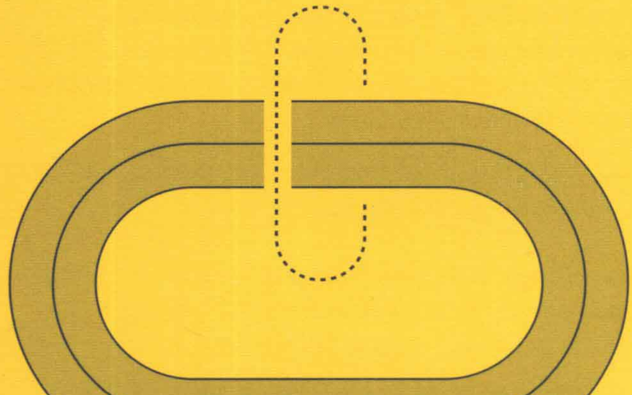
Lecture Notes in Mathematics

1669

Uwe Kaiser

Link Theory in Manifolds



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TO MY FATHER

INTRODUCTION

Invariants of links in 3-manifolds have been defined and studied classically through algebraic topology. In the last decade ideas from singularity theory, quantum field theory and statistical mechanics gave rise to the new theories of Vassiliev- and quantum invariants. But the resulting *combinatorial* context and the *topology of links* are not nicely combined.

The Conway polynomial $\nabla_K(z)$ of links K in S^3 is an invariant, which is well understood from both the classical and the modern viewpoint. It is combinatorially characterized by $\nabla_{\text{unknot}} = 1$ and the Conway relation

$$\nabla_{K_+} - \nabla_{K_-} = z\nabla_{K_0}.$$

Here

K_+ K_- K_0

are three links, which differ only in a 3-ball in the indicated way.

While this characterization is simple it is extremely difficult to be used for an existence proof [Ka1]. This is quite in contrast to the Jones polynomial [Ka2], which is the prominent example of the recent invariants.

In the early seventies John Conway suggested to generalize the Conway polynomial for the case of oriented 3-manifolds M in the following way: Consider the quotient of the free $\mathbb{Z}[z]$-module on the set of isotopy classes of oriented links in M by the submodule generated by all elements $K_+ - K_- - zK_0$. Obviously the resulting *Conway skein module* is universal with respect to maps from the set of isotopy classes of oriented links in M to $\mathbb{Z}[z]$-modules satisfying the Conway relation. The Conway skein module of S^3 is free of rank 1 and generated by the unknot (with the image of a link in the module given by its Conway polynomial multiplied with the generator). Conway skein modules of 3-dimensional handlebodies have been computed in [Tu1] and [P] using diagrammatic tools. But there is no topological understanding of the resulting invariants like in S^3. Conway skein modules are typical examples of *combinatorial universal constructions* using link theory in M. But in order to achieve explicit results concerning link theory, one needs to compute the modules and to understand the resulting link invariants *topologically*.

Our approach to link theory is motivated by the problems above.

In order to carry over the topological idea of the Conway polynomial in S^3 to the setting of oriented 3-manifolds, we propose to study *universal* constructions of linking numbers in 3-manifolds. In fact, linking numbers are the main ingredient in one of the standard existence proofs of the Conway polynomial in S^3 implying most of its topological properties. First one constructs the *Seifert pairing* of an oriented connected surface embedded in S^3 (see [Se], [Ka3] and [Ka4]) by taking the linking numbers, in S^3, of cycles on the surface with cycles pushed into the complement of the surface using a normal vector field. Then the Conway polynomial is defined by a *determinant construction* on a matrix representative of the Seifert pairing. It only depends on the equivalence class of the oriented link bounded by the surface. We will accomplish a *universal linking number map* construction for oriented 3-manifolds, with linking numbers crude enough to satisfy the standard properties. Then the Seifert pairing approach will generalize following the classical line of arguments.

The first essential property of linking numbers is the homology resp. bordism invariance. Let M be an oriented 3-manifold. Consider *link maps* (f_1, f_2) with $f_i : V_i \to M$ continuous maps of oriented compact closed 1-manifolds, $i = 1,2$, and $f_1(V_1) \cap f_2(V_2) = \emptyset$. The situation requires to study the bordism relation generated by oriented bordisms of f_1 in the complement of f_2 in M and vice versa. Let $\mathcal{I}(M)$ denote the resulting link bordism set. The structure of $\mathcal{I}(M)$ is easy to describe by the homology of M: For $a \in H_1(M)$ and $b \in H_2(M)$ let $ab \in \mathbb{Z}$ denote the oriented intersection number. Let $\mathfrak{r} : \mathcal{I}(M) \to H_1(M) \oplus H_1(M)$ be defined by the images of the fundamental classes of V_i by $(f_i)_*$ for $i = 1,2$.

Theorem 1. *Let M be a compact connected oriented 3-manifold. Then for all $a_1, a_2 \in H_1(M)$ there is a 1-1 correspondence between $\mathfrak{r}^{-1}(a_1, a_2)$ and the group $\mathbb{Z}/\Lambda(a_1, a_2)$, where $\Lambda(a_1, a_2)$ is the subgroup generated by all $a_ib \in \mathbb{Z}$ for $i = 1,2$ and all $b \in H_2(M)$.*

The link bordism set of the 3-ball is an infinite cyclic group under disjoint separated union, generated by the oriented Hopf link. This group acts transitively on the sets $\mathfrak{r}^{-1}(a_1, a_2)$ and the actions induce the bijection of theorem 1. This structure will be the main tool in further computations.

It follows that link bordism in those oriented 3-manifolds, where the intersection numbers of *closed* oriented surfaces with *closed* oriented curves always vanish, is different from the general case. We will show that intersection numbers vanish if and only if M embeds in a rational homology 3-sphere if and only if $2b_1(M) = b_1(\partial M)$, where b_1 is the first Betti-number. We call compact connected oriented 3-manifolds with this property *Betti-trivial.* Note that $2b_1(M) \geq b_1(\partial M)$ holds for all compact oriented 3-manifolds.

Next we let a *linking number map* for a 3-manifold be a map u from $\mathcal{I}(M)$ to an abelian group $\mathcal{A}$ with involution satisyfing (i) $u[f_1 \cup f_1', f_2] = u[f_1, f_2] + u[f_1', f_2]$ and (ii) equivariance with respect to $(f_1, f_2) \mapsto (f_2, f_1)$ and the involution on $\mathcal{A}$. Here $(f_1 \cup f_1', f_2)$ is an arbitrary link map and $[\ ,\ ]$ is the equivalence class in $\mathcal{I}(M)$. There is a universal linking number map $\mathcal{I}(M) \to \mathcal{A}_M$ for each oriented 3-manifold M, which is defined and constructed in standard terms. Let $\mathcal{A}_M$ be the *universal group* of the 3-manifold. We will prove the following result:

Theorem 2. *The universal group of a compact connected oriented 3-manifold is isomorphic to $H_1(M) \otimes H_1(M)$ if and only if M is not Betti-trivial. If M is Betti-trivial then there is the short exact sequence:*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\mu_M} \mathcal{A}_M \longrightarrow H_1(M) \otimes H_1(M) \longrightarrow 0,$$

where $\mu_M(1)$ is the image of the oriented Hopf link (see page 38) in a 3-ball in M under the universal map.

Linking number maps to abelian groups can be used to define bilinear Seifert pairings of compact oriented surfaces like in S^3. In order to derive invariants, which only depend on the "bordism class" of a given surface, we need determinants. This suggests to consider linking number maps with values in commutative rings and the definition of *ring lk-map.* This is a linking number map to an abelian group R as before, where R is a commutative ring with involution and identity, and moreover the Hopf link in M maps to the identity of the ring.

For M not Betti-trivial, *no* ring lk-maps exist. If M is Betti-trivial then a universal ring lk-map with values in a *universal ring* R_M of the manifold is defined. This ring can be computed from the homology of the manifold. The computation is non-trivial and involves a detailed analysis of the splitting classes of short exact sequences defined by the homology of knot exteriors in M.

Theorem 3. *The universal ring of a Betti-trivial manifold is isomorphic to the ring* $\mathbb{Z}[\frac{1}{k}][x_{ij}]$ *in* r^2 *indeterminates* x_{ij}*, where* k *is the least common multiple of the orders of non-trivial elements of the torsion subgroup of* $H_1(M)$ *and* r *is the rank of* $H_1(M)$.

For M a rational homology 3-sphere, $R_M \cong \mathbb{Z}[\frac{1}{k}] \subset \mathbb{Q}$ and the universal map $\mathcal{I}(M) \to R_M$ is the standard linking number map in $\mathbb{Q}$, which is defined using that $H_1(M)$ is torsion. In the general case of Betti-trivial manifolds, embeddings in rational homology 3-spheres define ring lk-maps to $\mathbb{Q}$. We will show that these determine the universal map if and only if $r \leq 1$.

Often, it is decidable now when ring lk-maps to specific rings exist. An important special case is the following.

Corollary 1. *Let* M *be Betti-trivial and let* k *be like in Theorem 3. Then there exists a ring lk-map to* $\mathbb{Z}_p$, $p \in \mathbb{Z}$, *if and only if* p *and* k *are coprime. In particular there exists a ring lk-map to* $\mathbb{Z}$ *if and only if* $k = 1$, *equivalently* $H_1(M)$ *is torsion-free.*

Finally, in order to apply our theory to define link invariants, we need to study the relation between links and oriented bounding surfaces. This is non-trivial in the situation of 3-manifolds. A study has been begun by R. Mandelbaum and B. Moishezon in [MM]. Their results are as follows: (i) An oriented link bounds an oriented surface if and only if its homology class in $H_1(M)$ is trivial, (ii) there is transitive action of $H_2(M)$ on the set of relative homology classes surfaces, which bound a fixed link. We will consider pairs (K, θ), called *Seifert structures*, where K is a link in M and θ is the relative homology class of an oriented bounding surface of K in the exterior of K. Equivalence of Seifert structures is defined by ambient isotopy in M. Then we will study the relation between equivalence of Seifert structures and links. We will show that each invariant of Seifert structures induces a natural invariant of oriented links with trivial homology class in M. Seifert structures very much behave like usual links. In particular, we will define a notion of skein triples $(K_+, \theta_+), (K_-, \theta_-), (K_0, \theta_0)$ of Seifert structures in M.

Theorem 4. *Let* M *be a Betti-trivial manifold. Then, for each equivalence class of Seifert structures* (K, θ) *in* M, *there is defined the Conway polynomial*

$$\nabla_{(K,\theta)} \in R_M[z]$$

satisfying the relation

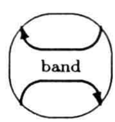
$$\nabla_{(K_+,\theta_+)} - \nabla_{(K_-,\theta_-)} = z\nabla_{(K_0,\theta_0)}$$

for skein triples of Seifert structures in M.

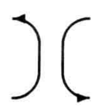
The Conway polynomial for Seifert structures induces the Conway polynomial for oriented links in M with values in a set of equivalence classes of maps $H_2(M) \to R_M[z]$. Equivalence of maps here is defined by the translation action of $H_2(M)$ in the argument. The Conway polynomial of Seifert structures has many properties known from S^3. So, if a Seifert structure contains a surface with more than one non-closed component then its Conway polynomial does vanish. Moreover it is possible to determine which matrices are realized as Seifert matrices of Seifert structures, thus deriving conclusions on the set of polynomials which are Conway polynomials of Seifert structures in M.

We like to point out that all the constructions above are *functorial* with respect to oriented embeddings of oriented 3-manifolds and *intrinsic*, i.e. without appealing to embeddings in rational homology 3-spheres.

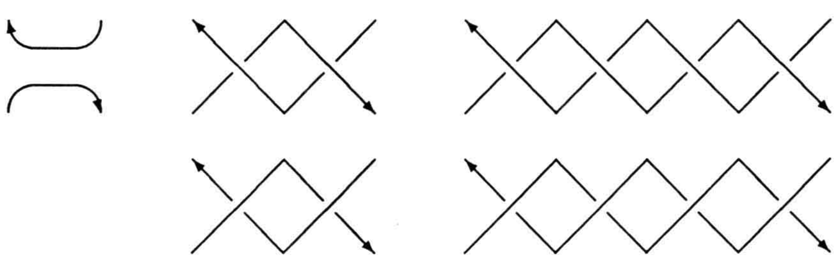
The topology of links in Betti-trivial manifolds is in many respects similar to that in S^3. We will discuss the *link characteristic* $\chi(K)$ of oriented links K in Betti-trivial 3-manifolds, which is defined by the maximal Euler characteristic of oriented surfaces without closed components bounding K. The link characteristic is comparable with the Thurston norm and the deep results of D. Gabai are available. These have previously been applied by M. Scharlemann and A. Thompson in the study of link genus, Conway moves and band operations in S^3. We prove a generalization of their result [ST] in the following form: Consider an oriented ball in M, which intersects the link in two parallel unlinked strings with opposite orientation, thus spanning a *band.* Then change the band by cutting and twisting:



oriented ball



K



twisted links K_b

This defines the sequence of links K_b, where b runs through twists of the given band.

The link K is defined by cutting the band.

Theorem 5. *Suppose M is irreducible Betti-trivial and with boundary components toral or two-spheres. Let K_b, K be the links, defined by twisting and cutting a link in a ball in M. Then either $\chi(K_b)$ does not depend on the twist and is $\leq \chi(K) - 1$, or there is a unique twist c with $\chi(K_c) > \chi(K_b) = \chi(K) - 1$ for all twists $b \neq c$.*

A corollary is that if K_+, K_-, K_0 are the three links in the Conway relation then two of $\chi(K_+)$, $\chi(K_-)$ and $\chi(K_0) - 1$ are equal and not larger than the third. Theorem 5, combined with a *duality principle* for bands, provides many non-triviality results for links in Betti-trivial manifolds.

The plan of these notes is as follows: We will start with a general discussion of link map bordism in manifolds and prove a classification result in a range of dimensions. This is mainly to show that the basis of our discussion is not specifically 3-dimensional. Generalizations of many concepts to high dimensional link theory are evident. In chapter 2 we discuss bordism relations in 3-manifolds and provide the classification of oriented resp. framed and embedded resp. singular bordism of r-component 1-dimensional links in compact connected oriented 3-manifolds. Chapter 3 contains the fundamental concepts of the theory including the computation of the universal rings of Betti-trivial 3-manifolds. In chapter 4 we set up the framework for invariants of oriented surfaces and determine the relation with link isotopy. Chapter 5 shows how, following the classical line of arguments, Seifert pairings defined by ring lk-maps give rise to polynomial invariants of links in 3-manifolds. In chapter 6, the characteristics of surfaces and links in 3-manifolds are defined in general. Then the discussion is specialized to Betti-trivial manifolds. We prove theorem 5 and discuss applications including the generalized band sum problem. Finally in chapter 7 we study the problem of Betti-trivial submanifolds of arbitrary connected compact oriented 3-manifolds. A process of cutting along oriented surfaces until the complement becomes Betti-trivial is considered. This gives rise to natural complexity invariants of oriented 3-manifolds, which are not Betti-trivial.

The appendix contains definitions and results on *inner homology* and *inner Betti-numbers*, important computational tools in the study of manifolds with boundary. Moreover, we have included a comprehensive treatment of the homology of link exteriors in 3-manifolds. General proofs of these results, which are indispensable for the understanding of the theory, do not seem available in the literature.

The chapters 1-3 are essentially independent with only superficial references. In particular the main result of chapter 1 is not applied in the discussion of bordism in 3-manifolds in chapter 2. The notions of chapter 4 are needed both in chapters 5 and 6. Chapter 7 is independent of the other parts of the notes.

We like to thank Michael Heusener, Ulrich Koschorke, Jerry Levine, Vladimir Nezhinskii and Oleg Viro for helpful and encouraging discussions. Furthermore we thank Ronnie Lee for bringing the work of Mandelbaum and Moishezon [MM] to our attention, and Marty Scharlemann for pointing out to us that theorem 5 for S^3 is implicitly contained in [ST].

Conventions:

Throughout we work in the smooth category. M will be a smooth manifold, in general compact and connected. ∂M will denote the boundary and $\mathrm{Int}(M) = M - \partial M$ the interior of a manifold M. For a submanifold resp. complex $V \subset M$ we let N_V denote a tubular neighbourhood resp. regular neighbourhood of V in M. We let $M_V = M - \mathrm{Int} N_V$ denote the exterior. For X a space or set, $|X|$ denotes the number of components or elements of X. For X, Y (based) spaces, $[X, Y]$ is the set of (based) homotopy classes of (based) continuous maps from X to Y. $\cong$ means diffeomorphism or isomorphism depending on the context. PD is Poincare duality.

Contents

Chapter 1

Link bordism in manifolds

We define bordism sets of link maps from manifolds (with structures on the tangent bundles) in a fixed manifold. We discuss properties of the bordism sets and the general structure. In a certain range of dimension, a classification is given by the bordism classes of the components and pairwise intersections. The main technical result is (1.3.4) with the application to classification given in (1.4).

1.1 Definitions and homotopy interpretation

We let $G = 1, SO, O$ denote the framed, oriented or unoriented situation. Let ε^k denote the trivial k-dimensional bundle over any space.

A *G-manifold M* is a manifold with a (homotopy class of) stable framing $\varepsilon^{n+2} \to TM \oplus \varepsilon^2$ for $G = 1$, and is oriented for $G = SO$. The boundary of a G-manifold V is a G-manifold using the inner normal isomorphism $TV|\partial V \cong T(\partial V) \oplus \varepsilon$. For each G-manifold V let $(-V)$ denote the G-manifold with the negative structure, which is defined by stabilization using a reflection. A diffeomorphism of G-manifolds is a diffeomorphism $g: V \to V'$ such that the derivative of g identifies the stable framings resp. orientations. Diffeomorphism of G-manifolds is denoted $\cong$.

Let $N \subset \partial M$ be a codimension-0 submanifold. Then a map $f: V \to M$ with V a manifold is *$N-$proper* if $f(\partial V) \subset N$. Note that, by using collars on ∂V, ∂N and ∂M, each proper map is homotopic through proper maps to a map with $f^{-1}(\partial M) = f^{-1}(\text{Int}N) = \partial V$. We call such maps *strictly $N-$proper.*

1.1.1 Definition. Let M be a manifold and let $N \subset \partial M$ be a codimension-0 submanifold.
(i) A *G-map in (M, N)* is an $N-$proper continuous map $f: V \to M$ with V a G-manifold.
(ii) The *boundary of a G-map* in (M, N) is the G-map in $N = (N, \emptyset)$ defined by the restriction to the boundary.
(iii) A diffeomorphism of G-maps $f: V \to M$ and $f': V' \to M$ in (M, N) is a diffeomorphism $g: V' \to V$ of G-manifolds such that $f' = f \circ g$.

Next let $p = (p_1, \ldots, p_r)$ be a sequence of non-negative integer numbers. For p a sequence and $k \in \mathbb{Z}$ let $p + k$ denote the sequence $(p_1 + k, \ldots, p_r + k)$.

1.1.2 Definition. A *G-link map in* (M, N) *of type* p is a sequence $f_i : V_i \to M$ of G-maps in (M, N) with $dim(V_i) = p_i$, $1 \leq i \leq r$, such that $f(V_i) \cap f(V_j) = \emptyset$ for $1 \leq i < j \leq r$. We briefly write $f : V \to M$ with $V = V_1 \cup \ldots \cup V_r$, $f|V_i$ is the *i-th part* of f.

Note that the boundary of a G-link map in (M, N) of type p is a G-link map in N of type $(p-1)$. In the following definition we smooth corners whenever necessary.

1.1.3 Definition. A *bordism* between G-link maps $f : V \to M$ and $f' : V' \to M$ of type p in (M, N) is a G-link map $F : W \to M \times I$ of type $(p+1)$ in the pair $(M \times I, (M \times \partial I) \cup N \times I)$ such that $\partial W \cong (-V) \cup Z \cup V'$ with the union along the boundaries and $\partial Z = (-\partial V) \cup \partial V'$. Moreover $F|V = f \times 0$ and $F|V' = f' \times 1$ (identified using the diffeomorphism), and $F|Z : Z \to N \times I$ is a G-link map in $(N \times I, N \times \partial I)$.

Let $\mathfrak{L}_p(M, N; G)$ denote the bordism set of G-link maps of type p in (M, N). Restriction defines the boundary map

$$\partial : \mathfrak{L}_p(M, N; G) \to \mathfrak{L}_{p-1}(N; G).$$

1.1.4 Example. The set $\mathfrak{L}_p(M; G) := \mathfrak{L}_p(M, \emptyset; G)$ is the bordism set of G-link maps in M (defined with closed G-manifolds and bordism is in usual sense).

We briefly discuss the Pontryagin–Thom construction to indicate how the geometric definitions fit into the framework of homotopy theory.

Let $MG(n)$ denote Thom space of the universal bundle over the classifying space $BG(n)$, where $G(n) = SO(n)$, $O(n)$ for $G = SO, O$, and $MG(n) = S^n$ is the Thom space of the trivial n-bundle over a point for $G = 1$. Let [,] denote based homotopy classes of based maps and let Ω be the loop functor.

If $p \in \mathbb{Z}$ then (1.1.3) reduces to the definition of the usual bordism group of G-maps (singular G-manifolds) in (M, N). In [A] and [Sw] the generalization $\Omega_j^G(X, Y)$ to pairs of complexes is discussed. In particular the Pontryagin–Thom construction defines an isomorphism:

$$\Omega_p^G(X, Y) \to \pi_{p+k}(X/Y \wedge MG(k))$$

for large k. It is known that the bordism groups of G-maps in (X, Y) form a generalized homology theory. For a compact G-manifold M , the Thom-Atiyah isomorphism holds

$$\pi_{p+k}(M/\partial M \wedge MG(k)) \to [M/\partial M, \Omega^k MG(k + m - p)],$$

(compare [C], (13.4)), where Ω^k denotes k-fold loop space. The homotopy group on the right hand side describes via Pontryagin–Thom construction the bordism group of continuous maps $f : V \to M$ with a stable framing $TV \cong f^*(TM)$ resp. an isomorphism of the underlying orientation bundles. The generalization to proper maps in (M, N) is obvious. This is what we call a *normal G-map in* (M, N) and