Almost periodic functions and differential equations

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Preface

The theory of almost periodic functions was mainly created and published during 1924–1926 by the Danish mathematician Harald Bohr. Bohr's work was preceded by the important investigations of P. Bohl and E. Esclangon. Subsequently, during the 1920s and 1930s, Bohr's theory was substantially developed by S. Bochner, H. Weyl, A. Besicovitch, J. Favard, J. von Neumann, V. V. Stepanov, N. N. Bogolyubov, and others. In particular, the theory of almost periodic functions gave a strong impetus to the development of harmonic analysis on groups (almost periodic functions, Fourier series and integrals on groups). In 1933 Bochner published an important article devoted to the extension of the theory of almost periodic functions to vector-valued (abstract) functions with values in a Banach space.

In recent years the theory of almost periodic equations has been developed in connection with problems of differential equations, stability theory, dynamical systems, and so on. The circle of applications of the theory has been appreciably extended, and includes not only ordinary differential equations and classical dynamical systems, but wide classes of partial differential equations and equations in Banach spaces. In this process an important role has been played by the investigations of L. Amerio and his school, which are directed at extending certain classical results of Favard, Bochner, von Neumann and S. L. Sobolev to differential equations in Banach spaces.

We survey briefly the contents of our book. In the first three chapters we present the general properties of almost periodic functions, including the fundamental approximation theorem. From the very beginning we consider functions with values in a metric or Banach space, but do not single out the case of a finite-dimensional Banach space and, in particular, the case of the usual numerical almost periodic functions. Of the known proofs of the approximation theorem we present just one: a proof based on an idea of Bogolyubov. However, it should be noted that another instructive proof due to Weyl and based on the theory of compact operators in a Hilbert space appears in many textbooks on functional analysis.

Chapter 4 is devoted to the theory of N-almost periodic functions. In comparison with the corresponding chapter of the book *Almost-Periodic Functions* by B. M. Levitan (Gostekhizdat, Moscow (1953)), we have added a proof of the fundamental lemma of Bogolyubov about the structure of a relatively dense set.

Chapter 5 is concerned with the theory of weakly almost periodic functions developed mainly by Amerio.

Chapter 6 contains, as well as traditionally fundamental questions (the theorem of Bohl–Bohr about the integral, and Favard's theorem about the integral), more refined ones, for instance, the theorem of M. I. Kadets about the integral.

We mention especially Chapter 7 whose title is Stability in the sense of Lyapunov and almost periodicity. The two chapters that follow it are formally based on it. Actually, we use only the simplest results, and when there is a need to refer to more difficult propositions we give independent proofs. Therefore, Chapters 6–11 can be read independently of one another.

Chapter 8 contains Favard theory, by which we mean the theory of almost periodic solutions of linear equations in a Banach space. In Chapter 9 the results from the theory of monotonic operators are applied to the problem of the almost periodicity of solutions of functional equations. In Chapter 10 we give another approach to the problem of almost periodicity. Finally, Chapter 11 is slightly outside the framework of the main theme of our book. In it we give one of the possible abstract versions of the classical averaging principle of Bogolyubov.

Chapters 1–5 were written mainly by B. M. Levitan, and Chapters 6–11 by V. V. Zhikov.

The authors thank K. V. Valikov for his assistance with the reading of the typescript.

Translator's note

This translation has been approved by Professor Zhikov, to whom I am grateful for correcting my mistranslations and some misprints in the original Russian version.

Professor Zhikov has asked me to mention that the theory of Besicovitch almost periodic functions is not reflected fully enough in the book, since this theory has recently been applied in spectral theory and in the theory of homogenisation of partial differential equations with almost periodic coefficients. The additional references are, in the main, concerned with this theme.

L. W. Longdon

Contents

We prove some of the simplest properties of almost periodic functions; these are straight-forward consequences of the definition.

Property 1. *An almost periodic function $f: J \to X$ is compact in the sense that the set $\bar{\mathcal{R}}_f$ is compact.*

Proof. It is sufficient to prove that for any $\varepsilon > 0$, $\mathcal{R}_f$ contains a finite ε-net for $\mathcal{R}_f$. Let $l = l(\varepsilon)$ be the length in Definition 3 corresponding to a given ε. We set

$$\mathcal{R}_{f;l} = \{x \in \mathcal{R}_f : x = f(t), -l/2 \leqslant t \leqslant l/2\}.$$

From the continuity of f it follows that the set $\mathcal{R}_{f;l}$ is compact; we show that it is an ε-net for the set $\mathcal{R}_f$. Let $t_0 \in J$ be chosen arbitrarily, and take an ε-almost period $\tau = \tau_\varepsilon$ such that $-l/2 \leqslant t_0 + \tau \leqslant l/2$, that is,

$$-t_0 - l/2 \leqslant \tau \leqslant -t_0 + l/2.$$

Then

$$\rho(f(t_0+\tau), f(t_0)) \leqslant \varepsilon.$$

Because $t_0 + \tau \in [-l/2, l/2]$, the set $\mathcal{R}_{f;l}$ is an ε-net for $\mathcal{R}_f$, as we required to prove.

Remark. For numerical almost periodic functions (that is, when $X = R^1$) and for almost periodic functions with values in a finite-dimensional Banach space, Property 1 reduces to the following: if f is an almost periodic function, then $\mathcal{R}_f$ is bounded.

Property 2. *Let $f: J \to X$ be a continuous almost periodic function. Then f is uniformly continuous on J.*

Proof. We take an arbitrary $\varepsilon > 0$ and set $\varepsilon_1 = \varepsilon/3$ and $l = l(\varepsilon_1)$. The function f is uniformly continuous in the closed interval $[-1, 1+l]$, that is, there is a positive number $\delta = \delta(\varepsilon_1)$ (without loss of generality we may assume that $\delta < 1$) such that

$$\rho(f(s''), f(s')) < \varepsilon_1 \tag{2}$$

whenever $|s'' - s'| < \delta$, $s', s'' \in J$. Now let t', t'' be any numbers from J for which $|t' - t''| < \delta$. We take a $\tau = \tau_{\varepsilon_1}$ with $0 \leqslant t' + \tau_{\varepsilon_1} \leqslant l$, that is, $-t' \leqslant \tau_{\varepsilon_1} \leqslant -t' + l$. Then $t'' + \tau_{\varepsilon_1} \in [-1, 1+l]$. We set $s' = t' + \tau_{\varepsilon_1}$ and $s'' = t'' + \tau_{\varepsilon_1}$. From (1), (2) and the triangle inequality we have

$$\rho(f(t''), f(t')) \leqslant \rho(f(t''), f(s'')) + \rho(f(s''), f(s'))$$
$$+ \rho(f(s'), f(t')) < \varepsilon.$$

Property 3. *Let $f_n : J \to X$, $n = 0, 1, 2, \ldots$, be a sequence of continuous almost periodic functions that converges uniformly on J to a function f. Then f is almost periodic.*

Proof. We take an arbitrary $\varepsilon > 0$ and let $n = n_\varepsilon$ be such that

$$\sup_{t \in J} \rho(f(t), f_{n_\varepsilon}(t)) \leq \varepsilon/3. \tag{3}$$

Let $\tau = \tau[f_{n_\varepsilon}]$ denote an $(\varepsilon/3)$-almost period of the function f_{n_ε}. Then it follows from (1), (3), and the triangle inequality that

$$\begin{aligned}\rho(f(t+\tau), f(t)) \leq \rho(f(t+\tau), f_{n_\varepsilon}(t+\tau))&\\ +\rho(f_{n_\varepsilon}(t+\tau), f_{n_\varepsilon}(t)) + \rho(f_{n_\varepsilon}(t), f(t))&\\ \leq \varepsilon&\end{aligned}$$

for all $t \in J$. This proves that f is almost periodic because the set of almost periods $\tau[f_{n_\varepsilon}]$ is relatively dense.

Property 4. *Let $x = f(t)$ be a continuous almost periodic function with values in a metric space X, and $y = g(x)$ be continuous on $\bar{\mathcal{R}}_f$ with values in a metric space X_1. Then $g[f(t)]$ is an almost periodic function with values in X_1.*

Proof. Since the set $\bar{\mathcal{R}}_f$ is compact and the function $g(x)$ is continuous on $\bar{\mathcal{R}}_f$, $g(x)$ is uniformly continuous on $\bar{\mathcal{R}}_f$. Therefore, for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $x', x'' \in \bar{\mathcal{R}}_f$ with $\rho(x', x'') \leq \delta$ we have

$$\rho_1(g(x''), g(x')) \leq \varepsilon.$$

Therefore, if τ is a δ-almost period for $f(t)$, then

$$\rho(f(t+\tau), f(t)) \leq \delta,$$

and so

$$\rho_1(g(f(t+\tau)), g(f(t))) \leq \varepsilon.$$

Corollary. *Let f be a continuous almost periodic function with values in a Banach space X. Then $\|f(t)\|^k$ is a continuous numerical almost periodic function for all $k > 0$.*

Property 5. *Suppose that f is an almost periodic function with values in a Banach space X. If the (strong) derivative f' exists and it is uniformly continuous on J, then f' is an almost periodic function.*

Proof. The proof uses the concept of an integral of a vector-valued function. In the case of continuous functions this is very simple because the Riemann integral exists with the usual fundamental properties (see, for example, G. E. Shilov, *Mathematical Analysis. Functions of a Single Variable*, Part 3, Ch. 12, § 12.5). By hypothesis, the derivative f' is uniformly continuous, and so for all $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $\|f'(t') - f'(t'')\| < \varepsilon$ whenever $|t' - t''| < \delta$. Therefore, if $1/n < \delta$, then

$$\left\| n\left[f\left(t + \frac{1}{n}\right) - f(t) \right] - f'(t) \right\| = \left\| n \int_0^{1/n} [f'(t+\eta) - f'(t)] \, d\eta \right\|$$

$$\leqslant n \int_0^{1/n} \|f'(t+\eta) - f'(t)\| \, d\eta < \varepsilon.$$

Consequently, the sequence of almost periodic functions $\phi_n(t) = n[f(t + 1/n) - f(t)]$ converges uniformly on J to $f'(t)$. Now we only need to use Property 3.

2 Bochner's criterion

The main results of this section are also valid for almost periodic functions with values in an arbitrary metric space X. But for simplicity we shall assume that X is a Banach space. We shall use the following notation:

X denotes a complex Banach space; $x, y, z, \ldots$ are elements of X, and $\|x\|$ is the norm of $x \in X$. $C(X)$ denotes the Banach space of continuous bounded functions $f: J \to X$ with the norm

$$\|f(t)\|_{C(X)} = \sup_{t \in J} \|f(t)\|,$$

and $\mathring{C}(X)$ is the subspace of $C(X)$ consisting of almost periodic functions. Let us note that the spaces $C(X)$ and $\mathring{C}(X)$ are invariant under translations, that is, $C(X)$ ($\mathring{C}(X)$) contains together with $f = f(s)$ the function $f^t(s) = f(s+t)$ for all $t \in J$.

1. Bochner's theorem. *Let $f: J \to X$ be a continuous function. For f to be almost periodic it is necessary and sufficient that the family of functions $H = \{f^h\} = \{f(t+h)\}$, $-\infty < h < \infty$, is compact in $C(X)$.*

Proof. (*a*) *Necessity.* We assume that f is an almost periodic function (see § 1, Definition 3). We denote by $\{r\}$ the set of all rational points on J and let $\{f^{h_n}\} = \{f(t+h_n)\}$ be an arbitrary sequence of functions from H. By using Property 1 and applying the diagonal process, we can select from the sequence $\{f(t+h_n)\}$ a subsequence (we denote it again by $\{f(t+h_n)\}$) which converges for any $r \in \{r\}$. We prove that the sequence $\{f(t+h_n)\}$ converges in $C(X)$. We take an arbitrary $\varepsilon > 0$ and let $l = l_\varepsilon$ be the corresponding length. Let

$\delta = \delta(\varepsilon)$ be chosen in accordance with Property 2. We subdivide the segment $[0, l]$ into p segments Δ_k $(k = 1, 2, \ldots, p)$ of length not greater than δ, and in each Δ_k we choose a rational point r_k. Suppose that $n = n_\varepsilon$ is chosen so that

$$\|f(r_k + h_n) - f(r_k + h_m)\| < \varepsilon \tag{4}$$

for $n, m \geqslant n_\varepsilon$ and $k = 1, 2, \ldots, p$. For every $t_0 \in J$ we find a $\tau = \tau_0$ such that

$$0 \leqslant t_0 + \tau \leqslant l \Leftrightarrow -t_0 \leqslant \tau \leqslant -t_0 + l.$$

Suppose that the number $t'_0 = t_0 + \tau$ falls in the interval Δ_{k_0} and that $r_{k_0} \in \Delta_{k_0}$ is the rational point chosen earlier. Then by our choice of δ we have

$$\begin{aligned} &\|f(t'_0 + h_n) - f(r_{k_0} + h_n)\| < \varepsilon, \\ &\|f(t'_0 + h_m) - f(r_{k_0} + h_m)\| < \varepsilon. \end{aligned} \tag{5}$$

It follows from (4) and (5) that

$$\begin{aligned} &\|f(t_0 + h_n) - f(t_0 + h_m)\| \\ &\quad \leqslant \|f(t_0 + h_n) - f(t'_0 + h_n)\| + \|f(t'_0 + h_n) - f(r_{k_0} + h_n)\| \\ &\qquad + \|f(r_{k_0} + h_n) - f(r_{k_0} + h_m)\| + \|f(r_{k_0} + h_m) - f(t'_0 + h_m)\| \\ &\qquad + \|f(t'_0 + h_m) - f(t_0 + h_m)\| < 5\varepsilon. \end{aligned}$$

Since $t_0 \in J$ was chosen arbitrarily, the last inequality implies that the sequence $\{f(t + h_n)\}$ converges in $C(X)$, that is, the set H is compact in $C(X)$.

(*b*) *Sufficiency.* We assume that the family $\{f(t + h)\}$, $-\infty < h < \infty$, is compact in $C(X)$ and prove that $f(t)$ is almost periodic (in the sense of Definition 3, § 1). First of all we show that f is a bounded function. For if this were not the case, then we could find a sequence of numbers h_n for which $\|f(h_n)\| \to \infty$. But then neither the sequence $\{f(t + h_n)\}$ nor any subsequence of it would be convergent at $t = 0$. From the boundedness of f it follows that the family of functions $\{f^h\} = \{f(t + h)\}$, $-\infty < h < \infty$ can be regarded as a set in $C(X)$.

By a criterion of Hausdorff, for all $\varepsilon > 0$ there are numbers $h_1, h_2, \ldots, h_p$ such that for all $h \in J$ there is a $k = k(h)$ such that

$$\sup_{t \in J} \|f(t + h) - f(t + h_k)\| < \varepsilon. \tag{6}$$

From (6) we have

$$\sup_{t \in J} \|f(t + h - h_k) - f(t)\| < \varepsilon,$$

that is, the numbers $h-h_k(h)$ $(k=1,2,\ldots,p)$ are ε-almost periods for $f(t)$. Now we only need to prove that the set of numbers $h-h_k$ is relatively dense. We set

$$L=\max_{1\leq k\leq p}|h_k|.$$

Then

$$h-L\leq h-h_k\leq h+L,$$

and since h is arbitrary this inequality implies that every interval of length $2L$ contains an ε-almost period for f.

2. Now we are going to deduce further properties of almost periodic functions that are obtained more simply from Bochner's criterion than from our definition.

Property 6. *The sum $f(t)+g(t)$ of two almost periodic functions is almost periodic. The product of an almost periodic function $f(t)$ and a numerical almost periodic function $\phi(t)$ is almost periodic.*

Proof. Let $\{h_n\}$ be an arbitrary sequence of real numbers. Firstly we extract from it a subsequence $\{h'_n\}$ such that the sequence of functions $\{f(t+h'_n)\}$ converges, and then a subsequence $\{h''_n\}$ of $\{h'_n\}$ for which the subsequence of functions $\{g(t+h''_n)\}$ is convergent. Then, clearly, the subsequence $\{f(t+h''_n)+g(t+h''_n)\}$ is convergent. Similarly, the product can be proved to be an almost periodic function.

Let $X_1, X_2, \ldots, X_n$ be Banach spaces, and let $X=\prod_k X_k$ be their cartesian product, that is, the Banach space with elements $x=(x_1, x_2, \ldots, x_n)$ and the norm

$$\|x\|=\sum_{k=1}^{n}\|x_k\|.$$

It follows easily from Bochner's criterion that if $f_1(t), f_2(t), \ldots, f_n(t)$ are almost periodic functions from J into $X_1, X_2, \ldots, X_n$, then the function $f(t)=(f_1(t), f_2(t), \ldots, f_n(t))$ is an almost periodic function from J into X. The next property is easily deduced from this remark.

Property 7. *Let $f_1(t), f_2(t), \ldots, f_n(t)$ be almost periodic functions from J into Banach spaces $X_1, X_2, \ldots, X_n$, respectively. Then for every $\varepsilon>0$, all the functions $f_1(t), f_2(t), \ldots, f_n(t)$ have a common relatively dense set of ε-almost periods.*

Proof. Suppose that τ is an ε-almost period for $f(t)=(f_1(t), f_2(t),\dots,f_n(t))$, that is,

$$\|f(t+\tau)-f(t)\|_X=\sum_{k=1}^{n}\|f_k(t+\tau)-f_k(t)\|_{X_k}<\varepsilon$$

for all $t\in J$. Obviously, for this τ we have

$$\|f_k(t+\tau)-f_k(t)\|<\varepsilon \quad (k=1,2,\dots,n),$$

as we required to prove.

3. The next property gives a condition for the compactness of a set of functions from $\mathring{C}(X)$, and is known as Lyusternik's theorem.

Lyusternik's theorem. *A set $M\subset\mathring{C}(X)$ is compact if and only if the following three conditions are satisfied:*

(1) *For every fixed $t_0\in J$ the set*

$$E_{t_0}=\{x\in X\colon x=f(t_0), f\in M\}\subset X$$

is compact.

(2) *The set M is equicontinuous, that is, for every $\varepsilon>0$ there is a $\delta=\delta(\varepsilon)$ such that $\|f(t')-f(t'')\|<\varepsilon$ whenever $|t'-t''|<\delta$ for all $f\in M$.*

(3) *The set M is equi-almost periodic, that is, for every $\varepsilon>0$ there is an $l=l_\varepsilon$ such that every interval $(\alpha,\alpha+l)\subset J$ contains a common ε-almost period for all $f\in M$.*

Proof. (*a*) ***Sufficiency.*** The proof is exactly the same as that of the necessity for the conditions in Bochner's theorem.

(*b*) *Necessity.* By the criterion of Hausdorff, for every $\varepsilon>0$ M contains a finite ε-net: $f_1, f_2,\dots,f_n$. Therefore, for all $f\in M$ there is a k_0, $1\leq k_0\leq n$, such that

$$\sup_{t\in J}\|f(t)-f_{k_0}(t)\|<\varepsilon. \tag{7}$$

For any $t_0\in J$, from (7) we obtain

$$\|f(t_0)-f_{k_0}(t_0)\|<\varepsilon,$$

and so the finite set of elements $f_1(t_0), f_2(t_0),\dots,f_n(t_0)$ forms a finite ε-net for the set E_{t_0}. Consequently, E_{t_0} is compact in X, that is, condition (1) of Lyusternik's theorem holds. Condition (2) follows from the uniform continuity of each $f_k(t)$ $(k=1,2,\dots,n)$ on J and from (7). Finally, condition (3) follows from (7) and Property 7.

Remark. For numerical almost periodic functions, condition (1) of Lyusternik's theorem can be restated as follows: the set E_{t_0} is bounded.

3. The connection with stable dynamical systems

Suppose that we are given a 1-parameter group of homeomorphisms of a metric space X, $S(t): X \to X (t \in J)$. If for any $x \in X$ the corresponding trajectory $x^t = S(t)x$ is a continuous function $J \to X$ we shall call $S(t)$ a *dynamical system* or *flow*.

A flow $S(t)$ is called *two-sidedly stable* or *equicontinuous* if the transformations $S(t)$ $(t \in J)$ are equicontinuous on every compact set from X.

The next property is obtained from Bochner's criterion.

Property 8. *Every compact trajectory of a two-sidedly stable flow is an almost periodic function.*

Proof. We set $f(t) = S(t)x$. Since a trajectory is compact, we can extract from any sequence $\{f(t_n)\}$ a fundamental subsequence $\{f(t'_n)\}$. The transformations $S(t)$ are equicontinuous on the set $\bar{\mathcal{R}}_f$, and so

$$\sup_{t \in J} \rho(f(t + t'_m), f(t + t'_n)) \leq \varepsilon$$

whenever $\rho(f(t'_m), f(t'_n)) \leq \delta$, that is, Bochner's criterion holds.

The converse holds in a certain sense: with each almost periodic function $f: J \to X$ can be associated a compact trajectory of a two-sidedly stable dynamical system. For if we consider in $C(X)$ a system of translates, then the trajectory $f^t = f(s + t)$ is compact. Since the distance between two elements of $C(X)$ is invariant under a translation, we have an isometric and so two-sidedly stable flow. It is worth noting that the difference between isometry and two-sided stability is essentially insignificant; if a two-sidedly stable flow is defined on a compact space X, then it can be made isometric by choosing the following metric

$$d(x_1, x_2) = \sup_{t \in J} \rho(S(t)x_1, S(t)x_2).$$

It is easy to see that the metric d is invariant under translation and topologically equivalent to the original metric ρ.

Let $f: J \to X$ be an almost periodic function. We denote by $\mathcal{H} = \mathcal{H}(f)$ the closure of the trajectory $f^t = f(s + t)$ in $C(X)$, and are going to show that $\mathcal{H}$ is minimal in the sense that any trajectory is everywhere dense in it. Suppose that $\hat{f} = \hat{f}(s)$ is any element from $\mathcal{H}$. Then for some sequence $\{t_m\} \subset J$ we have

$$\sup_{s \in J} \rho(f(s + t_m), \hat{f}(s)) \leq 1/m.$$

Therefore,

$$\sup_{s\in J} \rho(f(s), \hat{f}(s-t_m)) \leq 1/m,$$

that is, $\hat{f}(s-t_m) \to f(s)$ uniformly with respect to $s \in J$. The closure of the trajectory $\hat{f}^t$ contains f, and so it coincides with $\mathcal{H}$.

4 Recurrence

The minimal property of an almost periodic function proved in the last section is in fact a very simple property of abstract trajectories.

1. Let X be a Hausdorff topological space.

We shall call a 1-parameter semigroup of continuous operators $S(t): X \to X$ $(t \geq 0)$ simply a semigroup, and shall use the symbols x^t, $x(t)$ to denote the semitrajectory $S(t)x$ $(x \in X, t \geq 0)$. A function $x(t)$ is called a *trajectory* of a semigroup $S(t)$ if $x(t+\tau)$ $(t \geq 0)$ is a semitrajectory for every $\tau \in J$. A set $X_0 \subset X$ is called *invariant* if through each of its points passes at least one trajectory that is entirely contained in X_0. An example of a closed invariant set is the closure of a trajectory.

A set $X_0 \subset X$ is called *minimal* if it is closed, invariant, and does not contain proper closed invariant subsets.

Birkhoff's theorem. *If a semigroup has a compact semitrajectory, then there exists a compact minimal set.*

Proof. Let X_1 denote the closure of a compact semitrajectory. Obviously, the set $\bigcap_{t\geq 0} S(t)X_1$ is compact and invariant. We order the compact invariant sets by inclusion and apply Zorn's lemma, thus proving the existence of a minimal compact invariant set.

The trajectories that belong to a compact minimal set are conventionally called *recurrent* (in the sense of Birkhoff); an example of a recurrent trajectory is an almost periodic trajectory.

2. Suppose that we are given two semigroups defined on X and Y, respectively. Then there is an obvious semigroup on the cartesian product $X \times Y$ (the 'semigroup product').

Two trajectories $x(t)$, $y(t)$ are called *compatibly recurrent* if the trajectory $\{x(t), y(t)\}$ is recurrent in $X \times Y$. Clearly, compatible recurrence implies the recurrence of each component, but the converse does not hold.