MATHEMATICAL METHODS OF STATISTICS

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PREFACE.

During the last 25 years, statistical science has made great progress, thanks to the brilliant schools of British and American statisticians, among whom the name of Professor R. A. Fisher should be mentioned in the foremost place. During the same time, largely owing to the work of French and Russian mathematicians, the classical calculus of probability has developed into a purely mathematical theory satisfying modern standards with respect to rigour.

The purpose of the present work is to join these two lines of development in an exposition of the mathematical theory of modern statistical methods, in so far as these are based on the concept of probability. A full understanding of the theory of these methods requires a fairly advanced knowledge of pure mathematics. In this respect, I have tried to make the book self-contained from the point of view of a reader possessing a good working knowledge of the elements of the differential and integral calculus, algebra, and analytic geometry.

In the first part of the book, which serves as a mathematical introduction, the requisite mathematics not assumed to be previously known to the reader are developed. Particular stress has been laid on the fundamental concepts of a distribution, and of the integration with respect to a distribution. As a preliminary to the introduction of these concepts, the theory of Lebesgue measure and integration has been briefly developed in Chapters 4—5, and the fundamental concepts are then introduced by straightforward generalization in Chapters 6—7.

The second part of the book contains the general theory of random variables and probability distributions, while the third part is devoted to the theory of sampling distributions, statistical estimation, and tests of significance. The selection of the questions treated in the last part is necessarily somewhat arbitrary, but I have tried to concentrate in the first hand on points of general importance. When these are fully mastered, the reader will be able to work out applications to particular problems for himself. In order to keep the volume of the book within reasonable limits, it has been necessary to exclude certain topics of great interest, which I had originally intended to treat, such as the theory of random processes, statistical time series and periodograms.

The theory of the statistical tests is illustrated by numerical examples borrowed from various fields of application. Owing to considerations of space, it has been necessary to reduce the number of these examples rather severely. It has also been necessary to restrain from every discussion of questions concerning the practical arrangement of numerical calculations.

It is not necessary to go through the first part completely before studying the rest of the book. A reader who is anxious to find himself *in medias res* may content himself with making some slight acquaintance with the fundamental concepts referred to above. For this purpose, it will be advisable to read Chapters 1—3, and the paragraphs 4.1—4.2, 5.1—5.3, 6.1—6.2, 6.4—6.6, 7.1—7.2, 7.4—7.5 and 8.1—8.4. The reader may then proceed to Chapter 13, and look up the references to the first part as they occur.

The book is founded on my University lectures since about 1930, and has been written mainly during the years 1942—1944. Owing to war conditions, foreign scientific literature was during these years only very incompletely and with considerable delay available in Sweden, and this must serve as an excuse for the possible absence of quotations which would otherwise have been appropriate.

The printing of the Scandinavian edition of the book has been made possible by grants from the Royal Swedish Academy of Science, and from Stiftelsen Lars Hiertas Minne. I express my gratitude towards these institutions.

My thanks are also due to the Editors of the Princeton Mathematical Series for their kind offer to include the book in the Series, and for their permission to print a separate Scandinavian edition.

I am further indebted to Professor R. A. Fisher and to Messrs Oliver and Boyd for permission to reprint tables of the t- and χ^2 distributions from »Statistical methods for research workers».

A number of friends have rendered valuable help during the preparation of the book. Professors Harald Bohr and Ernst Jacobsthal, taking refuge in Sweden from the hardships of the times, have read parts of the work in manuscript and in proof, and have given stimulating criticism and advice. Professor Herman Wold has made a very careful scrutiny of the whole work in proof, and I have greatly profited

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Table of Contents.

First Part.

MATHEMATICAL INTRODUCTION.

Chapters 1—3. Sets of Points.

Chapters 4—7. Theory of Measure and Integration in R_1.

Chapters 8—9. Theory of Measure and Integration in R_n.

Chapters 10—12. Various Questions.

Second Part.

RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS.

CHAPTERS 13—14. FOUNDATIONS.

CHAPTERS 15—20. VARIABLES AND DISTRIBUTIONS IN R_1.

Chapters 21—24. Variables and Distributions in R_n.

Third Part.

STATISTICAL INFERENCE.

Chapters 25—26. Generalities.

Chapters 35—37. Tests of Significance, II.

FIRST PART

MATHEMATICAL INTRODUCTION

CHAPTERS 1–3. SETS OF POINTS.

CHAPTER 1.

GENERAL PROPERTIES OF SETS.

1.1. Sets. — In pure and applied mathematics, situations often occur where we have to consider the collection of all possible objects having certain specified properties. Any collection of objects defined in this way will be called a *set*, and each object belonging to such a set will be called an *element of the set.*

The elements of a set may be objects of any kind: points, numbers, functions, things, persons etc. Thus we may consider e. g. 1) the set of all positive integral numbers, 2) the set of all points on a given straight line, 3) the set of all rational functions of two variables, 4) the set of all persons born in a given country and alive at the end of the year 1940. In the first part of this book we shall mainly deal with cases where the elements are points or numbers, but in this introductory chapter we shall give some considerations which apply to the general case when the elements may be of any kind.

In the example 4) given above, our set contains a finite, though possibly unknown, number of elements, whereas in the three first examples we obviously have to do with sets where the number of elements is not finite. We thus have to distinguish between *finite* and *infinite* sets.

An infinite set is called *enumerable* if its elements may be arranged in a *sequence:* $x_1, x_2, \ldots, x_n, \ldots$, such that a) every x_n is an element of the set, and b) every element of the set appears at a definite place in the sequence. By such an arrangement we establish a *one-to-one correspondence* between the elements of the given set and those of the set containing all positive integral numbers $1, 2, \ldots, n, \ldots$, which forms the simplest example of an enumerable set.

We shall see later that there exist also infinite sets which are *non-enumerable.* If, from such a set, we choose any sequence of elements $x_1, x_2, \ldots$, there will always be elements left in the set which do not appear in the sequence, so that a non-enumerable set may be said to represent a higher order of infinity than an enumerable set. It will be shown later (cf 4. 3) that the set of all points on a given straight line affords an example of a non-enumerable set.

1.2. Subsets, space. — If two sets S and S_1 are such that every element of S_1 also belongs to S, we shall call S_1 a *subset* of S, and write

$$S_1 < S \quad \text{or} \quad S > S_1.$$

We shall sometimes express this also by saying that S_1 is *contained in* S or *belongs to* S. — When S_1 consists of one single element x, we use the same notation $x < S$ to express that x *belongs to* S.

In the particular case when both the relations $S_1 < S$ and $S < S_1$ hold, the sets are called *equal*, and we write

$$S = S_1.$$

It is sometimes convenient to consider a set S which does not contain any element at all. This we call the *empty set*, and write $S = 0$. The empty set is a subset of any set. If we regard the empty set as a particular case of a finite set, it is seen that *every subset of a finite set is itself finite, while every subset of an enumerable set is finite or enumerable.* Thus the set of all integers between 20 and 30 is a finite subset of the set 1, 2, 3, . . ., while the set of all odd integers 1, 3, 5, . . . is an enumerable subset of the same set.

In many investigations we shall be concerned with the properties and the mutual relations of various subsets of a given set $\boldsymbol{S}$. The set $\boldsymbol{S}$, which thus contains the totality of all elements that may appear in the investigation, will then be called the *space* of the investigation. If, e. g., we consider various sets of points on a given straight line, we may choose as our space the set $\boldsymbol{S}$ of all points on the line. Any subset S of the space $\boldsymbol{S}$ will be called briefly a *set in* $\boldsymbol{S}$.

1.3. Operations on sets. — Suppose now that a space $\boldsymbol{S}$ is given, and let us consider various sets in $\boldsymbol{S}$. We shall first define the operations of *addition*, *multiplication* and *subtraction* for sets.

The *sum* of two sets S_1 and S_2,

$$S' = S_1 + S_2,$$

is the set S' of all elements *belonging to at least one* of the sets S_1 and S_2. — The *product*

$$S'' = S_1 S_2$$

is the *common part* of the sets, or the set S'' of all elements *belonging to both* S_1 and S_2. — Finally, the *difference*

$$S''' = S_1 - S_2$$

will be defined only in the case when S_2 is a subset of S_1, and is then the set S''' of all elements *belonging to* S_1 *but not to* S_2.

Thus if S_1 and S_2 consist of all points inside the curves C_1 and C_2 respectively (cf Fig. 1), $S_1 + S_2$ will be the set of all points inside at least one of the two curves, while $S_1 S_2$ will be the set of all points common to both domains.

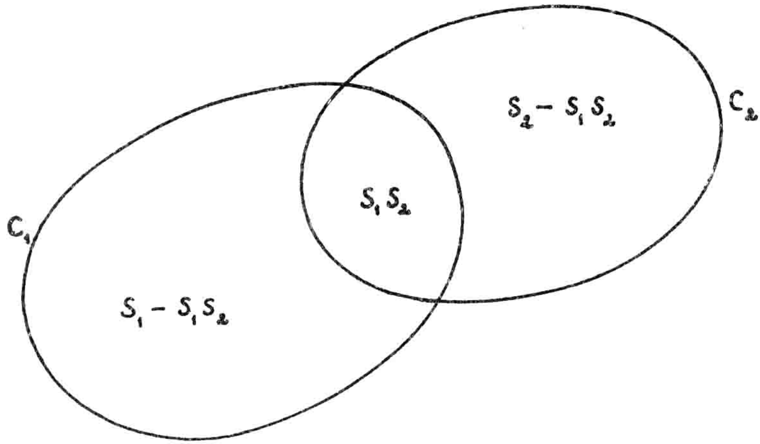

Fig. 1. Simple operations on sets.

The product $S_1 S_2$ is evidently a subset of both S_1 and S_2. The difference $S_n - S_1 S_2$, where n may denote 1 or 2, is the set of all points of S_n which do not belong to $S_1 S_2$.

In the particular case when S_1 and S_2 have no common elements, the *product* is empty, so that we have $S_1 S_2 = 0$. On the other hand, if $S_1 = S_2$ the *difference* is empty, and we have $S_1 - S_2 = 0$.

In the particular case when S_2 is a subset of S_1 we have $S_1 + S_2 = S_1$ and $S_1 S_2 = S_2$.

It follows from the symmetrical character of our definitions of the sum and the product that the operations of addition and multiplication are *commutative*, i. e. that we have

$$S_1 + S_2 = S_2 + S_1 \quad \text{and} \quad S_1 S_2 = S_2 S_1.$$

Further, a moment's reflection will show that these operations are also *associative* and *distributive*, like the corresponding arithmetic operations. We thus have

$$(S_1 + S_2) + S_3 = S_1 + (S_2 + S_3),$$
$$(S_1 S_2) S_3 = S_1 (S_2 S_3),$$
$$S_1 (S_2 + S_3) = S_1 S_2 + S_1 S_3.$$

It follows that we may without ambiguity talk of the sum or product of any finite number of sets:

$$S_1 + S_2 + \cdots + S_n \quad \text{and} \quad S_1 S_2 \cdots S_n,$$

where the order of terms and factors is arbitrary.

We may even extend the definition of these two operations to an enumerable sequence of terms or factors. Thus, given a sequence $S_1, S_2, \ldots$ of sets in $\boldsymbol{S}$, we define the sum

$$\sum_1^\infty S_\nu = S_1 + S_2 + \cdots$$

as the set of all elements *belonging to at least one* of the sets S_ν, while the product

$$\prod_1^\infty S_\nu = S_1 S_2 \cdots$$

is the set of all elements *belonging to all* S_ν. — We then have, e.g., $S(S_1 + S_2 + \cdots) = SS_1 + SS_2 + \cdots$.

Thus if S_ν denotes the set of all real numbers x such that $\frac{1}{\nu+1} \leqq x \leqq \frac{1}{\nu}$, we find that $\sum_1^\infty S_\nu$ will be the set of all x such that $0 < x \leqq 1$, while the product set will be empty, $\prod_1^\infty S_\nu = 0$. — On the other hand, if S_ν denotes the set of all x such that $0 \leqq x \leqq \frac{1}{\nu}$, the sum $\sum_1^\infty S_\nu$ will coincide with S_1, while the product $\prod_1^\infty S_\nu$ will be a set containing one single element, viz. the number $x = 0$.

For the operation of subtraction, an important particular case arises when S_1 coincides with the whole space $\boldsymbol{S}$. The difference

$$S^* = \mathbf{S} - S$$

is the set of all elements of our space which do not belong to S, and will be called the *complementary set* or simply the *complement* of S. We obviously have $S + S^* = \mathbf{S}$, $SS^* = 0$, and $(S^*)^* = S$.

It is important to observe that the complement of a given set S is relative to the space $\mathbf{S}$ in which S is considered. If our space is the set of all points on a given straight line L, and if S is the set of all points situated on the positive side of an origin O on this line, the complement S^* will consist of O itself and all points on the negative side of O. If, on the other hand, our space consists of all points in a certain plane P containing L, the complement S^* of the same set S will also include all points of P not belonging to L. — In all cases where there might be a risk of a mistake, we shall use the expression: S^* is the complement of S *with respect to* $\mathbf{S}$.

The operations of addition and multiplication may be brought into relation with one another by means of the concept of complementary sets. We have, in fact, for any finite or enumerable sequence $S_1, S_2, \ldots$ the relations

$$(S_1 + S_2 + \cdots)^* = S_1^* S_2^* \cdots, \qquad (1.3.1)$$
$$(S_1 S_2 \cdots)^* = S_1^* + S_2^* + \cdots.$$

The first relation expresses that *the complementary set of a sum is the product of the complements of the terms*. This is a direct consequence of the definitions. As a matter of fact, the complement $(S_1 + \cdots)^*$ is the set of all elements x of the space, of which it is not true that they occur in at least one S_ν. This is, however, the same thing as the set of all elements x which are absent from every S_ν, or the set of all x which belong to every complement S_ν^*, i.e. the product $S_1^* S_2^* \cdots$. The second relation is obtained from the first by substituting S_ν^* for S_ν. — For the operation of subtraction, we obtain by a similar argument the relation

$$S_1 - S_2 = S_1 S_2^*. \qquad (1.3.2)$$

The reader will find that the understanding of relations such as (1.3.1) and (1.3.2) is materially simplified by the use of figures of the same type as Fig. 1.

1.4. Sequences of sets. — When we use the word *sequence* without further specification, it will be understood that we mean a *finite or*