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Habib Ammari
Hyeonbae Kang

# Reconstruction of Small Inhomogeneities from Boundary Measurements

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Authors

Habib Ammari

Centre de Mathématiques Appliquées
École Polytechnique
UMR CNRS - 7641
Palaiseau 91128
France

*e-mail: ammari@cmapx.polytechnique.fr*

Hyeonbae Kang

School of Mathematical Sciences
Seoul National University
Seoul 151-747
Korea

*e-mail: hkang@math.snu.ac.kr*

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# Preface

Electrical impedance tomography (EIT) seeks to recover the electrical conductivity distribution inside a body from measurements of current flows and voltages on its surface. The vast and growing literature reflects many possible applications of EIT techniques, *e.g.*, for medical diagnosis or nondestructive evaluation of materials.

Since the underlying inverse problem is nonlinear and severely ill-posed, general purpose EIT reconstruction techniques are likely to fail. Therefore it is generally advisable to incorporate a-priori knowledge about the unknown conductivity. One such type of knowledge could be that the body consists of a smooth background containing a number of unknown small inclusions with a significantly different conductivity. This situation arises for example in breast cancer imaging or mine detection. In this case EIT seeks to recover the unknown inclusions. Due to the smallness of the inclusions the associated voltage potentials measured on the surface of the body are very close to the potentials corresponding to the medium without inclusion. So unless one knows exactly what patterns to look for, noise will largely dominate the information contained in the measured data. Furthermore, in applications it is often not necessary to reconstruct the precise values of the conductivity or geometry of the inclusions. The information of real interest is their positions and size.

The main purpose of this book is to describe fresh and promising techniques for the reconstruction of small inclusions from boundary measurements in a readable and informative form. These techniques rely on accurate asymptotic expansions of the boundary perturbations due to the presence of the inclusions. The general approach we will take to derive these asymptotic expansions is based on layer potential techniques. This allows us to handle inclusions with rough boundaries. In the course of deriving our asymptotic expansions, we introduce new concepts of generalized polarization tensors (GPT's). GPT's contain significant information on the inclusion which will be investigated. We then apply the asymptotic expansions for designing efficient direct reconstruction algorithms to detect the location, size, and/or orientation of the unknown inclusions.

This book would not have been possible without the collaborations and the conversations with a number of outstanding colleagues. We have not only profited from generous sharing of their ideas, insights and enthusiasm, but also from their friendship, support and encouragement. We feel specially indebted to Gen Nakamura, Jin Keun Seo, Gunther Uhlmann and Michael Vogelius. This book is dedicated to our friendship with them.

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*Habib Ammari*
*Hyeonbae Kang*

# Contents

# 1

# Introduction

Electrical Impedance Tomography (EIT) is designed to produce, efficiently and accurately, images of the conductivity distribution inside a body from measurements of current flows and voltages on the body's surface. Due to several merits of EIT such as safety, low cost, real time monitoring, EIT has received considerable attention for the last two decades; see for instance the review papers [255, 256, 78, 58] and the extensive list of references therein. The vast and growing literature reflects many possible applications of EIT, *e.g.*, for medical diagnosis or nondestructive evaluation of materials. In medical applications, EIT could potentially be used for monitoring lung problems, noninvasive monitoring of heart function and blood flow, screening for breast and prostate cancer, and improving electrocardiograms and electroencephalograms [78, 37, 82, 83, 163, 164, 143].

However, insensitivity of boundary measurements to any change of inner-body conductivity values has hampered EIT from providing accurate static conductivity images [4]. In practice captured current-to-voltage pairs must be limited by the number of electrodes attached on the surface of the body that confine the resolution of the image [153, 101]. We can definitely increase the resolution of the conductivity image by increasing the number of electrodes. However, it should be noticed that beyond a certain level, increasing the number of electrodes may not help in producing a better image for the inner-region of the body if we take into account the inevitable noise in measurements and the inherent insensitivity mentioned before.

In its most general form EIT is severely ill-posed and nonlinear. These major and fundamental difficulties can be understood by means of a mean value type theorem in elliptic partial differential equations. The value of the voltage potential at each point inside the region can be expressed as a weighted average of its neighborhood potential where the weight is determined by the conductivity distribution. In this weighted averaging way, the conductivity distribution is conveyed to the boundary potential. Therefore, the boundary data is entangled in the global structure of the conductivity distribution in a highly nonlinear way. This is the main obstacle to finding non-iterative reconstruction algorithms with limited data. If, however, in advance we have additional structural information about the medium, then we may be able to determine specific features about the conductivity distribution with a good resolution. One such type of knowledge could be that the body consists of a smooth background containing a number of unknown small inclusions with a significantly different conductivity. This situation arises for example in breast and prostate cancer imaging [37, 82, 83, 163, 164, 26, 243] or mine detection. In this case EIT seeks to recover the unknown inclusions. Due to the smallness of the inclusions the associated voltage potentials measured on the surface of the body are very close to the potentials corresponding to the medium without inclusions. So unless one knows exactly what patterns to look for, noise will largely dominate the information contained in the measured data. Furthermore, in applications it is often not necessary to reconstruct the precise values of the conductivity or the geometry of the inclusions. The information of real interest is their positions and size.

Taking advantage of the smallness of the inclusions, many promising reconstruction techniques have been designed since the pioneering works by Friedman and Vogelius [120, 121, 123]. It turns out that the method of asymptotic expansions of small volume inclusions provides a useful framework to reconstruct the location and geometric features of the inclusions in a stable way, even for moderately noisy data [73, 53, 28, 64, 191, 39, 261].

In this book we have made an attempt to describe these new and promising techniques for the reconstruction of small inclusions from boundary measurements in a readable and informative form. As we said, these techniques rely on accurate asymptotic expansions of the boundary perturbations due to the presence of the inclusions. The general approach we will take in this book is as follows. Based on layer potential techniques and decomposition formulae like the one due to Kang and Seo in [168] for the conductivity problem, we first derive complete asymptotic expansions. This allows us to handle inclusions with rough boundaries and those with extreme conductivities. In the course of deriving our asymptotic expansions, we introduce new concepts of generalized polarization tensors (GPT's). These concepts generalize those of classical Pólya–Szegö polarization tensors which have been extensively studied in the literature by many authors for various purposes [72, 24, 73, 212, 104, 105, 200, 198, 123, 180, 94, 186, 231, 232, 241, 95]. The GPT's appear naturally in higher-order asymptotics of the steady-state voltage potentials under the perturbation of conductor by conductivity inclusions of small diameter. GPT's contain significant information on the inclusion that will be investigated. We then apply the asymptotic expansions for designing efficient direct reconstruction algorithms to detect the location, size, and/or orientation of the unknown inclusions.

The book is intended to be self-contained. However, a certain familiarity with layer potential techniques is required. The book is divided into three parts that can be read independently.

Part I consists of four chapters dealing with the conductivity problem. It is organized as follows. In Chap. 2, we introduce the main tools for studying the conductivity problem and collect some notation and preliminary results regarding layer potentials. In Chap. 3, we introduce the GPT's associated with a Lipschitz bounded domain $B$ and a conductivity $0 < k \neq 1 < +\infty$. We prove that the knowledge of all the GPT's uniquely determines the domain $B$ and the conductivity $k$. We also provide important symmetric properties and positivity of the GPT's and derive isoperimetric inequalities satisfied by the tensor elements of the GPT's. These relations can be used to find bounds on the weighted volume. In Chap. 4, we provide a rigorous derivation of high-order terms in the asymptotic expansion of the output voltage potentials. The proofs of our asymptotic expansions are radically different from the ones in [123, 73, 259]. What makes the proofs particularly original and elegant is that the rigorous derivation of high-order terms follows almost immediately. In Chap. 5, we apply our accurate asymptotic formula for the purpose of identifying the location and certain properties of the shape of the conductivity inclusions. By improving the algorithm of Kwon, Seo, and Yoon [191] we first design two real-time algorithms with good resolution and accuracy. We then describe the variational algorithm introduced in [28] and review the interesting approach proposed by Brühl, Hanke, and Vogelius [64]. Their method is in the spirit of the linear sampling method of Colton and Kirsch [89].

In Part II we develop a method to detect the size and the location of an inclusion in a homogeneous elastic body in a mathematically rigorous way. Inclusions of small size are believed to be the starting point of crack development in elastic bodies. In Chap. 5, we review some basic facts on the layer potentials of the Lamé system. In Chap. 6, we give in a way analogous to GPT's, mathematical definitions of elastic moment tensors (EMT's) and show symmetry and positive-definiteness of the first-order EMT. The first-order EMT was introduced by Maz'ya and Nazarov [202]. In Chap. 7, we find a complete asymptotic formula of solutions of the linear elastic system in terms of the size of the inclusion. The method of derivation is parallel to that in Part I apart from some technical difficulties due to the fact that we are dealing with a system, not a single equation, and the equations inside and outside the inclusion are different. Based on this asymptotic expansion we derive in Chap. 8 formulae to find the location and the order of magnitude of the elastic inclusion. The formulae are explicit and can be easily implemented numerically.

The problem we consider in Part III is to detect unknown dielectric inclusions by means of a finite number of voltage-to-current pairs measured on the boundary. We consider solutions to the Helmholtz equation in two and three dimensions. We begin by proving in Chap. 9 existence and uniqueness of a solution to the Helmholtz equation. The proof, due to Vogelius and Volkov [259], uses the theory of collectively compact operators. Based on layer potential techniques and two new decomposition formulae of the solution to the Helmholtz equation, established in Chap. 10, we provide in Chap. 11 for such solutions a rigorous systematic derivation of complete asymptotic expansions of perturbations resulting from the presence of diametrically small inclusions with constitutive parameters different from those of the background medium. The leading-order term in these asymptotic formulae has been derived by Vogelius and Volkov in [259]. We then develop in Chap. 12 two effective algorithms for reconstructing small dielectric inclusions from boundary measurements at a fixed frequency. The first algorithm, like the variational method in Chap. 5, reduces the reconstruction problem of the small inclusions to the calculation of an inverse Fourier transform. The second one is the MUSIC (standing for MUltiple-Signal-Classification) algorithm. We explain how it applies to imaging of small dielectric inclusions. Another algorithm based on projections on three planes was proposed and successfully tested by Volkov in [261]. Results similar to those presented in this part have been obtained in the context of the full (time-harmonic) Maxwell equations in [31].

Finally, it is important to note that some of the techniques described in this book can be applied to problems in many fields other than inverse boundary value problems. In this connection we would particularly like to mention the mathematical theory of composite materials [84, 206, 186, 211, 24] and topological shape optimization [222, 196, 126, 132, 239].

# Part I

# Detection of Small Conductivity Inclusions

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with a connected Lipschitz boundary $\partial\Omega$. Let $\nu$ denote the unit outward normal to $\partial\Omega$. Suppose that $\Omega$ contains a finite number $m$ of small inclusions $D_s$, $s = 1, \ldots, m$, each of the form $D_s = \epsilon B_s + z_s$, where $B_s$, $s = 1, \ldots, m$, is a bounded Lipschitz domain in $\mathbb{R}^d$ containing the origin. We assume that the domains $D_s$, $s = 1, \ldots, m$, are separated from each other and from the boundary. More precisely, we assume that there exists a constant $c_0 > 0$ such that

$$|z_s - z_{s'}| \geq 2c_0 > 0 \quad \forall\, s \neq s' \quad \text{and} \quad \mathrm{dist}(z_s, \partial\Omega) \geq 2c_0 > 0 \quad \forall\, s\,,$$

that $\epsilon$, the common order of magnitude of the diameters of the inclusions, is sufficiently small and that these inclusions are disjoint. We also assume that the background is homogeneous with conductivity 1 and the inclusion $D_s$ has conductivity $k_s$, $0 < k_s \neq 1 < +\infty$, for $1 \leq s \leq m$.

Let $u$ denote the steady-state voltage potential in the presence of the conductivity inclusions $\bigcup_{s=1}^m D_s$, *i.e.*, the solution in $W^{1,2}(\Omega)$ to

$$\begin{cases} \nabla \cdot \left( \chi\left(\Omega \setminus \bigcup_{s=1}^{m} \overline{D_s}\right) + \sum_{s=1}^{m} k_s \chi(D_s) \right) \nabla u = 0 \quad \text{in } \Omega\,, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = g\,. \end{cases}$$

Let $U$ denote the "background" potential, that is, the solution to

$$\begin{cases} \Delta U = 0 \quad \text{in } \Omega\,, \\ \left.\dfrac{\partial U}{\partial \nu}\right|_{\partial\Omega} = g\,. \end{cases}$$

The function $g$ represents the applied boundary current; it belongs to $L^2(\partial\Omega)$ and has mean value zero. The potentials, $u$ and $U$, are normalized by

$$\int_{\partial\Omega} u \, d\sigma = \int_{\partial\Omega} U \, d\sigma = 0 \, .$$

The problem we consider in this part is to determine unknown inclusions $D_s$, $s = 1, \ldots, m$, by means of one or a finite number of current-to-voltage pairs $(g, u|_{\partial\Omega})$ measured on $\partial\Omega$.

This problem is called the inverse conductivity problem with one or finite boundary measurements (or Electrical Impedance Tomography) in contrast with the many measurements problem (or Calderón's problem) where an infinite number of boundary measurements are used. In many applied situations, it is the potential $u$ that is prescribed and the current $g$ that is measured on $\partial\Omega$. This makes some difference (not significant theoretically and computationally) in the case of finite boundary measurements but makes almost no difference in the case of many boundary measurements, since actually it is the set of Cauchy data $(g, u|_{\partial\Omega})$ that is given.

For the many measurements problem there is a well-established theory. We refer to the survey papers of Sylvester and Uhlmann [249], and of Uhlmann [255, 256], as well as to the book of Isakov [158], since this problem is out of the scope of our monograph. When $d \geq 2$, many boundary measurements provide much more information about the conductivity of $\Omega$ than a finite number of measurements. Thus, the inverse conductivity problem with finite measurements is more difficult than the one with many boundary measurements and not much was known about it until recently. Fortunately, there has been over the last few years a considerable amount of interesting work and new techniques dedicated to both theoretical and numerical aspects of this problem. It is the purpose of this part to describe some of these fresh and promising techniques, in particular, those for the reconstruction of diametrically small inclusions.

Let us very briefly emphasize our general methodology for solving our inverse conductivity problem (with finite measurements). We first derive an asymptotic expansion of the boundary voltage difference $u - U$ to any order in $\epsilon$. Then we apply this very explicit asymptotic behavior to the effective estimation of the location and some geometric features of the set of conductivity inclusions $\bigcup_{s=1}^m D_s$. To present these results we shall need a decomposition formula of $u$ into a harmonic part and a refraction part, the Neumann function associated with the background conductor $\Omega$, and the generalized polarization tensors (GPT's) associated with the scaled domains $B_s$ and their conductivities $k_s$. The GPT's are in fact the basic building blocks for our full asymptotic expansion of $u - U$ on $\partial\Omega$ and contain significant information on the domains $B_s$ and their conductivities $k_s$. Then it is important to precisely characterize these GPT's and derive their basic properties.

The problem we consider here occurs in many practical situations. The inclusions $\bigcup_{s=1}^m D_s$ might in a medical application represent potential tumors, in a material science application they might represent impurities, and finally in a war or post-war situation they could represent anti-personnel mines.

In medical applications, EIT is supported by the experimental evidence that different biological tissues have different electrical properties that change with cell concentration, cellular structure, molecular composition, and so on [165, 245]. Therefore, these properties manifest structural, functional, metabolic, and pathological conditions of the tissue providing valuable diagnostic information.

We conclude this introduction with a discussion of classical image reconstruction algorithms in EIT.

The most classical technique consists of a minimization approach. We assume an initial conductivity distribution for the model and iteratively update it until it minimizes the difference between measured and computed boundary voltages. This kind of method was first introduced in EIT by Yorkey, Webster, and Tompkins [266] following numerous variations and improvements. These include utilization of *a priori* information, various forms of regularization, and so on [264, 145, 257, 86]. Even though this approach is widely adopted for imaging by many researchers, it requires a large amount of computation time for producing images even with low spatial resolution and poor accuracy.

In the 1980's, Barber and Brown [45] introduced the back-projection algorithm for EIT that was the first fast and useful algorithm although it provides images with very low resolution. Since this algorithm is inspired from the computed tomography (CT) algorithm, it can be viewed as a generalized Radon Transform [240].

The third technique is the dynamical electrical impedance imaging. This interesting and sophisticated technique, developed by the Rensselaer impedance tomography group [78, 80, 79, 213, 244, 246, 108, 129, 124, 154, 153], is designed to produce images of a change of conductivity in the human body for purpose of applications in cardiac and respiratory imaging. The main idea is to decompose the conductivity into a static term, viewed as the background conductivity of human body, and a perturbing term, considered as the change of conductivity due to respiratory or heart function. The mathematical problem here is to visualize the perturbing term by an EIT system. Although this algorithm can provide accurate images when an initial guess of the background conductivity is reasonably good, it seems that new ideas are still needed to obtain good resolution images and completely satisfy practitioners, specially in screening for breast cancer.

Our main aim in this part is to propose a new mathematical direction of future EIT research mainly for biomedical applications. A new electronic system based on the mathematical modeling described in this book is being developed for breast cancer imaging at the Impedance Imaging Research Center by Jin Keun Seo and his group [26, 243, 192].