Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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S. Hildebrandt D. Kinderlehrer
M. Miranda (Eds.)

Calculus of Variations and Partial Differential Equations

Proceedings, Trento 1986

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continued on page 309

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CONTENTS

GLOBAL SOLVABILITY OF SECOND ORDER EVOLUTION EQUATIONS IN BANACH SCALES

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Dedicated to Hans Lewy

0. Introduction

Let us consider the abstract Cauchy problem

$$u'' + A(t)\, u = 0 \quad (t > 0) \ ,$$

(0.1)

$$u(0) = u_0, \quad u'(0) = u_1 \ ,$$

where $A(t)$ is a family of non-negative symmetric operators in a Hilbert space H.

Problem (0.1) is the abstract version of the mixed problem for some partial differential equations of evolutionary type, not necessarily of kovalevskian type; see [AS2].

In [AS2] it is proved that problem (0.1) is globally well-posed in the Banach scale generated by a suitable operator.

Here we pose the question of the global solvability of (0.1) in a general (not necessarily generated by an operator) Banach scale, and we give a first result in this direction (Theorem 3.3).

1. Global solvability of (0.1) in a Sobolev-type framework

We recall that a Hilbert triplet (V,H,V') is composed from a reflexive Banach space V, a Hilbert space H and a dense continuous embedding of V into H: denoting by V' the antidual of V and identifying H' with H via the Riesz isomorphism, we have the following chain of dense and continuous embeddings

$$V \subset H \subset V' \ .$$

We shall denote by $| \ |$ and $(,)$ respectively the norm and the inner product in H, and by $< , >$ the antiduality between V and V'. We note that $< , >$ coincides with $(,)$ on $H \times V$.

For each $t \geq 0$, we assume that $A(t) \in B(V,V')$, the space of bounded linear operators from V into V', and moreover that A(t) is symmetric and non-negative, i.e. that for each $v,w \in V$ we have

(1.1) $$<A(t)v,w> = \overline{<A(t)w,v>} \quad ,$$

(1.2) $$<A(t)v,v> \geq 0 \quad .$$

Thus, $V \times H$ is the natural space in which one can ask whether (0.1) is well-posed (the equation is then understood in a variational sense, cf. [LM]). The answer to such a question is in the affirmative under two assumptions. The first one has an algebraic character: A(t) must be coercive on V, uniformly on compact time intervals. The second one concerns the time regularity of the map $A(\bullet)$: it must be locally of bounded variation. More precisely, we have

Theorem 1.1 ([A2], cf. [DJ], [DT]). *Assume that* (1.1) *holds, and that for each* $T > 0$

(1.3) $$<A(t)v,v> \geq c(T) \, \|v\|_V^2 \quad (c(T) > 0), \ \forall \, v \in V, 0 \leq t \leq T \quad ,$$

(1.4) $$A \in BV([0,T], B(V,V')) \quad .$$

Then for every u_0 in V and u_1 in H there exists a unique solution of (0.1)

$$u \in C^0([0,+\infty[,V) \cap C^1([0,+\infty[,H) \quad .$$

Regularity results may also be proved: under suitable further assumptions the solution belongs in fact to $C^1([0,+\infty[,V) \cap C^2([0,+\infty[,H)$, see [A2], [A3] and the references quoted there. For further regularity, see [K].

If the assumptions (1.3) and (1.4) are somewhat relaxed, then even the *local* solvability in $V \times H$ may, in general, fail to be true. Actually any Sobolev-type framework is no more adequate, as it is shown by two delicate counterexamples, due to F. Colombini, E. De Giorgi and S. Spagnolo [CDS] and F. Colombini and S. Spagnolo [CS2], for the special case

(1.5) $$u'' + a(t) A_0 u = 0 \quad (t > 0) \quad ,$$
$$u(0) = u_0, \quad u'(0) = u_1 \quad ,$$

where A_0 is a positive, symmetric isomorphism of V onto V', and $a(t)$ is a continuous non-negative function. More precisely, [CDS] ([CS2]) exhibited a function $a(t) \geq 1$ ($a(t) \geq 0$) which is Hölder continuous for each exponent strictly less than 1 (infinitely differentiable), and a pair of initial data

$$u_0,\ u_1 \in D_\infty(A_0) \stackrel{\text{def}}{=} \bigcap_{k=1}^{\infty} D(A_0^k) \quad ,$$

such that no local solution to (1.5) exists.

2. A preliminary result: the global solvability of (1.5) in the analytic-type framework

The above counterexamples are straightforward abstract version of results proved by [CDS] and [CS2] for the hyperbolic equation

$$(2.1) \qquad \frac{\partial^2 u}{\partial t^2} - a(t)\frac{\partial^2 u}{\partial x^2} = 0 \quad (x \in \mathbf{R},\ t > 0) \quad ,$$

under the periodic boundary condition $u(x,t) = u(x + 2\pi, t)$.

On the other hand, [CDS] provided also a positive result (1): if the initial data are *analytic* 2π -periodic functions, then the Cauchy problem for (2.1) admits a classical (C^2) solution for all time.

In order to generalize such a result to the abstract Cauchy problem (1.5), we need to introduce the following definition, cf. [LM]:

Definition 2.1. Let $B : D(B) \subset H \rightarrow H$ be a closed linear operator. A vector v in $D_\infty(B)$ is a B-*analytic vector* iff there exists some $r > 0$ such that

$$\sum_{j=0}^{\infty} \left(|B^j v| \frac{r^j}{j!} \right)^2 < \infty \quad .$$

The supremum of admissible r in (2.2) is called the B-analyticity radius of v, and is denoted by $r_B(v)$.

Theorem 2.1 ([AS2], cf. [CDS], [A1], [AS1]). *Assume that* A_0 *is a positive symmetric isomorphism of* V *onto* V', *and that* $a(t)$ *is a continuous non-negative function.*

Then for each pair (u_0,u_1) of $A_0{}^{1/2}$-analytic vectors (2) there exists a unique global solution to (1.5). The solution and its derivative are $A_0{}^{1/2}$-analytic vectors for all time, with

$$\min \left\{ r_{A_0^{1/2}}(u(t)),\ r_{A_0^{1/2}}(u'(t)) \right\} = \text{constant} \quad (\forall\, t > 0) \quad .$$

Remark 2.1. The above theorem is a preliminary step to solve [AS1] for all time the Cauchy problem for the *nonlinear* equation

(2.2) $$u'' + m(\langle A_0u,u\rangle)\, A_0u = 0 \quad (t > 0) \quad ,$$

where the function $m(\bullet)$ is merely assumed to be continuous and non-negative, and A_0^{-1} is a compact operator in H. This improves the results of S. Bernstein [Be] and S.I. Pohozaev [P], and answers affirmatively a question raised by J.L. Lions [L]. Of course, uniqueness for the Cauchy problem for (2.2) may fail.

We note that some authors (G.F. Carrier [C], R. Narasimha [N]) interpreted (2.2) for $m(\rho) = c^2 + \varepsilon\rho$, $\varepsilon > 0$, as a nonlinear approximate model for the transversal vibrations of an elastic string.

3. The global solvability of (0.1) in the Banach scale generated by an operator

The above investigations show that if we want to relax the assumptions (1.3) and (1.4) in the study of (0.1), then we must introduce a richer framework: an analytic-type framework. Such a tool is provided by the notion of scale of Banach spaces, or Banach scale.

Definition 3.1. A *Banach scale* is a family of Banach spaces $(X_r)_{0<r<R}$ $(0 < R \leq \infty)$, with norms $|\ |_r$, such that

$$X_r \subset X_s \text{ with } |\ |_s \leq |\ |_r \text{ whenever } s \leq r \ .$$

We set

$$r(v) =^{\text{def}} \sup \{r : v \in X_r\} \ .$$

The number $r(v)$ is called the radius of v (in the scale (X_r)).

Definition 3.2. A Banach scale (X_r) is *dense* (in itself) iff X_r is dense in X_s whenever $s \leq r$.

Definition 3.3. A Banach scale (X_r) is *compatible* with the Hilbert triplet (V,H,V') iff $X_r \subset V$ with continuous inclusion for each r, and X_{r_0} is dense in H for some r_0.

The set of B-analytic vectors is, in an obvious way, a Banach scale:

Definition 3.4. Let $B : D(B) \subset H \to H$ be a closed linear operator. We set

$$X_r(B) =^{def} \left\{ v \in D_\infty(B) : |v|_r =^{def} \left(\sum_{j=0}^{\infty} \left(|B^j v| \frac{r^j}{j!} \right)^2 \right)^{1/2} < \infty \right\}.$$

We say that the scale $(X_r(B))$ is *generated* by the operator B. It may be proved (see [AS2]) that such a scale is dense and compatible with (V,H,V') if B is a self-adjoint operator in H with $D(B) \subset V$.

Definition 3.5. Let (X_r) be a dense Banach scale, compatible with (V,H,V'). A *radius function* for (0.1) is any continuous non-increasing function $y : [0,T(y)[\to]0,\infty[$, $0 < T(y) \le \infty$, with the following property: whenever the initial data u_0 and u_1 have a radius in (X_r) larger or equal than $y(0)$, then (0.1) admits a unique solution u in the class $C^0([0,T(y)[,V) \cap C^1([0,T(y)[,H)$, which moreover satisfies

$$(3.1) \qquad u \in \bigcap_{s<y(T)} W^{2,1}([0,T],X_s) \quad \text{for each} \quad T \in]0,T(y)[$$

(so the radius of $u(t)$ and $u'(t)$ are estimated from below by $y(t)$).

Definition 3.6. Let (X_r) be a dense Banach scale, compatible with (V,H,V'). The Cauchy problem (0.1) is *locally (globally) well-posed* in (X_r) iff for each $r_0 > 0$ there exists a radius function y for (0.1) with $y(0) = r_0$ (and y is defined for all time).

Theorem 2.1 may be rewritten in this terminology as follows:

Theorem 2.1'. *The Cauchy problem* (1.5) *is globally well-posed in the Banach scale* $(X_r(A_0^{1/2}))$, *with radius functions constant in time.*

The linear abstract version of the Cauchy-Kovalevskaya theorem, due to T. Yamanaka [Y] and L.V. Ovsjannikov [O], may be generalized to the case of second order equations, yielding the

Theorem 3.1 [C]. *Let* (X_r) *be a dense Banach scale, compatible with* (V,H,V'). *Assume that*

i) $A(t)$ *is of analytic order* ≤ 2 *in the scale* (X_r), *i.e.* $A(t)$ *maps* X_r *into* X_s *whenever* $s < r$, *with*

$$|A(t)v|_s \le c(T)(r-s)^{-2} \, |v|_r \quad (s < r, 0 \le t \le T) \;;$$

ii) *for each* $v \in X_r$, *the map* $A(\bullet)v$ *is a strongly measurable* X_s*-valued function, whenever* $s < r$.

Then (0.1) is *locally* well-posed in the scale (X_r), with radius functions which die down linearly.

We formulate the following

Open problem. To find compatibility assumptions between the operator map $A(\bullet)$ and a dense Banach scale (X_r) compatible with (V,H,V'), which ensure that (0.1) is *globally* well-posed in (X_r).

In [AS2] a partial answer has been provided for the case of the Banach scale generated by an operator:

Theorem 3.2 [AS2]. *Assume that* $A \in C^0([0,+\infty[,B(V,V'))$.

Let $B : V \to H$ *be a linear operator such that*

i) *the graph norm of* B *is equivalent to the initial one of* V;

ii) $A(t)$ *has Sobolev order* ≤ 2 *with respect to* B, *i.e. for each* $T > 0$

$$| A(t)v | \leq c(T) (| v | + | Bv | + | B^2 v |) \quad (0 \leq t \leq T) \ ;$$

iii) $A(t)$ *maps the space of* B-*analytic vectors into* $D_\infty(B)$;

iv) *for each* B-*analytic vector* v *and each* $j \in \mathbf{N}$, *the map* $B^j A(\bullet)v$ *is a strongly measurable* H-*valued function.*

Let $R > 0$ *be such that*

v) *the scale* $(X_r(B))_{0<r<R}$ *is dense, and compatible with* (V,H,V');

vi) $$|\left(B^j A(t) - A(t)B^j\right) v | \leq K j < A(t)B^j v, B^j v>^{1/2} + K^2 (j+2)!\, R^{-j} \sum_{h=0}^{j} \frac{R^h}{h!} |B^h v| \ .$$

Then (0.1) is globally well-posed in the scale $(X_r)_{0<r<R}$ with radius functions which die down exponentially $(\sim \exp(-3Kt))$.

The above theorem generalizes some of the results on hyperbolic partial differential equations of second order due to E. Jannelli [J], which improved the earlier results of F. Colombini and S. Spagnolo [CS1] and J.-M. Bony and P. Schapira [BS], which in turn had improved a classical result of S. Mizohata [M].

4. The global solvability of (0.1) in a general Banach scale

We provide here a second result in the direction of solving the above Open Problem, concerning the special case (1.5).

Theorem 3.3. *Assume that* A_0 *is a positive symmetric isomorphism of* V *onto* V', *and that* a(t) *is a continuous non-negative function.*

Let (X_r) *be a dense Banach scale, compatible with* (V,H,V'). *For each* r, *assume that*

i) $A_0^{1/4}$ *maps* X_r *into* X_s *whenever* $s < r$;

ii) X_s *is a Hilbert space with inner product* $(\ ,\ )_s$, *and* $A_0^{1/4}$ *is symmetric with respect to* $(\ ,\ )_s$ *on the subspace* X_r, *whenever* $s < r$;

iii) *there exists* $\frac{d}{ds}|v|_s^2$ *as a continuous function on* $X_r \times]0,r[$, *with*

$$|A_0^{1/4}v|_s^2 \leq c(r)\frac{d}{ds}|v|_s^2 \quad on \quad]0,r[\quad (\forall\, v \in X_r) \ .$$

Then (1.5) is *globally* well-posed in the scale (X_r), with radius functions constant in time.

Proof. Let $r_0 =^{\text{def}} \min\{r(u_0),r(u_1)\}$, and let $r_1 > r_0$.

If $s < r < r_1$, integrating iii) we get

$$\begin{aligned}(r-s)\ |A_0^{1/4}v|_s^2 &\leq \int_s^r |A_0^{1/4}v|_\rho^2\, d\rho \\ &\leq c(r_1)\left(|v|_r^2 - |v|_s^2\right) \\ &\leq c(r_1)\ |v|_r^2\end{aligned}$$

for each $v \in X_{r_1}$ hence for each $v \in X_r$ since the scale (X_r) is assumed to be dense. Then condition i) of Theorem 3.1 holds true in the subscale $(X_r)_{0<r<r_1}$ whence (1.5) is locally well-posed in such a scale.

Moreover, there exists $T = T(r_0,r_1) > 0$ with the property that if the radii of the initial data are larger than or equal to r_1, then the solutions belong to $C^2([0,T],X_{r_0})$. If we provide a suitable a priori estimate of such solutions, then, using in an essential way the fact that the scale (X_r) is assumed to be dense in itself, the local existence result of Theorem 3.1 may be improved to a *global* one. Such an estimate will be deduced from the study of a certain energy function, which we now introduce.

Let $b \in C^1([0,T])$, $b(t) > 0$, and let y(t) be an absolutely continuous non-increasing function on [0,T], $0 < y(t) < r_0$ (we will choose b and y later on). Let us define the energy function E(t) by setting

$$E(t) \stackrel{def}{=} b(t)\ |A_0^{1/2}u(t)|^2_{y(t)} + |u'(t)|^2_{y(t)} \quad (0 \le t \le T) \ .$$

Since $A_0{}^{1/4}$ is $(\ ,\ )_r$-symmetric, we have

(3.2) $$E'(t) = b'\ |A_0^{1/2}u|^2_y + 2\mathrm{Re}(bA_0u + u'',u')_y + y'\partial E/\partial y$$

$$\le |b'|\,b^{-1}E + 2\,|b - a|\ |(A_0u,u')_y| + y'\partial E/\partial y$$

$$\le |b'|\,b^{-1}E + |b - a|\,b^{-1/2}\left(b\,|A_0^{3/4}u|^2_y + |A_0^{1/4}u'|^2_y\right) + y'\partial E/\partial y$$

$$\le |b'|\,b^{-1}E + \left(c(r_0)\,|b - a|\,b^{-1/2} + y'\right)\partial E/\partial y \ .$$

We can now choose the functions b and y. For a fixed $\varepsilon > 0$, one can choose [CDS] the function b in such a way that

$$c(r_0)\int_0^T |b - a|\,b^{-1/2}dt < \varepsilon \ .$$

If we denote by $y(t)$ the function defined by

$$y(t) \stackrel{def}{=} r_0 - \varepsilon - c(r_0)\int_0^t |b - a|\,b^{-1/2}dx \quad (0 \le t \le T) \ ,$$

it follows from (3.2) that

(3.3) $$E'(t) \le c(r_0,T,\varepsilon)\,E(t) \quad (0 \le t \le T) \ ,$$

(3.4) $$y(t) \ge r_0 - 2\varepsilon \qquad (0 \le t \le T) \ .$$

By the classical argument of Gronwall, (3.3) gives an a priori estimate of the energy function E, hence of $|u'|_y$, and, in turn, of $|u|_y$. Then arguing as in Step 6 of Theorem 3.2 of [AS2], we can deduce, using the assumption that the scale (X_r) is dense in itself, that the solution to (1.5) exists on the whole interval $[0,T]$ as an $X_{r_0 - 2\varepsilon}$-valued $W^{2,1}$ function.

Since $\varepsilon > 0$ is arbitrary, we have proved that the constant map r_0 is a radius function for (1.5) on $[0,T]$. By considering (1.5) as a sequence of Cauchy problems on $[T_{n-1}, T_n]$, one can complete the proof.

Remark 3.1. As a simple application of Theorem 3.3, we get an easier proof of Theorem 2.1'. Indeed if we choose

$$X_r = D(\exp(rA_0^{1/2})), \quad |v|_r = \left(\sum_{j=0}^{\infty} |A_0^{j/4} v|^2 \frac{(2s)^j}{j!} \right)^{1/2} ,$$

then the scale (X_r) is dense and compatible with (V,H,V'), since $A_0{}^{1/2}$ is a positive self-adjoint operator with $D(A_0{}^{1/2}) = V$ [AS2], and it is easy to check the other assumptions of Theorem 3.3. Now, such a scale is equivalent (with respect to (3.1)) to the scale generated by $A_0{}^{1/2}$.

Notes

(1) Since we are not interested in generalizing results based on the phenomenon of dependence domains, we report here from [CDS], and later on from [CS1], [CS2], [J], [BS], [M], only results relevant to the topic.

(2) The realization of A_0 in H happens to be a positive self-adjoint operator in H, hence it admits a unique positive self-adjoint square root $A_0{}^{1/2}$.

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