

Lecture Notes in Mathematics

Gianfranco Capriz
Pasquale Giovine
Paolo Maria Mariano (Eds.)

Mathematical Models of Granular Matter

1937



Springer

Gianfranco Capriz · Pasquale Giovine
Paolo Maria Mariano (Eds.)

Mathematical Models of Granular Matter

With Contributions by:

A. Barrat · A.V. Bobylev · C. Cercignani · I.M. Gamba

R. Garcia-Rojo · J.D. Goddard · H.J. Herrmann

S. McNamara · A. Puglisi · T. Ruggeri · G. Toscani · E. Trizac

P. Visco · F. van Wijland

 Springer

Editors

Gianfranco Capriz
Department of Mathematics
University of Pisa
largo Bruno Pontecorvo 5
56127 Pisa
Italy
and
Accademia Nazionale dei Lincei
via della Lungara 10
00165 Roma
Italy
gianfranco.capriz@mac.com

Pasquale Giovine
Dipartimento di Meccanica e Materiali
Università "Mediterranea" di Reggio
Calabria
via Graziella, località Feo di Vito
89060 Reggio Calabria
Italy
giovine@unirc.it

Paolo Maria Mariano
DICEA
University of Florence
via Santa Marta 3
50139 Firenze
Italy
paolo.mariano@unifi.it

ISBN 978-3-540-78276-6 e-ISBN 978-3-540-78277-3
DOI 10.1007/978-3-540-78277-3

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2008921365

Mathematics Subject Classification (2000): 76T25, 74E20, 74E25, 82D30

© 2008 Springer-Verlag Berlin Heidelberg

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Cover design: WMXDesign GmbH

Coverart "Without title", xilo-picture by Luigi Mariano, 1973, private collection

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Editors:

J.-M. Morel, Cachan

F. Takens, Groningen

B. Teissier, Paris

Preface

The adjective ‘granular’ is attributed to materials when they are made of sets of unfastened discrete solid particles (granules) of a size larger than one micron, a length scale above which thermal agitation is negligible. In fact, the dominant energy scale in granular materials is the one of a single grain under gravity. Granular matter is common and we meet it everyday. Examples range from the dust settled on the books of our libraries, to the sand in the desert, to the meal used in cooking, itself obtained from grain, often stored in silos. Granular matter displays a variety of peculiarities that distinguish it from other substances studied by condensed matter physics and renders its overall mathematical modeling arduous. In a review paper of 1999 [dG] P.G. de Gennes writes: “granular matter is a new type of condensed matter, as fundamental as a liquid or a solid and showing in fact two states: one fluidlike, one solidlike. But there is as yet no consensus on the description of these two states!”

Almost all preconceptions on which the standard theory of continua is based seem to fail. The standard concept that the material element is well identified (even in the statistical conception common in gas dynamics) fails and, with it, the current mathematical picture assigning to it a precise placement. Even useful results of the standard kinetic theory of gases can be called upon confidently, in general. The populations of grains are far less profuse than the molecular ones in gases and far more crowded. The constraints imposed by grains on one another are generally too conspicuous to rigidify the lot. Also, boundary conditions are far from the simple classical scheme suggested by the divergence theorem and need separate critical modeling.

Heaps of granules do not sustain tension unless (at least a small) cohesion is present. They are, in general, in anisotropic metastable states. Such states last indefinitely unless external perturbations occur. Contrary to common solids and fluids, no thermal average among nearby states arises (see [JNB]). Interactions between granules are exerted through contacts occurring along graphs with topology depending on the way in which granules are packaged, on the distribution of the sizes of the granules themselves, on the boundary

conditions and, above all, on the sources of external disturbances. Subsets of granules may self-organize in order to sustain and distribute tensions along arcs: When granules are stored in a silo the pressure at the bottom of the silo does not increase indefinitely as the height of the stored material grows, rather it reaches a maximum value if the lateral walls of the silo are sufficiently tall. If the silo is shaken and grains with different sizes are stored within it, size segregation occurs instead of mixing. The phenomenon does not fit the entropic effects that one recognizes in standard liquids. Both traveling and standing waves accrue in the surface layer of grains, and slip appears along the walls of the silo. It is not clear yet how inelastic interactions and local disorder or segregation contribute to the acoustic propagation.

Layer dynamics is present also in ‘avalanches’. The example is the addition of grains to a heap from the top: the surface layer moves, the core persists. The description of the connection between the surface flow and the core at rest forces one to account for the transition between two phases, if one wants to propose a global picture of the phenomenon. As for the contact stresses in granular materials at rest, even for rapid flows the stresses induced by the collisions depend on the local numerosity of granules, in other words on the ‘degree of clustering’. Segregation in dense granular flows also has an influence. The overall mechanical behavior seems to be history-dependent.

Plastic effects may be prominent in the quasi-static regime: shear bands appear and inelastic deformations may be accumulated by cyclic loading processes. Inelastic collisions usually play a decisive role in dynamics, as in the fall of avalanches and in the walk of desert dunes. Chaotic agitation of granules leads them far from thermodynamic equilibrium, so fluctuations may be prominent. Microscopic slip friction between granules induces relaxation analogous to that of some solids with complex microstructure. Macroscopic friction is induced by earthquakes which may induce fluidization of granular matter.

Critical reviews such as [JNB], [dG], [K], and [AT] provide an adequate description of phenomena occurring in granular matter. They give a picture of the scenario.

A typical approach to the dynamics of polydisperse granular systems is based on the ‘inelastic’ Boltzmann equation which also provides the starting point for a plethora of hydrodynamic models. Assumptions on the types of contacts occurring must be chosen carefully: relative rotations may need to be accounted for, depending on the circumstances. Closures are obtained by assuming specific forms of the distribution function. A critical review of the results along these lines can be found in [V].

Even in a disperse state, a granular flow is dissipative as a consequence of the presence of inelastic collisions. Absence of equilibrium is the source of difficulties when one applies the Chapman-Enskog perturbative method to derive a (hydrodynamic) continuum picture from the kinetic description based on Boltzmann equation. Since collisions are inelastic, a double expansion in Knudsen number K and degree of inelasticity ϵ has to be used. Moreover, since

$\epsilon \propto K^3$ in the steady state, the expansion has to be extended up to Burnett or super-Burnett regimes to assure consistency, the latter regime being of degree $O(K^3)$. The appearance of unphysical instabilities under short-wave perturbations, typical of Burnett and super-Burnett regimes, requires viscosity regularizations of the field equations or the use of other possible techniques.

Attempts have been made to construct continuum models of granular matter from first principles, without resorting to kinetic justifications based on Boltzmann equation, especially in statics. Non-trivial difficulties arise, as already mentioned: at a gross scale some standard paradigms of traditional continuum mechanics need to be modified accurately. We have already mentioned for example the loss of the possibility of identifying perfectly a generic material element (i.e. even to define it). This difficulty is generated by the occurrence of segregation and also by the general lack of coherence due to the absence of cohesion between neighboring granules. Each material element must be then considered as an open system (as in [M]) from which granules may migrate from it to neighboring ones. Standard balances of interactions have to be then supplemented by an equation ruling the rate of local numerosity of granules. Such a rate, which is the rate of migration of granules, generates loss of information about the local texture of the granular matter, and increases the configurational entropy, although segregation is opposite to the entropic mixing (as mentioned above).

The definition of measures of deformation and stresses in terms of granular geometries and grain-to-grain interactions may be non-trivial. As regards the standard stress tensor, for example, a typical definition is made by the sum of the tensor product between the intergranular force and the vector indicating in a local frame the contact point between neighboring granules, a sum extended to all granules in contact with a given granule. However, although such a definition has a physical meaning, one does not know point by point (or better, element by element) the local distribution of granules, their geometry and the type of contact which may be inelastic and not even punctual. All these information characterize each model that can be constructed and have also constitutive nature.

By looking at deeper details, it is natural to consider a granular material as a complex body (in the sense of a body in which the material texture influences strongly the gross behavior). So, multifield descriptions of granular matter need to be called upon, as explained in some chapters of this book.

In any case, no accepted general consensus about the overall description of granular bodies exists. Such an absence of a unified description of the mechanics of the granular matter and the lack of preference for one or another approach, for aesthetic or experimental reasons, has pushed us to collect advances in the various prominent directions of research. Our aim is to furnish a panorama clarifying the state of current researches, solving problems, discussing critically points of view and opening new questions. Contributions range, in fact, from the kinetic approaches to granular flows to the continuum description of static and quasi-static behaviors. A non-trivial tentative of a

connection between the kinetic approach and a continuum modeling based on first principles is also present (see Chap. 4).

At the present state of knowledge and in the absence of a unitary point of view, one can only say that the variety and the peculiarities of behavior of granular matter render arduous the task of its overall mathematical modeling. Mathematical and physical questions of an intricate nature appear and tools additional to the traditional ones are needed. Some of these questions and tools are discussed in the subsequent chapters.

Motivated by experimental results on shear bands due to (unstable) plastic behavior of granular bodies, in Chap. 1, *Joe D. Goddard* discusses critically various techniques for the homogenization of granular media in a quasi-static regime, media that are seen here as multipolar continua in the sense of the mechanics of complex bodies. An energy-based method of homogenization is proposed as an improvement on previous approaches.

If one describes granular flows from the point of view of the kinetic theory, inelastic collisions play a role as mentioned above. The Maxwell model of binary collisions is a typical scheme adopted and is based on the assumption that the collision frequency is independent of the velocity of colliding particles. Such a model can be translated from rarefied gases to the description of granular flows. In Chap. 2, *Alexander V. Bobylev*, *Carlo Cercignani* and *Irene Martinez Gamba* discuss, from a general point of view, variants of the Maxwell model of pairwise interactions and establish their key properties that lead to self-similar asymptotics. Existence and uniqueness issues are also analyzed.

The approach based on the dissipative Boltzmann equation is further analyzed in Chap. 3 by *Giuseppe Toscani*. Two paths toward the hydrodynamic limit are discussed. They account for two different methodologies for the closure of macroscopic equations: (i) low inelasticity in the system, namely a perturbation in a precise sense that allows the local resort to Maxwellian functions, (ii) small spatial variations implying the use of a homogeneous cooling state.

One of the editors (**GC**) proposes in Chap. 4 (a chapter already mentioned above) further results on an earlier proposal for fast sparse flows. Complex bodies with kinetic substructure are considered. They are bodies in which each material element is a system in continuous agitation and are called *pseudofluids*. Granular flows fall naturally in this framework. The maelstrom within each material element and its influence on the neighboring fellows is governed by peculiar hydrodynamic balances: they are offspring of the microscopic interactions between granules.

The recursive analysis of higher momenta of the distribution function, even beyond Grad's 13 moments, has been the basic source of Extended Thermodynamics. The results on monoatomic gases provided by such an approach suggest its use in modeling granular flows. After summarizing the main modeling issues of Extended Thermodynamics, in Chap. 5 *Tommaso Ruggeri* analyzes the resulting hyperbolic system and discusses global existence questions and stability of constant state on the basis of Kawashima condition.

The construction of hydrodynamic models from first principles can find appropriate suggestions from detailed numerical simulations performed by looking directly at single granules and their contact interactions. The molecular dynamic approach is pursued in Chap. 6 by *Ramon García-Rojo*, *Sean McNamara* and *Hans J. Herrmann* who analyze (amid possible choices) the persistent elastic-plastic strain accumulation in compacted granular soils under cyclic stress conditions.

Since driven sets of granules (for example confined in a box) are in essence systems very far from thermodynamic equilibrium, as mentioned above, the effects of fluctuations are in general significant. Analysis of the injected power fluctuations is presented by *Alain Barrat*, *Andrea Puglisi*, *Emmanuel Trizac*, *Paolo Visco* and *Frederic van Wijland* in Chap. 7, a chapter divided into two parts: The first part deals with the way in which the probability density function of the fluctuations of the total energy is related to the characterization of energy correlations for both boundary and homogeneous driving. The second part contains an interpretation of some numerical and experimental results that seem to invalidate Gallavotti-Cohen symmetry [GC]. Such results appear contradictory to common analyses that seem to satisfy Gallavotti-Cohen fluctuation relation. By means of Lebowitz-Spohn approach to Markov processes, an approach applied to the inelastic Boltzmann equation, a functional satisfying a fluctuation relation is also introduced.

The last two editors collect their contributions in Chaps. 8 and 9. In particular, in Chap. 8 **PG** analyzes in the continuum limit the thermodynamics of a granular material modeled as a complex body endowed with a microstructure which is constrained and/or latent in the sense introduced by Capriz [C]. The consequences of grain rotations are described together with effects like dilatancy.

Finally, in Chap. 9, **PMM** considers granular matter in slow motion and describes it as a two-scale complex body for which each material element is considered as a grand-canonical ensemble of granules. The evolution equation of the numerosity of grains in each material element is derived in terms of grain-to-grain interactions.

February, 2008

Gianfranco Capriz
Pasquale Giovine
Paolo Maria Mariano

References

- [AT] Aranson, I.S., Tsimring, L.S.: Patterns and collective behavior in granular media: Theoretical concepts. *Rev. Modern Phys.*, **78**, 641–692 (2006)
- [C] Capriz, G.: Continua with latent microstructure. *Arch. Rational Mech. Anal.*, **90**, 43–56 (1985)
- [dG] de Gennes, P.G.: Granular matter, a tentative view. *Rev. Modern Phys.*, **71**, S375–S382 (1999)

- [G] Goldhirsch, J.: Kinetic and continuum descriptions of granular flows. In: Aref, H., Philips, J.W. (eds) *Mechanics for a New Millenium*. Philips Edts., Kluwer, Dordrecht, 345–358 (2001)
- [GC] Gallavotti, G., Cohen, E.G.D.: Dynamical ensembles in nonequilibrium statistical mechanics. *Phys. Rev. Letters*, **74**, 2694–2698 (1995)
- [K] Kadanoff, L.P.: Built upon sand: Theoretical ideas inspired by granular flows. *Rev. Modern Phys.*, **71**, 435–444 (1999)
- [JNB] Jaeger, H.M., Nagel, S.R., Behringer, R.P.: Granular solids, liquids, and gases. *Rev. Modern Phys.*, **68**, 1259–1273 (1996)
- [M] Mariano, P.M.: Migration of substructures in complex fluids. *J. Phys. A: Math. Teoretical*, **38**, 6823–6839 (2005)
- [V] Villani, C.: Mathematics of granular materials. *J. Stat. Phys.*, **124**, 781–822 (2007)

List of Contributors

Alain Barrat

Laboratoire de Physique Théorique
(CNRS UMR 8627),
Université Paris-Sud
Orsay, France
Alain.Barrat@th.u-psud.fr

Alexander V. Bobylev

Department of Mathematics,
Karlstad University
Universitetsgatan 2,
65188 Karlstad, Sweden
alexander.bobylev@kau.se

Gianfranco Capriz

Department of Mathematics,
University of Pisa
largo Bruno Pontecorvo 5,
56127 Pisa, Italy
gianfranco.capriz@mac.com

Carlo Cercignani

Dipartimento di Matematica,
Politecnico di Milano
via Bonardi 9,
20133 Milano, Italy
carcer@mate.polimi.it

Ramon García-Rojo

Física Teórica, Facultad de Física,
Universidad de Sevilla
Apartado de Correos 1065,
41080 Sevilla, Spain

Pasquale Giovine

Dipartimento di Meccanica
e Materiali,
Università "Mediterranea"
via Graziella,
Località Feo di Vito,
89127 Reggio Calabria,
Italy
giovine@unirc.it

Joe D. Goddard

Department of Mechanical
and Aerospace Engineering,
University of California
at San Diego - La Jolla
La Jolla, CA 92093-0411, USA
jgoddard@ucsd.edu

Hans J. Herrmann

Departamento de Física,
Universidade Federal do Ceará
Campus do Pici,
60455-900 Fortaleza
Brazil
hans@fisica.ufc.br

Paolo Maria Mariano

DICeA, University of Florence
via Santa Marta 3,
50139 Firenze, Italy
paolo.mariano@unifi.it

Irene Martinez Gamba
Department of Mathematics,
The University of Texas
at Austin
Austin, TX 78712, USA
gamba@math.utexas.edu

Sean McNamara
ICP, University of Stuttgart,
Pfaffenwaldring 27,
70569 Stuttgart, Germany
S.McNamara@icp.uni-stuttgart.de

Andrea Puglisi
Dipartimento di Fisica,
Università di Roma
“La Sapienza”
Piazzale Aldo Moro 2,
00185 Roma, Italy
andrea.puglisi@roma1.infn.it

Tommaso Ruggeri
Research Center for Applied
Mathematics, C.I.R.A.M.
University of Bologna
via Saragozza 8,
40123 Bologna, Italy
ruggeri@ciram.unibo.it

Giuseppe Toscani
Department of Mathematics,
University of Pavia
via Ferrata 1,
27100 Pavia, Italy
giuseppe.toscani@unipv.it

Emmanuel Trizac
Laboratoire de Physique,
Théorique et Modèles Statistiques
(CNRS UMR 8626),
Université Paris-Sud
Orsay, France
trizac@ipno.in2p3.fr

Frederic van Wijland
Laboratoire Matière et Systèmes
Complexes (CNRS UMR 7057),
Université Denis Diderot (Paris VII),
Paris, France
Frederic.Van-Wijland
@th.u-psud.fr

Paolo Visco
Laboratoire de Physique,
Théorique et Modèles Statistiques
(CNRS UMR 8626),
Université Paris-Sud
Orsay, France
Paolo.Visco@th.u-psud.fr

Lecture Notes in Mathematics

For information about earlier volumes
please contact your bookseller or Springer
LNM Online archive: springerlink.com

- Vol. 1755: J. Azéma, M. Émery, M. Ledoux, M. Yor (Eds.), *Séminaire de Probabilités XXXV* (2001)
- Vol. 1756: P. E. Zhidkov, Korteweg de Vries and Nonlinear Schrödinger Equations: Qualitative Theory (2001)
- Vol. 1757: R. R. Phelps, *Lectures on Choquet's Theorem* (2001)
- Vol. 1758: N. Monod, *Continuous Bounded Cohomology of Locally Compact Groups* (2001)
- Vol. 1759: Y. Abe, K. Kopfermann, *Toroidal Groups* (2001)
- Vol. 1760: D. Filipović, *Consistency Problems for Heath-Jarrow-Morton Interest Rate Models* (2001)
- Vol. 1761: C. Adelmann, *The Decomposition of Primes in Torsion Point Fields* (2001)
- Vol. 1762: S. Cerrai, *Second Order PDE's in Finite and Infinite Dimension* (2001)
- Vol. 1763: J.-L. Loday, A. Frabetti, F. Chapoton, F. Gochot, *Dialgebras and Related Operads* (2001)
- Vol. 1764: A. Cannas da Silva, *Lectures on Symplectic Geometry* (2001)
- Vol. 1765: T. Kerler, V. V. Lyubashenko, *Non-Semisimple Topological Quantum Field Theories for 3-Manifolds with Corners* (2001)
- Vol. 1766: H. Hennion, L. Hervé, *Limit Theorems for Markov Chains and Stochastic Properties of Dynamical Systems by Quasi-Compactness* (2001)
- Vol. 1767: J. Xiao, *Holomorphic Q Classes* (2001)
- Vol. 1768: M. J. Pflaum, *Analytic and Geometric Study of Stratified Spaces* (2001)
- Vol. 1769: M. Alberich-Carramiñana, *Geometry of the Plane Cremona Maps* (2002)
- Vol. 1770: H. Gluesing-Luerssen, *Linear Delay-Differential Systems with Commensurate Delays: An Algebraic Approach* (2002)
- Vol. 1771: M. Émery, M. Yor (Eds.), *Séminaire de Probabilités 1967-1980. A Selection in Martingale Theory* (2002)
- Vol. 1772: F. Burstall, D. Ferus, K. Leschke, F. Pedit, U. Pinkall, *Conformal Geometry of Surfaces in S^4* (2002)
- Vol. 1773: Z. Arad, M. Muzychuk, *Standard Integral Table Algebras Generated by a Non-real Element of Small Degree* (2002)
- Vol. 1774: V. Runde, *Lectures on Amenability* (2002)
- Vol. 1775: W. H. Meeks, A. Ros, H. Rosenberg, *The Global Theory of Minimal Surfaces in Flat Spaces*. Martina Franca 1999. Editor: G. P. Pirola (2002)
- Vol. 1776: K. Behrend, C. Gomez, V. Tarasov, G. Tian, *Quantum Comohology*. Cetraro 1997. Editors: P. de Bartolomeis, B. Dubrovin, C. Reina (2002)
- Vol. 1777: E. García-Río, D. N. Kupeli, R. Vázquez-Lorenzo, *Osserman Manifolds in Semi-Riemannian Geometry* (2002)
- Vol. 1778: H. Kiechle, *Theory of K-Loops* (2002)
- Vol. 1779: I. Chueshov, *Monotone Random Systems* (2002)
- Vol. 1780: J. H. Bruinier, Borchers Products on $O(2,1)$ and Chern Classes of Heegner Divisors (2002)
- Vol. 1781: E. Bolthausen, E. Perkins, A. van der Vaart, *Lectures on Probability Theory and Statistics*. Ecole d'Eté de Probabilités de Saint-Flour XXIX-1999. Editor: P. Bernard (2002)
- Vol. 1782: C.-H. Chu, A. T.-M. Lau, *Harmonic Functions on Groups and Fourier Algebras* (2002)
- Vol. 1783: L. Grüne, *Asymptotic Behavior of Dynamical and Control Systems under Perturbation and Discretization* (2002)
- Vol. 1784: L. H. Eliasson, S. B. Kuksin, S. Marmi, J.-C. Yoccoz, *Dynamical Systems and Small Divisors*. Cetraro, Italy 1998. Editors: S. Marmi, J.-C. Yoccoz (2002)
- Vol. 1785: J. Arias de Reyna, *Pointwise Convergence of Fourier Series* (2002)
- Vol. 1786: S. D. Cutkosky, *Monomialization of Morphisms from 3-Folds to Surfaces* (2002)
- Vol. 1787: S. Caenepeel, G. Militaru, S. Zhu, *Frobenius and Separable Functors for Generalized Module Categories and Nonlinear Equations* (2002)
- Vol. 1788: A. Vasil'ev, *Moduli of Families of Curves for Conformal and Quasiconformal Mappings* (2002)
- Vol. 1789: Y. Sommerhäuser, *Yetter-Drinfel'd Hopf algebras over groups of prime order* (2002)
- Vol. 1790: X. Zhan, *Matrix Inequalities* (2002)
- Vol. 1791: M. Knebusch, D. Zhang, *Manis Valuations and Prüfer Extensions I: A new Chapter in Commutative Algebra* (2002)
- Vol. 1792: D. D. Ang, R. Gorenflo, V. K. Le, D. D. Trong, *Moment Theory and Some Inverse Problems in Potential Theory and Heat Conduction* (2002)
- Vol. 1793: J. Cortés Monforte, *Geometric, Control and Numerical Aspects of Nonholonomic Systems* (2002)
- Vol. 1794: N. Pytheas Fogg, *Substitution in Dynamics. Arithmetics and Combinatorics*. Editors: V. Berthé, S. Ferenczi, C. Mauduit, A. Siegel (2002)
- Vol. 1795: H. Li, *Filtered-Graded Transfer in Using Non-commutative Gröbner Bases* (2002)
- Vol. 1796: J.M. Melenk, *hp-Finite Element Methods for Singular Perturbations* (2002)
- Vol. 1797: B. Schmidt, *Characters and Cyclotomic Fields in Finite Geometry* (2002)
- Vol. 1798: W.M. Oliva, *Geometric Mechanics* (2002)
- Vol. 1799: H. Pajot, *Analytic Capacity, Rectifiability, Menger Curvature and the Cauchy Integral* (2002)
- Vol. 1800: O. Gabber, L. Ramero, *Almost Ring Theory* (2003)
- Vol. 1801: J. Azéma, M. Émery, M. Ledoux, M. Yor (Eds.), *Séminaire de Probabilités XXXVI* (2003)
- Vol. 1802: V. Capasso, E. Merzbach, B. G. Ivanoff, M. Dozzi, R. Dalang, T. Mountford, *Topics in Spatial Stochastic Processes*. Martina Franca, Italy 2001. Editor: E. Merzbach (2003)

- Vol. 1803: G. Dolzmann, Variational Methods for Crystalline Microstructure – Analysis and Computation (2003)
- Vol. 1804: I. Cherednik, Ya. Markov, R. Howe, G. Lusztig, Iwahori-Hecke Algebras and their Representation Theory. Martina Franca, Italy 1999. Editors: V. Baldoni, D. Barbasch (2003)
- Vol. 1805: F. Cao, Geometric Curve Evolution and Image Processing (2003)
- Vol. 1806: H. Broer, I. Hoveijn, G. Lunther, G. Vegter, Bifurcations in Hamiltonian Systems. Computing Singularities by Gröbner Bases (2003)
- Vol. 1807: V. D. Milman, G. Schechtman (Eds.), Geometric Aspects of Functional Analysis. Israel Seminar 2000-2002 (2003)
- Vol. 1808: W. Schindler, Measures with Symmetry Properties (2003)
- Vol. 1809: O. Steinbach, Stability Estimates for Hybrid Coupled Domain Decomposition Methods (2003)
- Vol. 1810: J. Wengenroth, Derived Functors in Functional Analysis (2003)
- Vol. 1811: J. Stevens, Deformations of Singularities (2003)
- Vol. 1812: L. Ambrosio, K. Deckelnick, G. Dziuk, M. Mimura, V. A. Solonnikov, H. M. Soner, Mathematical Aspects of Evolving Interfaces. Madeira, Funchal, Portugal 2000. Editors: P. Colli, J. F. Rodrigues (2003)
- Vol. 1813: L. Ambrosio, L. A. Caffarelli, Y. Brenier, G. Buttazzo, C. Villani, Optimal Transportation and its Applications. Martina Franca, Italy 2001. Editors: L. A. Caffarelli, S. Salsa (2003)
- Vol. 1814: P. Bank, F. Baudoin, H. Föllmer, L.C.G. Rogers, M. Soner, N. Touzi, Paris-Princeton Lectures on Mathematical Finance 2002 (2003)
- Vol. 1815: A. M. Vershik (Ed.), Asymptotic Combinatorics with Applications to Mathematical Physics. St. Petersburg, Russia 2001 (2003)
- Vol. 1816: S. Albeverio, W. Schachermayer, M. Tala-grand, Lectures on Probability Theory and Statistics. Ecole d'Été de Probabilités de Saint-Flour XXX-2000. Editor: P. Bernard (2003)
- Vol. 1817: E. Koelink, W. Van Assche (Eds.), Orthogonal Polynomials and Special Functions. Leuven 2002 (2003)
- Vol. 1818: M. Bildhauer, Convex Variational Problems with Linear, nearly Linear and/or Anisotropic Growth Conditions (2003)
- Vol. 1819: D. Masser, Yu. V. Nesterenko, H. P. Schlickewe, W. M. Schmidt, M. Waldschmidt, Diophantine Approximation. Cetraro, Italy 2000. Editors: F. Amoroso, U. Zannier (2003)
- Vol. 1820: F. Hiai, H. Kosaki, Means of Hilbert Space Operators (2003)
- Vol. 1821: S. Teufel, Adiabatic Perturbation Theory in Quantum Dynamics (2003)
- Vol. 1822: S.-N. Chow, R. Conti, R. Johnson, J. Mallet-Paret, R. Nussbaum, Dynamical Systems. Cetraro, Italy 2000. Editors: J. W. Macki, P. Zecca (2003)
- Vol. 1823: A. M. Anile, W. Allegretto, C. Ringhofer, Mathematical Problems in Semiconductor Physics. Cetraro, Italy 1998. Editor: A. M. Anile (2003)
- Vol. 1824: J. A. Navarro González, J. B. Sancho de Salas, ∞^∞ – Differentiable Spaces (2003)
- Vol. 1825: J. H. Bramble, A. Cohen, W. Dahmen, Multiscale Problems and Methods in Numerical Simulations, Martina Franca, Italy 2001. Editor: C. Canuto (2003)
- Vol. 1826: K. Dohnen, Improved Bonferroni Inequalities via Abstract Tubes. Inequalities and Identities of Inclusion-Exclusion Type. VIII, 113 p, 2003.
- Vol. 1827: K. M. Pilgrim, Combinations of Complex Dynamical Systems. IX, 118 p, 2003.
- Vol. 1828: D. J. Green, Gröbner Bases and the Computation of Group Cohomology. XII, 138 p, 2003.
- Vol. 1829: E. Altman, B. Gaujal, A. Hordijk, Discrete-Event Control of Stochastic Networks: Multimodularity and Regularity. XIV, 313 p, 2003.
- Vol. 1830: M. I. Gil', Operator Functions and Localization of Spectra. XIV, 256 p, 2003.
- Vol. 1831: A. Connes, J. Cuntz, E. Guentner, N. Higson, J. E. Kaminker, Noncommutative Geometry, Martina Franca, Italy 2002. Editors: S. Doplicher, L. Longo (2004)
- Vol. 1832: J. Azéma, M. Émery, M. Ledoux, M. Yor (Eds.), Séminaire de Probabilités XXXVII (2003)
- Vol. 1833: D.-Q. Jiang, M. Qian, M.-P. Qian, Mathematical Theory of Nonequilibrium Steady States. On the Frontier of Probability and Dynamical Systems. IX, 280 p, 2004.
- Vol. 1834: Yo. Yomdin, G. Comte, Tame Geometry with Application in Smooth Analysis. VIII, 186 p, 2004.
- Vol. 1835: O.T. Izhboldin, B. Kahn, N.A. Karpenko, A. Vishik, Geometric Methods in the Algebraic Theory of Quadratic Forms. Summer School, Lens, 2000. Editor: J.-P. Tignol (2004)
- Vol. 1836: C. Năstăsescu, F. Van Oystaeyen, Methods of Graded Rings. XIII, 304 p, 2004.
- Vol. 1837: S. Tavaré, O. Zeitouni, Lectures on Probability Theory and Statistics. Ecole d'Été de Probabilités de Saint-Flour XXXI-2001. Editor: J. Picard (2004)
- Vol. 1838: A.J. Ganesh, N.W. O'Connell, D.J. Wischik, Big Queues. XII, 254 p, 2004.
- Vol. 1839: R. Gohm, Noncommutative Stationary Processes. VIII, 170 p, 2004.
- Vol. 1840: B. Tsirelson, W. Werner, Lectures on Probability Theory and Statistics. Ecole d'Été de Probabilités de Saint-Flour XXXII-2002. Editor: J. Picard (2004)
- Vol. 1841: W. Reichel, Uniqueness Theorems for Variational Problems by the Method of Transformation Groups (2004)
- Vol. 1842: T. Johnsen, A. L. Knutsen, K_3 Projective Models in Scrolls (2004)
- Vol. 1843: B. Jefferies, Spectral Properties of Noncommuting Operators (2004)
- Vol. 1844: K.F. Siburg, The Principle of Least Action in Geometry and Dynamics (2004)
- Vol. 1845: Min Ho Lee, Mixed Automorphic Forms, Torus Bundles, and Jacobi Forms (2004)
- Vol. 1846: H. Ammari, H. Kang, Reconstruction of Small Inhomogeneities from Boundary Measurements (2004)
- Vol. 1847: T.R. Bielecki, T. Björk, M. Jeanblanc, M. Rutkowski, J.A. Scheinkman, W. Xiong, Paris-Princeton Lectures on Mathematical Finance 2003 (2004)
- Vol. 1848: M. Abate, J. E. Forneaess, X. Huang, J. P. Rosay, A. Tumanov, Real Methods in Complex and CR Geometry, Martina Franca, Italy 2002. Editors: D. Zaitsev, G. Zampieri (2004)
- Vol. 1849: Martin L. Brown, Heegner Modules and Elliptic Curves (2004)
- Vol. 1850: V. D. Milman, G. Schechtman (Eds.), Geometric Aspects of Functional Analysis. Israel Seminar 2002-2003 (2004)
- Vol. 1851: O. Catoni, Statistical Learning Theory and Stochastic Optimization (2004)
- Vol. 1852: A.S. Kechris, B.D. Miller, Topics in Orbit Equivalence (2004)
- Vol. 1853: Ch. Favre, M. Jonsson, The Valutive Tree (2004)

- Vol. 1854: O. Saeki, Topology of Singular Fibers of Differential Maps (2004)
- Vol. 1855: G. Da Prato, P.C. Kunstmann, I. Lasiecka, A. Lunardi, R. Schnaubelt, L. Weis, Functional Analytic Methods for Evolution Equations. Editors: M. Iannelli, R. Nagel, S. Piazzera (2004)
- Vol. 1856: K. Back, T.R. Bielecki, C. Hipp, S. Peng, W. Schachermayer, Stochastic Methods in Finance, Bressanone/Brixen, Italy, 2003. Editors: M. Frittelli, W. Runggaldier (2004)
- Vol. 1857: M. Émery, M. Ledoux, M. Yor (Eds.), Séminaire de Probabilités XXXVIII (2005)
- Vol. 1858: A.S. Cherny, H.-J. Engelbert, Singular Stochastic Differential Equations (2005)
- Vol. 1859: E. Letellier, Fourier Transforms of Invariant Functions on Finite Reductive Lie Algebras (2005)
- Vol. 1860: A. Borisyuk, G.B. Ermentrout, A. Friedman, D. Terman, Tutorials in Mathematical Biosciences I. Mathematical Neurosciences (2005)
- Vol. 1861: G. Benettin, J. Henrard, S. Kuksin, Hamiltonian Dynamics – Theory and Applications, Cetraro, Italy, 1999. Editor: A. Giorgilli (2005)
- Vol. 1862: B. Helffer, F. Nier, Hypoelliptic Estimates and Spectral Theory for Fokker-Planck Operators and Witten Laplacians (2005)
- Vol. 1863: H. Führ, Abstract Harmonic Analysis of Continuous Wavelet Transforms (2005)
- Vol. 1864: K. Efstathiou, Metamorphoses of Hamiltonian Systems with Symmetries (2005)
- Vol. 1865: D. Applebaum, B.V. R. Bhat, J. Kustermans, J. M. Lindsay, Quantum Independent Increment Processes I. From Classical Probability to Quantum Stochastic Calculus. Editors: M. Schürmann, U. Franz (2005)
- Vol. 1866: O.E. Barndorff-Nielsen, U. Franz, R. Gohm, B. Kümmerer, S. Thorbjørnsen, Quantum Independent Increment Processes II. Structure of Quantum Lévy Processes, Classical Probability, and Physics. Editors: M. Schürmann, U. Franz, (2005)
- Vol. 1867: J. Sneyd (Ed.), Tutorials in Mathematical Biosciences II. Mathematical Modeling of Calcium Dynamics and Signal Transduction. (2005)
- Vol. 1868: J. Jorgenson, S. Lang, $\text{Pos}_n(\mathbb{R})$ and Eisenstein Series. (2005)
- Vol. 1869: A. Dembo, T. Funaki, Lectures on Probability Theory and Statistics. Ecole d'Été de Probabilités de Saint-Flour XXXIII-2003. Editor: J. Picard (2005)
- Vol. 1870: V.I. Gurarii, W. Lusky, Geometry of Müntz Spaces and Related Questions. (2005)
- Vol. 1871: P. Constantin, G. Gallavotti, A.V. Kazhikhov, Y. Meyer, S. Ukai, Mathematical Foundation of Turbulent Viscous Flows, Martina Franca, Italy, 2003. Editors: M. Cannone, T. Miyakawa (2006)
- Vol. 1872: A. Friedman (Ed.), Tutorials in Mathematical Biosciences III. Cell Cycle, Proliferation, and Cancer (2006)
- Vol. 1873: R. Mansuy, M. Yor, Random Times and Enlargements of Filtrations in a Brownian Setting (2006)
- Vol. 1874: M. Yor, M. Émery (Eds.), In Memoriam Paul-André Meyer – Séminaire de Probabilités XXXIX (2006)
- Vol. 1875: J. Pitman, Combinatorial Stochastic Processes. Ecole d'Été de Probabilités de Saint-Flour XXXII-2002. Editor: J. Picard (2006)
- Vol. 1876: H. Herrlich, Axiom of Choice (2006)
- Vol. 1877: J. Steuding, Value Distributions of L -Functions (2007)
- Vol. 1878: R. Cerf, The Wulff Crystal in Ising and Percolation Models, Ecole d'Été de Probabilités de Saint-Flour XXXIV-2004. Editor: Jean Picard (2006)
- Vol. 1879: G. Slade, The Lace Expansion and its Applications, Ecole d'Été de Probabilités de Saint-Flour XXXIV-2004. Editor: Jean Picard (2006)
- Vol. 1880: S. Attal, A. Joye, C.-A. Pillet, Open Quantum Systems I, The Hamiltonian Approach (2006)
- Vol. 1881: S. Attal, A. Joye, C.-A. Pillet, Open Quantum Systems II, The Markovian Approach (2006)
- Vol. 1882: S. Attal, A. Joye, C.-A. Pillet, Open Quantum Systems III, Recent Developments (2006)
- Vol. 1883: W. Van Assche, F. Marcellàn (Eds.), Orthogonal Polynomials and Special Functions, Computation and Application (2006)
- Vol. 1884: N. Hayashi, E.I. Kaikina, P.I. Naumkin, I.A. Shishmarev, Asymptotics for Dissipative Nonlinear Equations (2006)
- Vol. 1885: A. Telcs, The Art of Random Walks (2006)
- Vol. 1886: S. Takamura, Splitting Deformations of Degenerations of Complex Curves (2006)
- Vol. 1887: K. Habermann, L. Habermann, Introduction to Symplectic Dirac Operators (2006)
- Vol. 1888: J. van der Hoeven, Transseries and Real Differential Algebra (2006)
- Vol. 1889: G. Osipenko, Dynamical Systems, Graphs, and Algorithms (2006)
- Vol. 1890: M. Bunge, J. Funk, Singular Coverings of Toposes (2006)
- Vol. 1891: J.B. Friedlander, D.R. Heath-Brown, H. Iwaniec, J. Kaczorowski, Analytic Number Theory, Cetraro, Italy, 2002. Editors: A. Perelli, C. Viola (2006)
- Vol. 1892: A. Baddeley, I. Bárány, R. Schneider, W. Weil, Stochastic Geometry, Martina Franca, Italy, 2004. Editor: W. Weil (2007)
- Vol. 1893: H. Hanßmann, Local and Semi-Local Bifurcations in Hamiltonian Dynamical Systems, Results and Examples (2007)
- Vol. 1894: C.W. Groetsch, Stable Approximate Evaluation of Unbounded Operators (2007)
- Vol. 1895: L. Molnár, Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces (2007)
- Vol. 1896: P. Massart, Concentration Inequalities and Model Selection, Ecole d'Été de Probabilités de Saint-Flour XXXIII-2003. Editor: J. Picard (2007)
- Vol. 1897: R. Doney, Fluctuation Theory for Lévy Processes, Ecole d'Été de Probabilités de Saint-Flour XXXV-2005. Editor: J. Picard (2007)
- Vol. 1898: H.R. Beyer, Beyond Partial Differential Equations, On linear and Quasi-Linear Abstract Hyperbolic Evolution Equations (2007)
- Vol. 1899: Séminaire de Probabilités XL. Editors: C. Donati-Martin, M. Émery, A. Rouault, C. Stricker (2007)
- Vol. 1900: E. Bolthausen, A. Bovier (Eds.), Spin Glasses (2007)
- Vol. 1901: O. Wittenberg, Intersections de deux quadriques et pincesaux de courbes de genre 1, Intersections of Two Quadrics and Pencils of Curves of Genus 1 (2007)
- Vol. 1902: A. Isaev, Lectures on the Automorphism Groups of Kobayashi-Hyperbolic Manifolds (2007)
- Vol. 1903: G. Kresin, V. Maz'ya, Sharp Real-Part Theorems (2007)
- Vol. 1904: P. Giesl, Construction of Global Lyapunov Functions Using Radial Basis Functions (2007)

Vol. 1905: C. Prévôt, M. Röckner, A Concise Course on Stochastic Partial Differential Equations (2007)

Vol. 1906: T. Schuster, The Method of Approximate Inverse: Theory and Applications (2007)

Vol. 1907: M. Rasmussen, Attractivity and Bifurcation for Nonautonomous Dynamical Systems (2007)

Vol. 1908: T.J. Lyons, M. Caruana, T. Lévy, Differential Equations Driven by Rough Paths, Ecole d'Été de Probabilités de Saint-Flour XXXIV-2004 (2007)

Vol. 1909: H. Akiyoshi, M. Sakuma, M. Wada, Y. Yamashita, Punctured Torus Groups and 2-Bridge Knot Groups (I) (2007)

Vol. 1910: V.D. Milman, G. Schechtman (Eds.), Geometric Aspects of Functional Analysis. Israel Seminar 2004-2005 (2007)

Vol. 1911: A. Bressan, D. Serre, M. Williams, K. Zumbrun, Hyperbolic Systems of Balance Laws. Cetraro, Italy 2003, Editor: P. Marcati (2007)

Vol. 1912: V. Berinde, Iterative Approximation of Fixed Points (2007)

Vol. 1913: J.E. Marsden, G. Misiolek, J.-P. Ortega, M. Perlmutter, T.S. Ratiu, Hamiltonian Reduction by Stages (2007)

Vol. 1914: G. Kutniok, Affine Density in Wavelet Analysis (2007)

Vol. 1915: T. Bıyıkoglu, J. Leydold, P.F. Stadler, Laplacian Eigenvectors of Graphs. Perron-Frobenius and Faber-Krahn Type Theorems (2007)

Vol. 1916: C. Villani, F. Rezakhanlou, Entropy Methods for the Boltzmann Equation. Editors: F. Golse, S. Olla (2008)

Vol. 1917: I. Veselić, Existence and Regularity Properties of the Integrated Density of States of Random Schrödinger (2008)

Vol. 1918: B. Roberts, R. Schmidt, Local Newforms for $\mathrm{GSp}(4)$ (2007)

Vol. 1919: R.A. Carmona, I. Ekeland, A. Kohatsu-Higa, J.-M. Lasry, P.-L. Lions, H. Pham, E. Taflin, Paris-Princeton Lectures on Mathematical Finance 2004. Editors: R.A. Carmona, E. Çinlar, I. Ekeland, E. Jouini, J.A. Scheinkman, N. Touzi (2007)

Vol. 1920: S.N. Evans, Probability and Real Trees. Ecole d'Été de Probabilités de Saint-Flour XXXV-2005 (2008)

Vol. 1921: J.P. Tian, Evolution Algebras and their Applications (2008)

Vol. 1922: A. Friedman (Ed.), Tutorials in Mathematical BioSciences IV. Evolution and Ecology (2008)

Vol. 1923: J.P.N. Bishwal, Parameter Estimation in Stochastic Differential Equations (2008)

Vol. 1924: M. Wilson, Littlewood-Paley Theory and Exponential-Square Integrability (2008)

Vol. 1925: M. du Sautoy, L. Woodward, Zeta Functions of Groups and Rings (2008)

Vol. 1926: L. Barreira, V. Claudia, Stability of Nonautonomous Differential Equations (2008)

Vol. 1927: L. Ambrosio, L. Caffarelli, M.G. Crandall, L.C. Evans, N. Fusco, Calculus of Variations and Non-Linear Partial Differential Equations. Cetraro, Italy 2005. Editors: B. Dacorogna, P. Marcellini (2008)

Vol. 1928: J. Jonsson, Simplicial Complexes of Graphs (2008)

Vol. 1929: Y. Mishura, Stochastic Calculus for Fractional Brownian Motion and Related Processes (2008)

Vol. 1930: J.M. Urbano, The Method of Intrinsic Scaling. A Systematic Approach to Regularity for Degenerate and Singular PDEs (2008)

Vol. 1931: M. Cowling, E. Frenkel, M. Kashiwara, A. Valette, D.A. Vogan, Jr., N.R. Wallach, Representation Theory and Complex Analysis. Venice, Italy 2004. Editors: E.C. Tarabusi, A. D'Agnolo, M. Picardello (2008)

Vol. 1932: A.A. Agrachev, A.S. Morse, E.D. Sontag, H.J. Sussmann, V.I. Utkin, Nonlinear and Optimal Control Theory. Cetraro, Italy 2004. Editors: P. Nistri, G. Stefani (2008)

Vol. 1933: M. Petkovic, Point Estimation of Root Finding Methods (2008)

Vol. 1934: C. Donati-Martin, M. Émery, A. Rouault, C. Stricker (Eds.), Séminaire de Probabilités XLI (2008)

Vol. 1935: A. Unterberger, Alternative Pseudodifferential Analysis (2008)

Vol. 1936: P. Magal, S. Ruan (Eds.), Structured Population Models in Biology and Epidemiology (2008)

Vol. 1937: G. Capriz, P. Giovine, P.M. Mariano (Eds.), Mathematical Models of Granular Matter (2008)

Vol. 1938: D. Auroux, F. Catanese, M. Manetti, P. Seidel, B. Siebert, I. Smith, G. Tian, Symplectic 4-Manifolds and Algebraic Surfaces. Cetraro, Italy 2003. Editors: F. Catanese, G. Tian (2008)

Vol. 1939: D. Boffi, F. Brezzi, L. Demkowicz, R.G. Durán, R.S. Falk, M. Fortin, Mixed Finite Elements, Compatibility Conditions, and Applications. Cetraro, Italy 2006. Editors: D. Boffi, L. Gastaldi (2008)

Vol. 1940: J. Banasiak, V. Capasso, M.A.J. Chaplain, M. Lachowicz, J. Miękiś, Multiscale Problems in the Life Sciences. From Microscopic to Macroscopic. Będlewo, Poland 2006. Editors: V. Capasso, M. Lachowicz (2008)

Vol. 1941: S.M.J. Haran, Arithmetical Investigations. Representation Theory, Orthogonal Polynomials, and Quantum Interpolations (2008)

Vol. 1942: S. Albeverio, F. Flandoli, Y.G. Sinai, SPDE in Hydrodynamic. Recent Progress and Prospects. Cetraro, Italy 2005. Editors: G. Da Prato, M. Röckner (2008)

Vol. 1943: L.L. Bonilla (Ed.), Inverse Problems and Imaging. Martina Franca, Italy 2002 (2008)

Vol. 1944: A. Di Bartolo, G. Falcone, P. Plaumann, K. Strambach, Algebraic Groups and Lie Groups with Few Factors (2008)

Vol. 1945: F. Brauer, P. van den Driessche, J. Wu (Eds.), Mathematical Epidemiology (2008)

Recent Reprints and New Editions

Vol. 1702: J. Ma, J. Yong, Forward-Backward Stochastic Differential Equations and their Applications. 1999 – Corr. 3rd printing (2007)

Vol. 830: J.A. Green, Polynomial Representations of GL_n , with an Appendix on Schensted Correspondence and Littelmann Paths by K. Erdmann, J.A. Green and M. Schöcker 1980 – 2nd corr. and augmented edition (2007)

Vol. 1693: S. Simons, From Hahn-Banach to Monotonicity (Minimax and Monotonicity 1998) – 2nd exp. edition (2008)

Vol. 470: R.E. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. With a preface by D. Ruelle. Edited by J.-R. Chazottes. 1975 – 2nd rev. edition (2008)

Vol. 523: S.A. Albeverio, R.J. Høegh-Krohn, S. Mazzucchi, Mathematical Theory of Feynman Path Integral. 1976 – 2nd corr. and enlarged edition (2008)

Contents

From Granular Matter to Generalized Continuum

<i>J.D. Goddard</i>	1
1 Introduction	1
1.1 Mathematical Preliminaries	3
1.2 Balances	4
2 Micromechanics	5
2.1 Granular Microstructure and Rotation	5
2.2 Graph Theory for Extrinsic Modes	7
2.3 Extrinsic Power	12
3 Energy-Based Homogenization	13
3.1 Intrinsic Moments and Continuum Fields	15
4 Conclusions	17
Appendix: Simplex and Edge-Complex Gradients	17
References	20

Generalized Kinetic Maxwell Type Models of Granular Gases

<i>A.V. Bobylev, C. Cercignani, and I.M. Gamba</i>	23
1 Introduction	23
2 Maxwell Models of the Boltzmann Equation	26
3 Isotropic Maxwell Model in the Fourier Representation.....	28
4 Models with Multiple Interactions	30
4.1 Statement of the General Problem	31
5 The General Problem in Fourier Representation	33
5.1 Existence and Uniqueness of Solutions	33
5.2 Large Time Asymptotics	34
5.3 Existence of Self-Similar Solutions	41
5.4 Properties of Self-Similar Solutions	42