ANALYTIC FUNCTIONS of SEVERAL COMPLEX VARIABLES

ROBERT C. GUNNING
HUGO ROSSI

PRENTICE-HALL SERIES in MODERN ANALYSIS

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PREFACE

The general theory of analytic functions of several complex variables was formulated considerably later than the more familiar theory of analytic functions of a single complex variable. Some of the principal function-theoretic problems were attacked and a basic foundation for the subject was laid late in the nineteenth century by Weierstrass, and around the turn of the century by Cousin, Hartogs and Poincaré. Certain central problems, either trivial in one variable or peculiar to several variables, were left open. Significant work in many directions was achieved by Bergman, Behnke, Bochner and others, in papers appearing from about the mid-1920's until the present time. The peculiarities of several complex variables were well exposed and the central difficulties clearly stated by the time of the appearance of the book of Behnke and Thullen, but the main problems were still there. Then K. Oka brought into the subject a brilliant collection of new ideas based primarily in the earlier work of H. Cartan and, in a series of papers written between 1936 and 1953, systematically eliminated these problems. But Oka's work had a far wider scope, and it was H. Cartan who realized this and developed the algebraic basis in the theory. This was essentially put into its present form in the seminars of Cartan in Paris (1951–52 and 1953–54) and the vastly useful tools provided by sheaf theory were first systematically employed there. The deep and extensive work of Grauert and Remmert on complex analytic spaces was built upon this foundation, and the same is of course true for the impressive works of many others during the last decade and at the present time.

The intention of the present volume is to provide an extensive introduction to the Oka-Cartan theory and some of its applications, and to the general theory of analytic spaces. We have neither attempted to write an encyclopedia of the subject of analytic functions of several complex variables, nor even tried to cover everything that is known today in the two areas of principal emphasis. Many fascinating aspects of this broad and active field that might have been encompassed by a book of the same title have been omitted almost entirely; the reader must look elsewhere for the differential-geometric and algebraic-geometric sides of the subject, for the theory of automorphic functions and complex symmetric spaces, for the Bergman kernel function, and for applications to mathematical physics. An attempt has been made to append a rather complete bibliography of books and papers in the two areas on which this introduction concentrates, so that the reader can pursue these topics further at will.

This book has been written with the prospective student of several complex variables in mind. In fact, the main reason for writing this book has been the untenable lack of an adequate introduction to one of the most active mathematical fields of the day. Further, there have been many recent results which cast a new light on much of the introductory material, and these results should properly be exposed early in the development of the subject. We have tried both to arrange this book so that the fundamental techniques will be exposed as soon as possible, and at the same time to give a firm foundation for their use. Of course, as a very active field, several complex variables is still in a state of flux. There are many different approaches that an introduction such as this could take, and one's choice of the ideal organization of the material varies from year to year. Indeed, were we to rewrite this book from scratch starting today it would probably turn out to be a quite different book.

The prerequisites for reading this book are, essentially, a good undergraduate training in analysis (principally the classical theory of functions of one complex variable), algebra, and topology; references have been provided for any important bits of mathematical lore which we did not consider standard minimum equipment for beginning graduate students in mathematics or their equivalent. The book is divided into nine chapters numbered with Roman numerals; each chapter is subdivided into sections indicated by capital letters. The definitions, lemmas, theorems, jokes, etc., are numbered in one sequence within each Section; the principal formulas are numbered similarly in a separate sequence. An expression such as "Theorem III C2l" indicates a reference to the second numbered entity (in this case, a theorem) in Section C of Chapter III; for references within the same Chapter the Roman numeral will be dropped, and for references within the same Section the letter will often be dropped as well. References given in other forms will be left to the reader to decipher, with our best wishes for his success.

In somewhat more detail, the outline of the contents is as follows. Chapter I is in itself an introductory course in the subject (perhaps one semester in length). It presents, in outline, the essentials of the problems and some approaches to their solutions, and, in some special cases, includes the complete solutions. The discussion also shows the necessity for developing further techniques for tackling the problems, and thus motivates the remainder of the work. Except for Sections G and H, which are optional, the contents of this Chapter are prerequisite for what follows. However, to get into the subject most rapidly, bypassing the motivational portions, the reader may pass directly to Chapter II after reading Sections A, B, and C of Chapter I; Sections D through G are not needed until Chapter VI, and reading them can be postponed until specific references are given to them. Chapter II contains the local theory of analytic functions and varieties, and is the natural sequel to Chapter I. Beyond this point, there are several paths which the reader may follow, one of which of course is the straightforward plodding through the chapters as they occur. The reader mostly interested in the sheaf-theoretic aspects, especially in Cartan's famous Theorems A and B, may proceed next to Chapters IV, VI, and VIII. The reader interested rather in complex analytic spaces and their properties may proceed directly to Chapters III and V. The sheaf-theoretic notation and terminology introduced in Chapter IV are used in Chapter V for convenience, but none of the deeper properties are really required; however the discussion in Chapter VII does require some of the results of Chapter VI. The final chapter consists of an exposition, from the point of view of the preceding material, of pseudoconvexity.

This book developed from joint courses in several complex variables given by the authors at Princeton University during the academic years 1960–61 and 1962–63. It is a deep pleasure to both of us to be able to record here the debts of gratitude we owe to those who helped make this book possible. Lutz Bungart and Robin Hartshorne wrote and organized the lecture notes for our first course; these notes formed the kernel of the present work, and their reception encouraged us to proceed with the task. Thomas Bloom, William Fulton, Michael Gilmartin, and David Prill at Princeton aided in the revision of these notes and contributed many corrections and improvements. We are deeply indebted to Errett Bishop and Kenneth Hoffman, who read our tentative drafts with many helpful and inspiring comments. Finally, our thanks go to the typists of these various drafts; Caroline Browne, Eleanor Clark, Patricia Clark, and Elizabeth Epstein; and to the staff of Prentice-Hall, Inc.

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CONTENTS

CHAPTER I

HOLOMORPHIC FUNCTIONS

A. The Elementary Properties of Holomorphic Functions

The field of real numbers will be denoted by $\mathbb{R}$, and the field of complex numbers by $\mathbb{C}$; both are topological fields with the familiar structures. In studying the theory of functions of several complex variables, we are particularly interested in the space $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$, the Cartesian product of n copies of the complex plane. For the points of $\mathbb{C}^n$ we shall use the notation $z = (z_1, \ldots, z_n)$, where $z_j = x_j + iy_j \in \mathbb{C}$ and x_j, y_j are real numbers (and i is a square root of -1). The absolute value of a complex number z_1 will be denoted by $|z_1|$, and for $z \in \mathbb{C}^n$, we define

$$|z| = \max\{|z_j|;\ 1 \le j \le n\}.$$

An **open polydisc** (or **open polycylinder**) in $\mathbb{C}^n$ is a subset $\Delta(w; r) \subset \mathbb{C}^n$ of the form

$$(1) \qquad \Delta(w; r) = \Delta(w_1, \ldots, w_n; r_1, \ldots, r_n) = \{z \in \mathbb{C}^n \mid |z_j - w_j| < r_j, 1 \le j \le n\};$$

the point $w \in \mathbb{C}^n$ is called the **center** of the polydisc, and

$$r = (r_1, \ldots, r_n) \in \mathbb{R}^n, \quad (r_j > 0),$$

is called the **polyradius.** The closure of $\Delta(w; r)$ will be called the **closed polydisc** with center w and polyradius r, and will be denoted by $\bar{\Delta}(w; r)$. More generally, if $D_j \subset \mathbb{C}$ are any subdomains (connected open subsets) of the complex plane, the product set $D = D_1 \times \cdots \times D_n \subset \mathbb{C}^n$ will be called an **open polydomain.** A polydisc is the special case in which the sets D_j are discs; similarly, an **open polysquare** is the special case in which the sets D_j are open squares in the plane. The open polydiscs form a basis for the collection of open sets in the Cartesian product topology on $\mathbb{C}^n$. Considered only as a topological space (or as a real vector space), $\mathbb{C}^n$ is of course just the same as $\mathbb{R}^{2n}$, the ordinary Euclidean space of $2n$ dimensions. Thus we can impose on $\mathbb{C}^n$ in a natural manner any of the structures of $\mathbb{R}^{2n}$; for instance, the Lebesgue measure on $\mathbb{R}^{2n}$ becomes a measure on $\mathbb{C}^n$, which will be denoted by dV.

A complex-valued function f on a subset $D \subset \mathbb{C}^n$ is merely a mapping from D into the complex plane; the value of the function f at a point $z \in D$ will be denoted by $f(z)$, as usual.

1. Definition. *A complex-valued function* f *defined on an open subset* $\mathrm{D} \subset \mathbb{C}^n$ *is called* **holomorphic in** D *if each point* $\mathrm{w} \in \mathrm{D}$ *has an open neighborhood* U, $\mathrm{w} \in \mathrm{U} \subset \mathrm{D}$, *such that the function* f *has a power series expansion*

$$(2) \qquad f(z) = \sum_{\nu_1 \cdots \nu_n = 0}^{\infty} a_{\nu_1 \cdots \nu_n}(z_1 - w_1)^{\nu_1} \cdots (z_n - w_n)^{\nu_n}$$

which converges for all $\mathrm{z} \in \mathrm{U}$. *The set of all functions holomorphic in* D *will be denoted by* $\mathcal{O}_\mathrm{D}$.

Notice that polynomials in the functions $z_1, \ldots, z_n$ are holomorphic in all of $\mathbb{C}^n$. It is a familiar result from elementary analysis that a power series expansion of the form (2) is absolutely uniformly convergent in all suitably small open polydiscs $\Delta(w; r)$ centered at the point w. A first consequence of this observation is that the function f is continuous in such polydiscs $\Delta(w; r)$; and hence, any function holomorphic in D is also continuous in D. A second consequence is that the power series (2) can be rearranged arbitrarily and will still represent the function f. In particular, if the coordinates $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$ are given any fixed values $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n$, then this power series can be arranged as a convergent power series in the variable z_j alone, for z_j sufficiently close to w_j; and this holds for any values a_i sufficiently near w_i. That is to say, the function f is **holomorphic in each variable separately** throughout the domain in which it is analytic; thus the ordinary complex derivative with respect to one of the variables z_j is well-defined, and will be denoted by $\partial/\partial z_j$. A converse to the boldface statement is also true, as follows.

2. Theorem (Osgood's Lemma). *If a complex-valued function* f *is continuous in an open set* $\mathrm{D} \subset \mathbb{C}^n$, *and is holomorphic in each variable separately, then it is holomorphic in* D.

Proof: Select any point $w \in D$, and any closed polydisc $\bar{\Delta}(w; r) \subset D$. Since f is holomorphic in each variable separately in an open neighborhood of $\bar{\Delta}(w; r)$, a repeated application of the Cauchy integral formula for functions of one variable leads to the formula

$$(3) \qquad f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|w_1 - \zeta_1| = r_1} \frac{d\zeta_1}{\zeta_1 - z_1} \int_{|w_2 - \zeta_2| = r_2} \frac{d\zeta_2}{\zeta_2 - z_2} \cdots \int_{|w_n - \zeta_n| = r_n} \frac{d\zeta_n}{\zeta_n - z_n} f(\zeta),$$

for all $z \in \Delta(w; r)$. For any fixed point z, the integrand in (3) is continuous on the compact domain of integration; hence the iterated integral in (3) can be replaced by the single multiple integral

$$(4) \qquad f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|w_j - \zeta_j| = r_j} \frac{f(\zeta)\, d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}.$$

But now, again for a fixed point $z \in \Delta(w; r)$, the series expansion

$$\frac{1}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} = \sum_{\nu_1 \ldots \nu_n = 0}^{\infty} \frac{(z_1 - w_1)^{\nu_1} \cdots (z_n - w_n)^{\nu_n}}{(\zeta_1 - w_1)^{\nu_1 + 1} \cdots (\zeta_n - w_n)^{\nu_n + 1}}$$

is absolutely uniformly convergent for all points ζ on the domain of integration in (4); consequently, after substituting this expansion into (4) and interchanging the orders of summation and integration, it follows immediately that the function f has a power series expansion of the form (2), with

$$(5) \qquad a_{\nu_1 \ldots \nu_n} = \left(\frac{1}{2\pi i}\right)^n \int_{|w_j - \zeta_j| = r_j} \frac{f(\zeta)\, d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - w_1)^{\nu_1 + 1} \cdots (\zeta_n - w_n)^{\nu_n + 1}}.$$

Therefore f is a holomorphic function, as desired.

Remark: The hypothesis that the function f be continuous in D is actually inessential; but this stronger theorem (Hartogs' theorem) is surprisingly much more difficult. This result will not be needed in the present book, so we shall not include a proof; the reader interested in pursuing this question is referred to Bochner-Martin [46, VII].

Some of the observations made during the course of the preceding proof merit separating out for special attention. First, any function f holomorphic in an open neighborhood of a closed polydisc $\bar{\Delta}(w; r)$ has a **Cauchy integral representation** of the form (4); that formula is the natural generalization of the **Cauchy integral formula** for holomorphic functions of one complex variable. By differentiating (4), it follows that

$$(6) \qquad \frac{\partial^{k_1 + \cdots + k_n} f(z)}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} = \frac{(k_1!) \cdots (k_n!)}{(2\pi i)^n} \int_{|w_j - \zeta_j| = r_j} \frac{f(\zeta)\, d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1)^{k_1 + 1} \cdots (\zeta_n - z_n)^{k_n + 1}}.$$

Upon then comparing (5) and (6), it further follows that the coefficients in the power series expansion (2) of f are given by

$$(7) \qquad (\nu_1!) \cdots (\nu_n!) a_{\nu_1 \ldots \nu_n} = \frac{\partial^{\nu_1 + \cdots + \nu_n} f}{\partial z_1^{\nu_1} \cdots \partial z_n^{\nu_n}}(w).$$

As a further consequence of these observations, it follows that the power series expansion of a holomorphic function at w is uniquely determined by that function and converges within any polydisc $\Delta(w; r)$ contained in the region of analyticity of that function; for the proof of Theorem 2 exhibited a power series expansion convergent within any fixed compact subset of $\Delta(w; r)$, and by (7) all of these series expansions must actually coincide.

One corollary which can be drawn from Osgood's lemma is an extension of the familiar Cauchy-Riemann equations, as a criterion for analyticity. As a convenient notation, introduce the first-order linear partial differential operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad\text{and}\quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \tag{8}$$

where x_j, y_j are the underlying real coordinates in $\mathbb{C}^n$, and $z_j = x_j + iy_j$. It should perhaps be remarked that the left-hand sides in (8) are defined by that equation, and have no separate meaning. However, note that

$$\frac{\partial}{\partial z_j} z_j = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)(x_j + iy_j) = 1,$$

and hence that

$$\frac{\partial}{\partial z_j} z_j^n = n z_j^{n-1};$$

therefore, when applied to holomorphic functions, the operator $\partial/\partial z_j$ coincides with the familiar complex derivative of a holomorphic function.

3. Theorem (Cauchy-Riemann Criterion). *A complex-valued function* f, *which is defined in an open subset* $\mathrm{D} \subset \mathbb{C}^n$ *and which is continuously differentiable in the underlying real coordinates of* $\mathbb{C}^n$, *is holomorphic in* D *if and only if it satisfies the system of partial differential equations*

$$\frac{\partial}{\partial \bar{z}_j} f(z) = 0, \qquad j = 1, 2, \ldots, n. \tag{9}$$

Proof: At any point of D, consider $f(z)$ as a function of the single variable z_j, holding the other variables constant. Decomposing f into its real and imaginary parts by writing $f(z) = u(z) + iv(z)$, note that

$$2\frac{\partial}{\partial \bar{z}_j} f(z) = \left(\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial y_j}\right) + i\left(\frac{\partial u}{\partial y_j} + \frac{\partial v}{\partial x_j}\right).$$

Therefore (9) is equivalent to the classical Cauchy-Riemann equations for each variable separately; and, as is well-known, this in turn is equivalent to the function f being holomorphic in each variable separately. The desired theorem then follows immediately from Osgood's lemma.

The following facts follow easily from the Cauchy-Riemann criterion.

4. Theorem. *Let* D *be an open set in* $\mathbb{C}^n$. *Then:*

(i) $\mathcal{O}_D$ *is a ring under the operations* $(f+g)(z) = f(z) + g(z)$, $(fg)(z) = f(z)g(z)$.

(ii) *If* f *is in* $\mathcal{O}_D$ *and is nowhere zero, then* $1/f$ *is in* $\mathcal{O}_D$.

(iii) *If* f *is in* $\mathcal{O}_D$, *and is real-valued or has constant modulus, then* f *is constant.*

Proof: (i) By direct computation,

$$\frac{\partial}{\partial \bar{z}_j}(f+g) = \frac{\partial f}{\partial \bar{z}_j} + \frac{\partial g}{\partial \bar{z}_j},$$

$$\frac{\partial}{\partial \bar{z}_j}(fg) = \frac{\partial f}{\partial \bar{z}_j}\, g + f \frac{\partial g}{\partial \bar{z}_j}; \tag{10}$$

hence the assertion (i) follows from Theorem 3.

(ii) Apply (10) with g replaced by f^{-1}. We find that

$$0 = f \cdot \frac{\partial f^{-1}}{\partial \bar{z}_j}.$$

(iii) If $f \in \mathcal{O}_D$ is real-valued, $\partial f/\partial x_j$ and $\partial f/\partial y_j$ are also real-valued. But $\partial f/\partial x_j = i\, \partial f/\partial y_j$, so both are zero for all j, $1 \leq j \leq n$. Thus f is constant. If f has constant modulus, then for any $w \in D$ we can write $f = \rho e^{i\theta(z)}$, where θ is a well-defined real-valued function in a neighborhood of w. Then, in U,

$$0 = \frac{\partial f}{\partial \bar{z}_j} = if \cdot \frac{\partial \theta}{\partial \bar{z}_j}.$$

Thus θ is holomorphic, so is also constant.

One of the fundamental properties of holomorphic functions of one complex variable is that the composition of two holomorphic functions is also holomorphic; the Cauchy-Riemann criterion now permits us to extend this property to functions of several complex variables, as follows. Suppose that $D \subset \mathbb{C}^n$ and that $D' \subset \mathbb{C}^m$ are two open domains; the variables in D will be written $z = (z_1, \ldots, z_n)$ and the variables in D' will be written $w = (w_1, \ldots, w_m)$. Any mapping $G\colon D \longrightarrow D'$ can be described by m functions

$$w_1 = g_1(z_1, \ldots, z_n), \quad \ldots, \quad w_m = g_m(z_1, \ldots, z_n). \tag{11}$$

The mapping G will be called a **holomorphic mapping** if the m functions $g_1, \ldots, g_m$ are holomorphic functions in D. If $f(w_1, \ldots, w_m) = f(w)$ is any function defined in D', the composite $f(G(z))$ is then a well-defined function in D.

5. Theorem (Composition Theorem). *If* f(w) *is a holomorphic function in* D′ *and if* G: D ⟶ D′ *is a holomorphic mapping, then the composition* f(G(z)) *is a holomorphic function in* D.

Proof: Separate the functions (11) into their real and imaginary parts by writing $g_j(z) = u_j(z) + iv_j(z)$. Since all the mappings involved are differentiable in the underlying real coordinates, the usual chain rule for differentiation can be applied as follows:

$$\frac{\partial f(G(z))}{\partial \bar{z}_j} = \sum_{k=1}^{m}\left(\frac{\partial f}{\partial u_k}\frac{\partial u_k}{\partial \bar{z}_j} + \frac{\partial f}{\partial v_k}\frac{\partial v_k}{\partial \bar{z}_j}\right) \tag{12}$$

$$= \sum_{k=1}^{m}\frac{1}{2}\left(\frac{\partial f}{\partial u_k} - i\frac{\partial f}{\partial v_k}\right)\frac{\partial g_k}{\partial \bar{z}_j} + \sum_{k=1}^{m}\frac{1}{2}\left(\frac{\partial f}{\partial u_k} + i\frac{\partial f}{\partial v_k}\right)\frac{\partial \bar{g}_k}{\partial \bar{z}_j}$$

$$= \sum_{k=1}^{m}\left(\frac{\partial f}{\partial w_k}\frac{\partial g_k}{\partial \bar{z}_j} + \frac{\partial f}{\partial \bar{w}_k}\frac{\partial \bar{g}_k}{\partial \bar{z}_j}\right).$$

(This is the **complex form of the chain rule.**) If the function f and the mapping G are both holomorphic, then $\partial f/\partial \bar{w}_k = 0$ and $\partial g_k/\partial \bar{z}_j = 0$ for all k; so by the above formula, $\partial f(G(z))/\partial \bar{z}_j = 0$ for all j. It then follows from the Cauchy-Riemann criterion that the function $f(G(z))$ is holomorphic, as desired.

Many other familiar results from the theory of holomorphic functions of one complex variable also have easy extensions to functions of several complex variables.

6. Theorem (Identity Theorem). *If* f(z) *and* g(z) *are holomorphic functions in a connected open set* $D \subset \mathbb{C}^n$, *and if* f(z) = g(z) *for all points* z *in a nonempty open subset* $U \subset D$, *then* f(z) = g(z) *for all points* $z \in D$.

Proof: Let E be the interior of the set consisting of all points z for which $f(z) = g(z)$; thus E is an open subset of D, and is nonempty since $U \subset E$. It clearly suffices to show that E is relatively closed in D as well; for it will then follow from the connectedness of D that $E = D$, and the theorem is therewith demonstrated. Therefore, consider any point $w \in D \cap \bar{E}$, where $\bar{E}$ is the point set closure of E; and select a number $r > 0$ sufficiently small that the polydisc $\Delta(w; r, \ldots, r) \subset D$. Since $w \in \bar{E}$, there must exist a point w' such that $|w'_j - w_j| < r/2$, $(j = 1, \ldots, n)$, and that $w' \in E$; note that $w \in \Delta(w'; r/2, \ldots, r/2)$. The function $f(z) - g(z)$ is holomorphic in

$$\Delta(w'; r/2, \ldots, r/2),$$

hence has a power series expansion centered at w' and converging throughout this small polydisc. Now since $w' \in E$, this function vanishes identically in an open neighborhood of w', and so by (7) all the coefficients in this power series expansion are zero; but then $f(z) - g(z) \equiv 0$ throughout

$$\Delta(w'; r/2, \ldots, r/2),$$

and thus $w \in E$. This shows that E is relatively closed in D, as desired.

7. Theorem (Maximum Modulus Theorem). *If* $\mathrm{f(z)}$ *is holomorphic in a connected open set* $\mathrm{D} \subset \mathbb{C}^n$, *and if there is a point* $\mathrm{w} \in \mathrm{D}$ *such that* $|\mathrm{f(z)}| \leq |\mathrm{f(w)}|$ *for all points* z *in some open neighborhood of* w, *then* $\mathrm{f(z)} \equiv \mathrm{f(w)}$ *for all points* $\mathrm{z} \in \mathrm{D}$.

Proof: Following the pattern of one of the customary proofs of the maximum modulus theorem for functions of one complex variable, we begin by observing that as a consequence of the Cauchy integral formula (4), for any polydisc $\Delta(w; r) \subset D$,

$$V(\Delta) f(w) = \int_{\Delta(w;r)} f(\zeta)\, dV(\zeta),$$

where $dV(\zeta)$ is the Euclidean volume element and $V(\Delta) = \int_{\Delta(w;r)} dV(\zeta)$ is the volume of $\Delta(w; r)$. As a consequence of this formula,

$$V(\Delta)\, |f(w)| \leq \int_{\Delta(w;r)} |f(\zeta)|\, dV(\zeta).$$

Now select a polycylinder $\Delta(w; r)$ such that $|f(w)| - |f(z)| \geq 0$ for all points $z \in \Delta(w; r)$; then

$$0 \leq \int_{\Delta(w;r)} (|f(w)| - |f(\zeta)|)\, dV(\zeta)$$

$$= V(\Delta)\, |f(w)| - \int_{\Delta(w;r)} |f(\zeta)|\, dV(\zeta) \leq 0,$$

so that $|f(w)| - |f(z)| = 0$ for all $z \in \Delta(w; r)$. Then, by Theorem 4, f must be constant in $\Delta(w; r)$; indeed $f(z) = f(w)$ for all $z \in \Delta(w; r)$. The desired result follows immediately from the identity theorem.

Since the power series expansion (2) of a function holomorphic in a neighborhood of w converges absolutely, we may regroup terms into a series of homogeneous polynomials:

$$f(z) = \sum_{k=0}^{\infty} \left(\sum_{\nu_1 + \cdots + \nu_n = k} a_{\nu_1 \ldots \nu_n} (z_1 - w_1)^{\nu_1} \cdots (z_n - w_n)^{\nu_n} \right). \tag{13}$$