Lecture Notes in Mathematics 1582

Werner Balser

From Divergent Power Series to Analytic Functions

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Theory and Application of Multisummable Power Series

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Preface

Since the second half of the last century, asymptotic expansions have been an important and very successful tool to understand the structure of solutions of ordinary and partial differential (or difference) equations. The by now classical part of this theory has been presented in many books on differential equations in the complex plane or related topics, by such distinguished authors as Wolfgang Wasow [Wa], Yasutaka Sibuya [Si], and many others. In my opinion, the most important result in this context is (in Wasow's terminology) the Main Asymptotic Existence Theorem: it states that to every formal solution of a differential equation, and every sector (in the complex plane) of sufficiently small opening, one can find a solution of the equation having the formal one as its asymptotic expansion. This solution, in general, is not uniquely determined, and the proofs given for this theorem (in various degrees of generality) do not provide a truly efficient way to compute such a solution, say, in terms of the formal solution. In fact, to prove this result, even for linear, but in particular non-linear equations, and to determine sharp bounds for the opening of the sector (or more generally, determine size and location of all sectors for which the theorem holds, for a given equation with "generic Stokes phenomenon") is not an easy task and has kept researchers busy until very recently; see, e.g., Ramis and Sibuya's paper on Hukuhara domains [RS 1] of 1989, or Wolfgang Jurkat's discussion of Asymptotic Sectors [Ju 1].

In the general theory of asymptotic expansions, the analogue to the Main Asymptotic Existence Theorem is usually called Ritt's Theorem, and is much easier to prove: Given any formal power series and any sector of arbitrary (but finite) opening (on the Riemann surface of the Logarithm), there exists a function, analytic in this sector and having the formal power series as its asymptotic expansion. This function is never uniquely determined — not even when the power series converges. To overcome this non-uniqueness, G.N. Watson [Wt 1/2] in 1911/12, and F. Nevanlinna [Ne] in 1918, introduced a special kind of asymptotic expansions, now commonly called *of Gevrey order* $k > 0$. These have the property that the analogue to Ritt's Theorem holds for sectors of opening up to π/k, in which cases the function again is not uniquely determined. If the opening is larger than π/k, however, a function which has a given formal power series as expansion of Gevrey order $k > 0$ may not exist, *but if it does, then it is uniquely determined.* In case of existence, the function can be represented as Laplace Transform of another function, which is analytic at the origin, and whose power series expansion is explicitly given in terms of the formal power series.

This achievement in the general theory of asymptotic expansions obviously escaped the attention of specialists for differential equations in the complex domain for quite some time: In a series of papers [Ho 1-3], J. Horn showed for linear systems of ODE, if the leading term of the coefficient matrix (at a singularity of second kind) has all distinct eigenvalues, and if the sector is large enough, then one has uniqueness in the Main Asymptotic Existence Theorem, and the function can be represented as a Laplace integral, or equivalently, in terms of (inverse) factorial series; however, he did not relate his observations to the general results of Watson and Nevanlinna. Later, Trjitzinsky [Tr] and Turrittin [Tu] treated somewhat more general situations, and they also pointed out the limitation of this approach to special cases.

In 1978/80, J.-P. Ramis [Ra 1/2] introduced his notion of k-summability of formal power series, which may best be interpreted as a formalization of the ideas of Watson and Nevanlinna. Applying this to linear systems of (meromorphic) ODE, he proved that every formal (matrix) solution to every such equation can be factored into a finite (matrix-) product of power series (times some explicit functions), so that each factor is k-summable, with k depending upon the factor. (In my treatment of *first level formal solutions*, [Ba 3-6] and [BJL], I had, more or less by accident, independently obtained the same result.) This factorization of formal solutions is not truly effective, so that this result did not really give a way to compute the resulting (matrix) function from the formal series.

More recently, J. Ecalle [Ec 1/2] presented a way to achieve this computation, introducing his definition of *multisummability*. In a way, his method differs from Ramis' definition of k-summability by cleverly enlarging the class of functions to which Laplace Transform, in some weak form, can be applied. He stated without proofs a large number of results concerning properties and applications of multisummability to formal solutions of (non-linear) systems of ODE. Based upon the described factorization of formal solutions of *linear* equations, it was more or less evident that multisummability applied to all formal solutions of linear equations. However, in the non-linear situation, the first complete proof for this was only very recently given by B.L.J. Braaksma [Br 1]. In which form this result carries over to formal solutions of difference, or other functional equations, is still an open problem upon which much work is done at present. Other directions of activity are the analysis of the Stokes phenomenon for (non-linear) systems, based on the theory of multisummability. Again, Ecalle has done some pioneering work in this direction, but has not given detailed proofs.

As is common in a rapidly growing field, it is difficult for a newcomer to appreciate the results achieved, because in the research papers and monographs at hand, every author chooses his/her own notations and has his/her own ideas of what is elementary or needs to be proved. Concerning notation, I do the same in this text, but at least I am consistent throughout the book, and I have included all proofs — or stated parts of them in form of exercises which I feel readers with some background in Complex Variables should be able to do, if they so wish. In winter semester of 1991/92, I taught a course in Ulm on multisummability. The material covered in this course has become the nucleus of this text, but was considerably expanded and, in some cases, presented in a more elegant form.

I am grateful to Y. Sibuya and B.L.J. Braaksma, who introduced me to the beautiful theory of multisummability in a joint seminar at the University of Minneapolis, in Spring of 1990, and I pray that they will consider me a good student. I also owe thanks to my student Andreas Beck, who went over the proofs and did all exercises, and to Sabine Lebhart for carefully typing the manuscript. Finally, I wish to apologize to my wife Christel, for spending parts of our vacation on a Dutch island on writing this text.

Ulm, 1994 Werner Balser

Contents

Chapter 1

Asymptotic Power Series

This chapter is to set the framework for the remaining ones: We define the notions of *asymptotic expansions*, and in particular, *Gevrey asymptotics*, and we show their main properties. Despite of the fact that we frequently speak of *differential algebras*, a reader is not required to know more than their definition: For our purpose, a differential algebra A is an algebra (over the field of complex numbers), together with a linear mapping d of A into itself which obeys the product rule, i.e. for every $a_1, a_2 \in A$ we have

$$d(a_1 a_2) = d(a_1)a_2 + a_1 d(a_2) .$$

1.1 Sectors

Throughout this text, we will deal with analytic functions which generally have a branch point at the origin. Therefore, it is convenient to think of these functions as defined in sectorial regions on the Riemann surface of the (natural) Logarithm. Consequently, complex numbers $z = re^{i\varphi}$ $(r > 0)$ will *not* be the same once their arguments φ differ by integer multiples of 2π. Strictly speaking, instead of complex numbers we deal with *pairs* (r, φ), but there is little risk of confusion in writing $re^{i\varphi}$ instead of (r, φ).

A *sector* (on the Riemann surface of the Logarithm) is defined to be a set of the form

$$S = S(d, \alpha, \rho) = \Big\{ z = re^{i\varphi} \mid 0 < r < \rho,\ d - \alpha/2 < \varphi < d + \alpha/2 \Big\} ;$$

where d is an arbitrary real number, α is a positive real, and ρ either is a positive real number or $+\infty$. We shall refer to d, resp. α, resp. ρ, as *the bisecting direction*, resp. *the opening*, resp. *the radius* of S. In particular, if $\rho = +\infty$, resp. $\rho < +\infty$, we will speak of S having *infinite*, resp. *finite, radius*. It should be kept in mind that we do *not* consider sectors of infinite opening, nor an empty sector. If we write $S(d, \alpha, \rho)$, then it shall go without saying that d, α, ρ are as above. In case $\rho = +\infty$, we mostly write $S(d, \alpha)$ instead of $S(d, \alpha, +\infty)$. A *closed sector* is a set of the form

$$\overline{S} = \overline{S}(d, \alpha, \rho) = \Big\{ z = re^{i\varphi} \mid 0 < r \le \rho,\ d - \alpha/2 \le \varphi \le d + \alpha/2 \Big\} ,$$

with d and α as before, but ρ a positive real number (i.e. never equal to $+\infty$). *Hence closed sectors always are of finite radius, and they never contain the origin.*

1.2 Analytic Functions in Sectors

Let S be a given sector, and let f be a function analytic in S (hence f may be multi-valued if S has opening larger than 2π). We say that f *is bounded at the origin*, if for every closed subsector $\overline{S_1}$ of S there exists a positive real constant c (depending upon $\overline{S_1}$) such that

$$|f(z)| \le c \quad \text{for every} \quad z \in \overline{S_1}\ .$$

If a complex constant, denoted by $f(0)$, exists such that

$$f(0) = \lim_{\substack{z\to 0\\ z\in S}} f(z)\ ,$$

uniformly in every closed subsector, we say that f *is continuous at the origin.*

If S has opening more than 2π, and f is analytic in S, we say that f is *single-valued*, provided that

$$f(z) = f(ze^{2\pi i}) \quad \text{whenever} \quad z, ze^{2\pi i} \in S\ .$$

We say that f (analytic in some sector S) is *analytic at the origin*, if f can be analytically continued to a sector $\tilde S$ of opening more than 2π, and if f, moreover, is single-valued and bounded at the origin (in $\tilde S$); a well-known result on removable singularities then implies that f has a convergent power series expansion about the origin.

Let $S = S(d, \alpha)$ be a sector of infinite radius, and let f be analytic in S, or at least analytic for all $z \in S$ with $|z| > \rho$. Suppose that $k > 0$ exists such that the following holds true:

To every φ with $|d - \varphi| < \alpha/2$ and every $r_0 > \rho$ there exist $\varepsilon, c_1, c_2 > 0$ such that for every $z = re^{i\tau}$ with $r \ge r_0$, $|\varphi - \tau| < \varepsilon$,

$$|f(z)| \le c_1 \exp\{c_2 |z|^k\}\ .$$

Then we shall say that f *is of exponential size at most* k (in S). This notion compares to that of (exponential) order as follows: If f is of exponential size at most k (in S), then it either is of order less than k, or of order equal to k and of finite type, and vice versa (see, e.g., [Bo] for the definition of order and type, and formulas relating both to the coefficients of an entire function).

Example: Mittag-Leffler's function

$$E_\alpha(z) = \sum_{n=0}^{\infty} z^n/\Gamma(1+\alpha n)\ , \quad \alpha > 0\ ,$$

is an entire function of exponential order $k = 1/\alpha$ and finite type, hence is of exponential size (at most) k in every sector of infinite radius. More generally, if $(f_n)_{n=0}^{\infty}$ is a sequence of complex numbers such that for some $c > 0$

$$|f_n| \leq c^n, \quad n \geq 0,$$

then

$$f(z) = \sum_{n=0}^{\infty} f_n z^n / \Gamma(1 + n/k)$$

is bounded by $E_{1/k}(c|z|)$, and therefore $f(z)$ is of exponential size at most k in every sector of infinite radius.

Exercises.

1. Show that if an analytic function f is of exponential size at most $k > 0$ in $S(d, \alpha)$, then the constants c_1, c_2 in the above estimate can be chosen independent of φ, provided φ is restricted to a closed subinterval of

$$(d - \alpha/2, d + \alpha/2).$$

For the following exercises, define

$$g(z) = \int_0^\infty e^{zt} t^{-t} dt = \int_0^\infty \exp[t(z - \log t)] dt$$

(integrating along the positive real axis); compare also [Nm].

2. Show that g is an entire function and compute its power series expansion.

3. For $\operatorname{Im} z = \frac{\pi}{2} + c$, $c > 0$, show that Cauchy's theorem allows to replace integration along the real axis by integration along the positive imaginary axis. Use this to show (for these z)

$$|g(z)| \leq \frac{1}{c}.$$

Prove a similar estimate for $\operatorname{Im} z = -(\frac{\pi}{2} + c)$.

4. For every sector S of infinite radius, not containing the positive real axis, show that $g(z)$ is of exponential size zero in S.

5. Use Phragmen-Lindelöf's theorem (see [SG]) or a direct lower estimate of $g(x)$ for $x > 0$, to show that in sectors S including the positive real axis, $g(z)$ cannot be of finite exponential size, hence g is of infinite order.

6. For c as above, let

$$f(z) = g(z + \frac{\pi}{2} + c).$$

Show that $f(z)$ remains bounded along every ray $\arg z = \varphi$ (for $|z| \to \infty$). Why does this not imply that f is of exponential order zero in every sector S?

1.3 Formal Power Series

Given a sequence $(f_n)_{n=0}^{\infty}$ of complex numbers, the formal object

$$\hat{f}(z) = \sum_0^\infty f_n z^n$$

is called a *formal power series* (in z). The set of all such formal power series is denoted by

$$\mathbb{C}[[z]] .$$

We say that $\hat{f}$ *converges*, or *is convergent*, if $\rho > 0$ exists so that the power series converges for all z with $|z| < \rho$, defining a function $f(z)$, analytic in a neighbourhood of the origin. We shall call f *the sum* of $\hat{f}$ (whenever $\hat{f}$ converges), and we write

$$f = \mathcal{S} \hat{f} .$$

The set of all convergent (formal) power series will be denoted by

$$\mathbb{C}\{z\} .$$

If $\hat{f}(z) = \sum f_n z^n$ is a formal power series so that for some positive C, K, and k we have

$$|f_n| \le C K^n \Gamma(1 + n/k) \quad \text{for every} \quad n \ge 0 ,$$

then we say that $\hat{f}$ is a formal power series *of Gevrey order* k^{-1}, and we write

$$\mathbb{C}[[z]]_{1/k}$$

for the set of all such formal power series. It is easily seen (compare the Exercises below) that $\mathbb{C}[[z]]_{1/k}$, under natural operations, forms a differential algebra.

Exercises. For $\hat{f}(z) = \sum f_n z^n$, $\hat{g}(z) = \sum g_n z^n$, and $\alpha \in \mathbb{C}$, define

$$\begin{aligned} (\hat{f} + \hat{g})(z) &= \sum_0^\infty (f_n + g_n) z^n , & (\hat{f}\hat{g})(z) &= \sum_0^\infty z^n \sum_{m=0}^n f_{n-m} g_m , \\ (\alpha \hat{f})(z) &= \sum_0^\infty (\alpha f_n) z^n , & \hat{f}'(z) &= \sum_0^\infty (n+1) f_{n+1} z^n . \end{aligned}$$

1. Show that $\mathbb{C}[[z]]$, with respect to addition and multiplication with scalars, as defined above, is a vector space over $\mathbb{C}$.

2. Show that $\mathbb{C}[[z]]$, with respect to multiplication of power series, as defined above, is a commutative algebra over $\mathbb{C}$.

3. Show that $\mathbb{C}[[z]]$, with respect to derivation as defined above, is a differential algebra over $\mathbb{C}$ (i.e. show that the map $\hat{f} \mapsto \hat{f}'$ is $\mathbb{C}$-linear and obeys the product rule $(\hat{f}\hat{g})' = \hat{f}'\hat{g} + \hat{f}\hat{g}'$).

4. Show that $\hat{e}$, the power series whose coefficients are all zero except for the constant term which equals one, acts as the unit element of $\mathbb{C}[[z]]$.

5. Show that the invertible elements of $\mathbb{C}[[z]]$, i.e. those $\hat{f}$ to which $\hat{g}$ exists such that $\hat{f}\hat{g} = \hat{e}$, are exactly those whose constant term is non-zero.

6. For arbitrary $k > 0$, show that $\mathbb{C}[[z]]_{1/k}$ again is a differential algebra over $\mathbb{C}$ (with respect to the same operations as above).

   Hint: Use the Beta Integral

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 (1-x)^{\alpha-1} x^{\beta-1} dx\,, \qquad \operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0\,,$$

   to show

$$\sum_{m=0}^{n} \Gamma(1+\tfrac{n-m}{k})\Gamma(1+\tfrac{m}{k}) \le (1+n)(1+n/k)\Gamma(1+n/k)\,.$$

   From this estimate, derive that $\mathbb{C}[[z]]_{1/k}$ is closed with respect to multiplication. Moreover, use Stirling's Formula to show

$$\frac{\Gamma(1+\frac{n+1}{k})}{\Gamma(1+\frac{n}{k})(n/k)^{1/k}} \longrightarrow 1 \quad (n \to \infty)\,,$$

   and from this, derive that $\mathbb{C}[[z]]_{1/k}$ is closed with respect to derivation.

7. For arbitrary $k > 0$, show $\hat{f} \in \mathbb{C}[[z]]_{1/k}$ invertible (in $\mathbb{C}[[z]]_{1/k}$) iff it is invertible in $\mathbb{C}[[z]]$, i.e. iff its constant term does not vanish.

   Hint: For

$$\hat{f}(z) = 1 + z\hat{h}(z)\,, \qquad \hat{h}(z) = \sum_{m=0}^{\infty} h_m z^m\,,$$

$$\hat{y}(z) = 1 + z\hat{x}(z)\,, \qquad \hat{x}(z) = \sum_{m=0}^{\infty} x_m z^m\,,$$

   show that $\hat{f}, \hat{y} \in \mathbb{C}[[z]]_{1/k}$ iff the series

$$\sum_{m=0}^{\infty} h_m z^{m/k}/\Gamma(1+m/k) \quad \text{and} \quad \sum_{m=0}^{\infty} x_m z^{m/k}/\Gamma(1+m/k)$$

   converge for $|z| < \rho$, with sufficiently small $\rho > 0$, defining functions $h(z)$, resp. $x(z)$, analytic in the variable $z^{1/k}$. Moreover, show that

$$\hat{f} \cdot \hat{y} = 1$$

is equivalent to the Volterra integral equation

$$x(z) + h(z) + \int_0^z k(z-t)x(t)dt \equiv 0 ,$$

with

$$k(z) = \sum_{m=0}^{\infty} h_m z^{(m+1)/k-1}/\Gamma((m+1)/k) ,$$

and use the theory of these equations to show that a (unique) solution $x(z)$ exists which has a convergent expansion of the above form, showing $\hat{y} \in \mathbb{C}[[z]]_{1/k}$ (for such $\hat{f}$ with constant term 1, but this is no restriction).

8. For arbitrary $k > 0$, show that for $\hat{f} = \sum f_n z^n \in \mathbb{C}[[z]]_{1/k}$ with $f_0 = 0$ we have
$$z^{-1}\hat{f}(z) := \sum_0^{\infty} f_{n+1} z^n \in \mathbb{C}[[z]]_{1/k} .$$

9. If we interprete $\mathbb{C}[[z]]_{1/k}$ for $k = \infty$ according to the convention $1/\infty = 0$, show that
$$\mathbb{C}[[z]]_{1/\infty} = \mathbb{C}\{z\} .$$
Check that the statements of Ex. 6–8 hold true for $k = \infty$.

1.4 Asymptotic Expansions

Given a function f, analytic in some sector S, and a formal power series $\hat{f}(z) = \sum f_n z^n \in \mathbb{C}[[z]]$, one says that $f(z)$ *asymptotically equals* $\hat{f}(z)$, as $z \to 0$ in S, or: $\hat{f}(z)$ *is the asymptotic expansion of* $f(z)$ in S, iff to every non-negative integer N and every closed subsector $\overline{S_1}$ of S there exists $C = C(N, \overline{S_1}) > 0$ such that for $z \in \overline{S_1}$

$$|z|^{-N} \left| f(z) - \sum_{n=0}^{N-1} f_n z^n \right| \leq C ;$$

in other words iff the functions

$$r_f(z, N) := z^{-N}\Big(f(z) - \sum_{n=0}^{N-1} f_n z^n\Big)$$

are bounded at the origin, for every $N \geq 0$. If this is so, we write for short

$$f(z) \cong \hat{f}(z) \quad \text{in} \quad S ,$$

and whenever we do, it will go without saying that S is a sector, f is analytic in S, and $\hat{f}$ is a formal power series.

For this type of asymptotics we refer the reader to standard texts as [Wa], [CL]. For our purposes, we only require the following

Proposition 1. a) *Given a sector* S *and a function* f, *analytic in* S *and*

$$f(z) \cong \hat{f}(z) \quad in \quad S$$

for some $\hat{f} = \sum f_n z^n \in \mathbb{C}[[z]]$, *the functions* $r_f(z,N)$ *(as above) are all continuous at the origin, and*

$$\lim_{\substack{z\to 0\\ z\in S}} r_f(z,N) = f_N \quad (N \geq 0)\,.$$

b) *Under the same assumptions as in a), suppose that the opening of* S *is larger than* 2π, *and that*

$$f(ze^{2\pi i}) = f(z) \quad whenever \quad z, ze^{2\pi i} \in S\,.$$

Then f *is analytic at the origin, and* $\hat{f}$ *converges and coincides with the power series expansion of* f *at the origin.*

Proof. a) Observe

$$r_f(z, N+1) = z^{-1}\Big(r_f(z,N) - f_N\Big)\,,$$

hence $r_f(z, N+1)$ bounded at the origin implies

$$\lim_{\substack{z\to 0\\ z\in S}} \Big(r_f(z,N) - f_N\Big) = 0\,.$$

b) Under our assumptions, $f(z)$ is a single-valued analytic function in a punctured disc around the origin, and remains bounded as $z \to 0$. Hence the origin is a removable singularity of f, i.e. $f(z)$ can be expanded into its power series about the origin. It follows right from the definition that the power series expansion is, at the same time, an asymptotic expansion, and from a) we conclude that an asymptotic expansion is uniquely determined by $f(z)$. This proves $\hat{f}(z)$ to converge and be the power series expansion for $f(z)$. □

Let $\boldsymbol{A}(S)$ be the set of all functions $f(z)$, analytic in the sector S and having an asymptotic expansion $\hat{f}(z)$. In view of Proposition 1 a), to every $f(z) \in \boldsymbol{A}(S)$ there is *precisely one* $\hat{f} \in \mathbb{C}[[z]]$ such that $f(z) \cong \hat{f}(z)$ in S. Therefore, we have a mapping

$$\begin{aligned} J:\ \boldsymbol{A}(S) &\longrightarrow \mathbb{C}[[z]] \\ f(z) &\longmapsto \hat{f}(z) = (Jf)(z)\,, \end{aligned}$$

mapping each f to its asymptotic expansion. Standard results on asymptotics (see [Wa], [CL]) show that $\boldsymbol{A}(S)$, under the natural operations, is a differential algebra, and J is a homomorphism between the two differential algebras $\boldsymbol{A}(S)$ and $\mathbb{C}[[z]]$. Moreover, Ritt's Theorem implies that J is surjective. However, J is not injective — even if we consider sectors of large opening. In the next section we are going to study another type of asymptotic expansions which are better suited to our purposes, since it will turn out that the corresponding map J, for sectors of sufficiently large opening, is injective (however, not surjective).