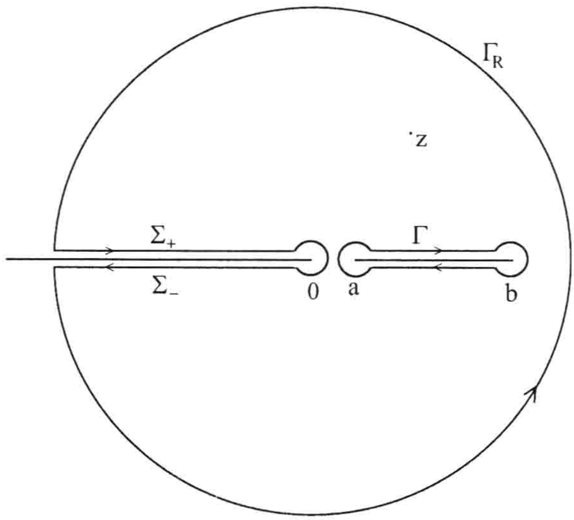

Fig. 1.1. Contour C.

$$I(z) = \frac{\pi}{\sqrt{(z-a)(z-b)}}\left\{\log\frac{1}{z} + \log\frac{[z+\sqrt{ab}+\sqrt{(z-a)(z-b)}]^2}{(\sqrt{a}+\sqrt{b})^2 z}\right\}, \tag{1.1.15}$$

where the branches of the square root and the logarithm are taken as in Lemma 1.1.

Another important consequence of Cauchy's theorem is the residue theorem [11, 13], which can quickly lead to interesting applications. The *residue* of a function $f(z)$ at an isolated singularity z_0 is the coefficient a_{-1} in its Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n. \tag{1.1.16}$$

In terms of an integral, we also have

$$\text{Res}\,(f, z_0) = \frac{1}{2\pi i}\int_\gamma f(z)dz, \tag{1.1.17}$$

where γ is a simple, closed, and positively oriented curve encircling z_0 and not any other singularity. The symbol on the left-hand side of (1.1.17) denotes the residue of f at z_0. Furthermore, from (1.1.16) we have

$$\text{Res}\,(f; z_0) = \lim_{z\to z_0}(z-z_0)f(z), \tag{1.1.18}$$

if the limit on the right-hand side exists.

Theorem 1.1.3. (Residue Theorem) *Let γ be the positively oriented piecewise smooth boundary of a compact subset K of an open set Ω, and let $z_1, \cdots, z_n$ be n distinct points in K. Let $f(z)$ be an analytic function in Ω except for isolated singularities at $z_1, \cdots, z_n$. Then,*

$$\int_\gamma f(z)dz = 2\pi i \sum_{i=1}^{n} \text{Res }(f, z_i). \tag{1.1.19}$$

As a simple example, let us evaluate the integral

$$I(\lambda) = \int_0^\infty \frac{\cos \lambda x}{1+x^2} dx, \qquad \lambda > 0, \tag{1.1.20}$$

which will be used as an illustration of "Exponential Asymptotics" in a later chapter. Since the integrand is an even function, we have

$$I(\lambda) = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos \lambda x}{1+x^2} dx. \tag{1.1.21}$$

Let γ be the curve shown in Figure 1.2, where $R > 1$. By the residue theorem,

$$\int_\gamma \frac{e^{i\lambda z}}{1+z^2} dz = 2\pi i \text{ Res}\left(\frac{e^{i\lambda z}}{1+z^2}, i\right).$$

Each of the integrals on the three lines not on the real-axis can easily be shown to be $O(1/R)$. Thus, as $R \to \infty$, we have

$$\int_{-\infty}^\infty \frac{e^{i\lambda x}}{1+x^2} dx = 2\pi i \text{ Res}\left(\frac{e^{i\lambda z}}{1+z^2}, i\right).$$

Since $1 + z^2 = (z - i)(z + i)$, the residue on the right-hand side is given by

$$\lim_{z \to i} (z - i) \frac{e^{i\lambda z}}{1+z^2} = \frac{e^{-\lambda}}{2i}.$$

Coupling the two results gives

$$\int_{-\infty}^\infty \frac{e^{i\lambda z}}{1+z^2} dz = \pi e^{-\lambda}.$$

Taking just the real part, we obtain from (1.1.21)

$$I(\lambda) = \frac{\pi}{2} e^{-\lambda}, \qquad \lambda > 0. \tag{1.1.22}$$

Sometimes for purposes of computation, it is convenient to formulate the residue theorem in a different form. Suppose $f(z)$ is an analytic function

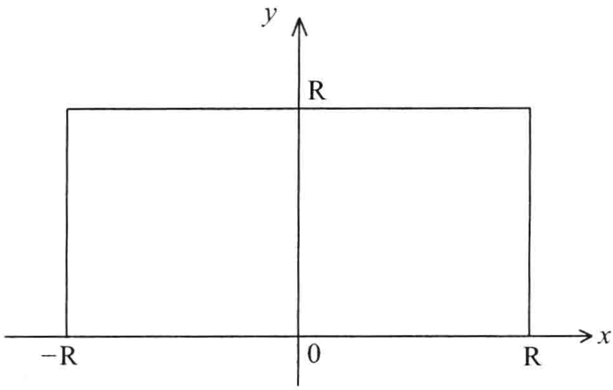

Fig. 1.2. Contour γ.

except for a finite number of isolated singularities in $\mathbb{C}$, and put $z = 1/z'$. Then,

$$f(z)dz = -\frac{1}{z'^2} f\left(\frac{1}{z'}\right) dz'.$$

In view of (1.1.16), the last equation suggests that we define the residue of f at infinity to be the residue of the function

$$g(z) = -\frac{1}{z^2} f\left(\frac{1}{z}\right) \tag{1.1.23}$$

at $z = 0$. If $\sum_{n=-\infty}^{\infty} a_n z^n$ is the Laurent expansion of $f(z)$ in a neighborhood of infinity, then the residue of f at ∞ is $-a_{-1}$. In terms of an integral, one can derive from (1.1.17) and (1.1.23)

$$\text{Res}\,(f, \infty) = -\frac{1}{2\pi i} \int_\Gamma f(\zeta) d\zeta, \tag{1.1.24}$$

where Γ can be a sufficiently large positively oriented circle not containing any isolated singularities of f. It can also be shown that

$$\text{Res}\,(f, \infty) = \lim_{z \to \infty} \{-zf(z)\}, \tag{1.1.25}$$

provided the limit exists. By applying the residue theorem (Theorem 1.1.3) to the curve in Figure 1.3, we obtain the alternative formulation

$$\int_\gamma f(z)dz = -2\pi i \sum \text{Residues of } f \text{ outside } \gamma \text{ including } \infty. \tag{1.1.26}$$

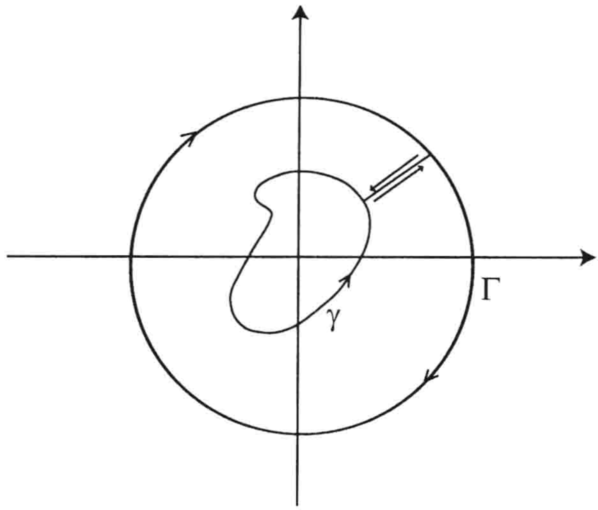

Fig. 1.3. Residues outside γ.

As an illustration, let us evaluate the integral

$$I(z) = \int_a^b \frac{\sqrt{(x-a)(b-x)}}{x} \frac{dx}{x-z}, \qquad 0 < a < b < \infty, \qquad (1.1.27)$$

where z is any complex number $\neq 0$ and $\notin [a, b]$. Put

$$R(z) := \sqrt{(z-a)(z-b)} \qquad (1.1.28)$$

for z in $\mathbb{C}$ cut along the line segment $[a, b]$, and define $R(x) = \sqrt{(x-a)(b-x)}e^{\pm i\pi/2}$ for x in (a, b) with $+$ and $-$ signs corresponding, respectively, to the upper and lower edges of the cut. Let $\widetilde{\gamma}$ be a clockwise oriented curve enclosing the interval $[a, b]$ but not 0 and z. Then, we can write $I(z)$ as

$$\frac{1}{\pi} I(z) = \frac{1}{2\pi i} \int_{\widetilde{\gamma}} \frac{R(\zeta)}{\zeta} \frac{d\zeta}{\zeta - z}. \qquad (1.1.29)$$

It is clear that outside $\widetilde{\gamma}$, there are poles at $0, z$ and ∞. Thus, by (1.1.26), we have

$$\frac{1}{\pi} I(z) = \operatorname*{Res}_{\zeta=0}\left(\frac{R(\zeta)}{\zeta}\frac{1}{\zeta - z}\right) + \operatorname*{Res}_{\zeta=z}\left(\frac{R(\zeta)}{\zeta}\frac{1}{\zeta - z}\right) + \operatorname*{Res}_{\zeta=\infty}\left(\frac{R(\zeta)}{\zeta}\frac{1}{\zeta - z}\right); \qquad (1.1.30)$$

that is,

$$I(z) = -\pi\left[1 - \frac{\sqrt{(z-a)(z-b)}}{z} - \frac{\sqrt{ab}}{z}\right]. \qquad (1.1.31)$$

Since $\sqrt{(z-a)(z-b)} = -\sqrt{ab} + \frac{1}{2}\frac{a+b}{\sqrt{ab}}z + \cdots$, taking the limit as $z \to 0$ gives

$$I(0) = \pi\left[\frac{1}{2} \cdot \frac{a+b}{\sqrt{ab}} - 1\right]. \tag{1.1.32}$$

To conclude this section, we mention an expansion of the form

$$\sum_{j=0}^{\infty} f(z,j) = \frac{1}{2i}\int_{\Gamma} \cot(\pi t) f(z,t)dt, \tag{1.1.33}$$

where $f(z,t)$ depends on a real or complex parameter z, and is an analytic function of the complex variable t, and where Γ is a loop contour enclosing the points $t = 0, 1, 2, \cdots$, but not enclosing $-1, -2, -3, \cdots$ or the singularities of $f(z,t)$. This result is known as the *Watson transformation* [24, pp. 34 & 44], and can be easily verified by observing that the residue of $\cot \pi t$ at $t = j$ is $1/\pi$. In a similar manner, one can also establish

$$\sum_{j=0}^{\infty}(-1)^j f(z,j) = \frac{1}{2i}\int_{\Gamma} \csc(\pi t) f(z,t)dt. \tag{1.1.34}$$

As an illustration, we consider the sum

$$S(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{z^2 + j^2}, \tag{1.1.35}$$

where z is a complex parameter $\neq 0, \pm 1, \pm 2, \cdots$. Direct application of (1.1.34) would not lead to the result we wish to derive; instead, we make a slight modification of the method. Clearly, $S(z)$ can also be expressed as

$$S(z) = \frac{1}{2z^2} + \frac{1}{2}\sum_{j=-\infty}^{\infty} (-1)^j f(z,j), \tag{1.1.36}$$

where

$$f(z,t) = \frac{1}{z^2 + t^2}. \tag{1.1.37}$$

Let J_n denote the contour shown in Figure 1.4, where n is an arbitrary positive integer bigger than $|\text{Re}(iz)|$ and c is an arbitrary positive number satisfying $c < \text{Im}(iz)$.

By the residue theorem,

$$\sum_{j=-n}^{n} (-1)^j f(z,j) = \frac{1}{2i}\int_{\Gamma_n} \frac{\csc(\pi t)}{z^2 + t^2}dt.$$