General Theory of Markov Processes

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To my wife Sheila and my son Colin

Preface

This work is intended to serve as a reference to the theory of *right processes*, a very general class of right continuous strong Markov processes. The use of the term *general theory* is meant to suggest both the absence of hypotheses of special type other than those for right processes, and the coordination of the methods with those of the general theory of processes, as exposed in the first two volumes of *Probabilités et Potentiel* by Dellacherie and Meyer. We do provide in the appendix a fairly extensive discussion and summary of the general techniques needed in the text, with hopes that it may lead the reader to a fuller appreciation of the Dellacherie-Meyer volumes.

The original definition of right process (*processus droit*) was set down twenty years ago by Meyer as an abstraction of certain properties possessed by standard Markov processes, which had been up to that time the largest class of strong Markov processes that could be shown to have an intimate connection with abstract potential theory. The hypotheses of Meyer were weakened in the subsequent lecture notes of Getoor (1975). Right processes in the sense of Meyer or Getoor do form a class large enough to encompass most right continuous Markov processes of practical interest such as Brownian motion, diffusions, Lévy processes (processes with stationary independent increments), Feller processes and so on, constructed from reasonable transition semigroups. However, the form of hypotheses discussed by Meyer and Getoor contains a serious flaw, in that their hypotheses are not invariant under the classical transformations of Markov processes such as killing, time-change, mappings of the state space, and Doob's h-transforms. Motivated by the wish to have a setting which is preserved by essentially all these transformations, we propose hypotheses for right processes weaker than those of either Meyer or Getoor, but which remain strong enough to guarantee a rich theory of sample path behavior and close links with potential theory.

The point of view of the book is chiefly to study the probabilistic structure of a given right process as expressed through such objects as its homogeneous functionals, its additive and multiplicative functionals, its associated stochastic calculus, and to consider the transformations of right processes that yield other right processes. It has been a constant goal to avoid imposing secondary hypotheses which would limit the domain of applicability. There is only one section concerning the construction of a right process from a nice (Ray) semigroup, and while adequate for constructing some classical examples, it is not of great generality. There is no discussion of construction of Markov processes by solving Stroock-Varadhan type martingale problems. The recent book of Ethier-Kurtz (1986) has much on these matters.

Explicit examples of right processes are discussed principally in the exercises. The connections between right processes and abstract potential theory are discussed though not always in full detail. For example, though there is a discussion of the Hunt-Shih identification of hitting operators and réduite of an excessive function on a set, we do not present a complete proof. The reader interested in questions of more direct potential theoretic type is referred to volumes III and IV of Dellacherie-Meyer.

The sections on multiplicative functionals and homogeneous random measures, the latter a generalization of additive functionals, bring up to date the older books of Meyer and Blumenthal-Getoor. Especially in the sections on Lévy systems and exit systems, there is a penalty to be paid for the breadth of the hypotheses, requiring us to construct kernels on spaces larger than the state space so that the statements of the results will look a bit unusual to experts familiar with their forms under restrictive measurability conditions. However, the applications of these constructions do not appear to be affected in any essential way by this complication.

It is a pleasure to thank those individuals whose comments on earlier versions have eliminated many inaccuracies, inconsistencies and irrelevancies. Marti Bachman, Ron Getoor, Joe Glover, Bernard Maisonneuve, Joanna Mitro, Wenchuan Mo, Art Pittenger, Phil Protter, Tom Salisbury and Michel Weil provided me with valuable feedback for which I am very grateful. Thanks are also due to Neola Crimmins, whose expert entry of part of the first draft simplified the task of assembling the final document in TEX format.

La Jolla, 1988.

Contents

I Fundamental Hypotheses

1. Markov Property, Transition Functions and Entrance Laws

A stochastic process indexed by a subset of the real line has the **Markov property** if, roughly speaking, the past and future are conditionally independent given the present, for every possible value of the present. See (1.1) below for the precise specification. In this definition, the state space is required to have only measurable structure—no algebra or topology is involved. Nevertheless, because of applications to special examples and our focus on path regularizations which would otherwise take a different form, we shall work exclusively with topological state spaces. The minimal hypothesis on every state space E shall be that E is a Radon topological space. See §A1. This is not a burdensome restriction. Every Polish (:=complete, separable, metrizable) space and every locally compact Hausdorff space with countable base (LCCB) is Radonian.

The notation $\mathcal{B}(E)$ stands for the Borel σ-algebra on E, but we shall use the simpler notation $\mathcal{E}$ in its place unless clarity dictates otherwise. The notation $\mathcal{E}^u$ will, following the pattern described in the Appendix, denote the σ-algebra of universally measurable subsets of E. Other σ-algebras intermediate to $\mathcal{E}$ and $\mathcal{E}^u$ will be introduced later. We shall always denote a generic such σ-algebra by $\mathcal{E}^\bullet$ with the superscript $^\bullet$ usually being one of $0, r, e$, referring to the σ-algebras on E generated by the Borel, Ray and excessive functions respectively. Thus $\mathcal{E}^0$ is just another name for $\mathcal{E}$. See §10. In later sections, we shall make a distinction between $\mathcal{E}^0$ and $\mathcal{E}$, identifying $\mathcal{E}$ with $\mathcal{E}^r$ instead of $\mathcal{E}^0$. This will require minor reinterpretation of some of the constructs in this chapter, but to do otherwise would lead to serious notational complications later.

The reader is now assumed to be familiar with the terminology established in the Appendix, especially in A0–A3. In particular, given a σ-algebra $\mathcal{M}$ on a space M, b$\mathcal{M}$ (resp., p$\mathcal{M}$) stands for the class of *bounded* (resp., *positive*) $\mathcal{M}$-measurable functions on M. (Positive always refers to values in $[0, \infty]$, rather than the positive reals).

Let $(\Omega, \mathcal{G}, \mathbf{P})$ be a probability space, I be an index set contained in the real line $\mathbf{R}$, and let $X = (X_t)_{t \in I}$ be a **stochastic process** indexed by I, with values in E. That is, $(X_t)_{t\in I}$ is a collection of measurable maps of $(\Omega, \mathcal{G})$ into $(E, \mathcal{E})$. In order to emphasize the dependence here on $\mathcal{E}$, we call X an $\mathcal{E}$-stochastic process. Similar definitions will apply when $\mathcal{E}$ is replaced with a larger σ-algebra $\mathcal{E}^\bullet$. It is, of course, a more demanding condition for X to be an $\mathcal{E}^\bullet$-stochastic process, as it is required in this case that for every $t \in I$, $\{X_t \in F\} := \{\omega \in \Omega : X_t(\omega) \in F\}$ be in $\mathcal{G}$ for every set F in $\mathcal{E}^\bullet$ rather than for every F in $\mathcal{E}$.

Corresponding to a fixed σ-algebra $\mathcal{E}^\bullet$ on E and a fixed $\mathcal{E}^\bullet$-stochastic process X on Ω, the natural σ-algebra $\mathcal{F}^\bullet_{\le t}$ (or, more simply, $\mathcal{F}^\bullet_t$) is defined as $\sigma\{f(X_r) : r \in I, r \le t, f \in \mathcal{E}^\bullet\}$. A similar definition specifies the σ-algebra $\mathcal{F}^\bullet_{\ge t}$ of the future from t. Thus, for example, $\mathcal{F}^0_t$ (resp., $\mathcal{F}^u_t$) denotes the σ-algebra generated by the maps $f(X_r)$ with $r \le t$ and f in $\mathcal{E}^0 (:= \mathcal{E})$ (resp., f in $\mathcal{E}^u$).

The process X has the $\mathcal{E}^\bullet$-**Markov property** if the σ-algebras $\mathcal{F}^\bullet_{\le t}$, $\mathcal{F}^\bullet_{\ge t}$ are conditionally independent given X_t, for every $t \in I$. That is, for $t \in I$, $A \in \mathcal{F}^\bullet_{\le t}$ and $B \in \mathcal{F}^\bullet_{\ge t}$,

$$\mathbf{P}\{A \cap B \mid X_t\} = \mathbf{P}\{A \mid X_t\} \cdot \mathbf{P}\{B \mid X_t\}. \tag{1.1}$$

The need for the prefix $\mathcal{E}^\bullet$ is only temporary, as we shall see after the discussion of augmentation procedures in §6. Under the condition (1.1), one may compute, using the well known properties of conditional expectations,

$$\begin{aligned}\mathbf{P}\{\,A \cap B\,\} &= \mathbf{P}\{\,\mathbf{P}\{A \cap B \mid X_t\}\,\} \\ &= \mathbf{P}\{\,\mathbf{P}\{A \mid X_t\}\,\mathbf{P}\{B \mid X_t\}\,\} \\ &= \mathbf{P}\{\,\mathbf{P}\{B \mid X_t\}\,; A\,\}.\end{aligned}$$

As $A \in \mathcal{F}^\bullet_t$ is arbitrary, it follows that (1.1) implies

$$\mathbf{P}\{B \mid \mathcal{F}^\bullet_t\} = \mathbf{P}\{B \mid X_t\} \tag{1.2}$$

for every $B \in \mathcal{F}^\bullet_{\ge t}$, $t \in I$. That is, prediction of future behavior of X based on the entire past is only as valuable as the predictor based on the present value X_t alone. Conversely, the condition (1.2) implies (1.1) by similar manipulations, and consequently (1.2) is also referred to as the $\mathcal{E}^\bullet$-Markov property of X. In many respects, (1.2) is more convenient to manipulate and generalize. In the first place, it is reasonable and useful to replace the filtration $(\mathcal{F}_t^\bullet)$ with a more general filtration $(\mathcal{G}_t)$ to which (X_t) is $\mathcal{E}^\bullet$-adapted. This leads us to say that (X_t) is **$\mathcal{E}^\bullet$-Markovian** with respect to $(\mathcal{G}_t)$ if X is $\mathcal{E}^\bullet$-adapted to $(\mathcal{G}_t)$, and if, for all $t \in I$ and all $B \in \mathcal{F}_{\geq t}^\bullet$,

$$\mathbf{P}\{B \mid \mathcal{G}_t\} = \mathbf{P}\{B \mid X_t\}. \tag{1.3}$$

In applications, (1.3) has a more convenient form

$$\mathbf{P}\{H \mid \mathcal{G}_t\} = \mathbf{P}\{H \mid X_t\}, \qquad H \in \mathrm{p}\mathcal{F}_{\geq t}^\bullet. \tag{1.4}$$

Formula (1.4) is an immediate consequence of (1.3), starting with the case $H = 1_B$, $B \in \mathcal{F}_{\geq t}^\bullet$, and making use of the Monotone Class Theorem (A0.1).

The definition above is too crude to be useful except when I is a discrete subset of $\mathbf{R}$. We bring more precision to bear by introduction of the notion of a transition function $(P_{s,t})$ for X.

(1.5) DEFINITION. *A family $(P_{s,t})$ of Markov kernels on $(E, \mathcal{E}^\bullet)$ indexed by pairs $s, t \in I$ with $s \leq t$ is a* **transition function** *on $(E, \mathcal{E}^\bullet)$ if, for all $r \leq s \leq t$ in I and all $x \in E$, $B \in \mathcal{E}^\bullet$*

$$P_{r,t}(x, B) = \int_E P_{r,s}(x, dy)\, P_{s,t}(y, B).$$

In accordance with the discussion of kernels in A3, $P_{s,t}(x, dy)$ is a kernel on $(E, \mathcal{E}^\bullet)$ provided that, for all $x \in E$, $P_{s,t}(x, dy)$ is a positive measure on $(E, \mathcal{E}^\bullet)$, and for every $B \in \mathcal{E}^\bullet$, $x \to P_{s,t}(x, B)$ is $\mathcal{E}^\bullet$ measurable. In addition, $P_{s,t}(x, dy)$ is a **Markov** kernel if $P_{s,t}(x, E) = 1$ for all $x \in E$. The equation in (1.5) is called the **Chapman-Kolmogorov equation.**

Define the action of the Markov kernel $P_{s,t}$ on $\mathrm{b}\mathcal{E}^\bullet$ (resp., $\mathrm{p}\mathcal{E}^\bullet$) by

$$P_{s,t} f(x) := \int P_{s,t}(x, dy)\, f(y), \qquad f \in \mathrm{p}\mathcal{E}^\bullet \cup \mathrm{b}\mathcal{E}^\bullet,$$

so that $P_{s,t} f \in \mathrm{b}\mathcal{E}^\bullet$ (resp., $\mathrm{p}\mathcal{E}^\bullet$.) See §A3. We say that a transition function $(P_{s,t})$ on $(E, \mathcal{E}^\bullet)$ is the transition function for a process $(X_t)_{t \in I}$ with values in E, and satisfying the Markov property (1.4) relative to $(\mathcal{G}_t)$ in case

$$\mathbf{P}\{f(X_t) \mid \mathcal{G}_s\} = P_{s,t} f(X_s), \quad s \leq t \in I, f \in \mathrm{b}\mathcal{E}^\bullet. \tag{1.6}$$

(1.7) THEOREM. *Let $(X_t)_{t \in I}$ be $\mathcal{E}^\bullet$-adapted to $(\mathcal{G}_t)$, and suppose that $(P_{s,t})$ is a transition function on $(E, \mathcal{E}^\bullet)$ such that (1.6) holds for every $s \leq t \in I$ and every $f \in \mathrm{b}\mathcal{E}^\bullet$. Then X has the Markov property (1.4).*

PROOF: The class $\mathcal{H}$ of random variables in $\mathrm{b}\mathcal{F}_{\geq t}^\bullet$ for which (1.4) holds is clearly an MVS (see A0) because of monotonicity properties of conditional expectations. By hypothesis, $\mathcal{H}$ contains every H of the form $f(X_t)$ with $f \in b\mathcal{E}^\bullet$. As $b\mathcal{F}^\bullet_{\geq t}$ is generated by the multiplicative class $\mathcal{V} = \cup_n \mathcal{V}_n$, where $\mathcal{V}_n$ is the collection of products $F_1 F_2 \cdots F_n$ with $F_j = f_j(X_{t_j})$, $t \leq t_1 \leq t_2 \leq \cdots \leq t_n$, $f_j \in b\mathcal{E}^\bullet$, it suffices by the Monotone Class Theorem to verify that $\mathcal{V} \subset \mathcal{H}$. Proceed by induction on n to get $\mathcal{V}_n \subset \mathcal{H}$ for all $n \geq 1$. By our first remarks above, $\mathcal{V}_1 \subset \mathcal{H}$. Suppose, inductively, that $\mathcal{V}_n \subset \mathcal{H}$ and let $G = F_1 \dots F_{n+1} \in \mathcal{V}_{n+1}$. Compute $\mathbf{P}\{G \mid \mathcal{G}_t\}$ by first conditioning relative to $\mathcal{G}_{t_n}$, so that

$$\mathbf{P}\{G \mid \mathcal{G}_t\} = \mathbf{P}\{\mathbf{P}\{G \mid \mathcal{G}_{t_n}\} \mid \mathcal{G}_t\} = \mathbf{P}\{F_1 \dots F_n P_{t_n,t_{n+1}} f_{n+1}(X_{t_n}) \mid \mathcal{G}_t\}.$$

However, the random variable being conditioned in the last term is clearly in $\mathcal{V}_n$, and thus, by inductive hypothesis, $\mathcal{G}_t$ may be replaced by X_t. The same calculation with $\mathcal{G}_t$ replaced throughout by X_t completes the inductive step by proving $\mathbf{P}\{G \mid \mathcal{G}_t\} = \mathbf{P}\{G \mid X_t\}$, which finishes the proof.

We shall be interested primarily in the case $I = \mathbf{R}^+ := [0, \infty[$ though the cases $]0, \infty[$, $]-\infty, \infty[$ and $]0, 1[$ also arise frequently in practice.

A family $(P_t)_{t \geq 0}$ of Markov kernels on $(E, \mathcal{E}^\bullet)$ is called a **Markov transition semigroup** or simply a transition semigroup in case

$$P_{t+s} f(x) = P_t(P_s f)(x), \quad t, s \geq 0, x \in E, f \in b\mathcal{E}^\bullet.$$

A transition function $(P_{s,t})$ indexed by $s \leq t \in \mathbf{R}^+$ is called **temporally homogeneous** if there is a transition semigroup (P_t) with $P_{s,t} = P_{t-s}$ for all $s \leq t$. Starting with a transition semigroup (P_t), $P_{s,t} := P_{t-s}$ defines a temporally homogeneous transition function.

A Markov process X satisfying (1.6) with a homogeneous transition function (P_t) has the characteristic property

$$\mathbf{P}\{f(X_{t+s}) \mid \mathcal{G}_t\} = P_s f(X_t), \quad t, s \geq 0, \quad f \in b\mathcal{E}^\bullet. \tag{1.8}$$

This is the **simple Markov property** of X relative to (P_t).

The Markov processes considered here will be temporally homogeneous for the most part. See however exercise (1.15), which deals with the so-called space-time process connected with a general Markov process.

Suppose now that $(X_t)_{t \geq 0}$ has the Markov property (1.8) relative to $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbf{P})$, with transition semigroup (P_t). The distribution μ_0 of X_0 is called the **initial law** of X, and the distribution μ_t of X_t then satisfies $\mu_t = \mu_0 P_t$ for all $t \geq 0$. That is, for $f \in b\mathcal{E}^\bullet$,

$$\mu_t(f) := \int f \, d\mu_t = \mathbf{P} f(X_t) = \mathbf{P}\, P_t f(X_0) = \mu_0(P_t f).$$

If the index set for X were instead $]0,\infty[$, there would be no initial law μ_0 definable as above. However, the μ_t would obviously satisfy the identities

$$\mu_{t+s} = \mu_t P_s, \qquad t, s > 0. \tag{1.9}$$

A family $(\mu_t)_{t>0}$ of positive measures on $(E, \mathcal{E}^\bullet)$ satisfying (1.9) is called an **entrance law** for the semigroup (P_t). It is called *finite* in case $\mu_t(E) < \infty$ $\forall t > 0$, *bounded* if $\sup_t \mu_t(E) < \infty$, *probability* if $\mu_t(E) = 1$ for all t. If there is a measure μ_0 such that $\mu_t = \mu_0 P_t$ for all $t > 0$, then μ_0 is said to **close** the entrance law $(\mu_t)_{t>0}$. A probability entrance law $(\mu_t)_{t>0}$ need not have a closing element μ_0. For example, let E be the open right half line $\mathbf{R}^{++}$ and let $P_t(x, dy) := \epsilon_{x+t}(dy)$ —unit mass a location $x + t$. Then, for $t > 0$, $\mu_t(dy) := \epsilon_t(dy)$ defines a probability entrance law for (P_t) without a closing element. See Chapter V for the compactification theory needed to permit the representation of a closing element for an arbitrary probability entrance law.

A (temporally homogeneous) Markov process $(X_t)_{t\geq 0}$ satifying (1.8) and having initial law μ_0 necessarily satisfies the more general identities

$$\begin{aligned} &\mathbf{P}\{f_1(X_{t_1}) f_2(X_{t_2}) \cdots f_n(X_{t_n})\} \\ &\qquad = \mu_0 \left(P_{t_1} \left(f_1 \cdot P_{t_2 - t_1} \left(f_2 \cdots (f_{t_n} \cdot P_{t_n - t_{n-1}} f_n) \cdots \right)\right)\right), \end{aligned} \tag{1.10}$$

for $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, $f_1, \cdots, f_n \in b\mathcal{E}^\bullet$. This is a simple consequence of (1.8) via an induction argument. The last formula is perhaps more intuitive in its differential version, which states that under the same conditions as above,

$$\begin{aligned} &\mathbf{P}\{X_0 \in dx_0, X_{t_1} \in dx_1, \cdots, X_{t_n} \in dx_n\} \\ &\qquad = \mu_0(dx_0) P_{t_1}(x_0, dx_1) \cdots P_{t_n - t_{n-1}}(x_{n-1}, dx_n). \end{aligned} \tag{1.11}$$

In this form, the Markov property corresponds to the Huygens principle in wave propagation—in order to compute $\mathbf{P}\{X_t \in dx\}$, one may imagine interposing a barrier at time $t_1 < t$ and, knowing the position X_{t_1}, perform calculations supposing that the process starts afresh at X_{t_1}. The integral version (1.10) asserts that the total probability that $X_t \in dx$ is obtained by adding the above probabilities over all possible positions X_{t_1}, weighted by the probabilities of reaching the points X_{t_1} in the first place.

(1.12) EXERCISE. *Formulate the appropriate versions of (1.10) and (1.11) in the case where X is homogeneous Markov with time parameter set $]0,\infty[$.*

The next pair of exercises is designed to give the reader a little practice with arguments involving completions. This kind of "sandwiching" will be used repeatedly in later sections. Exercise (1.14) will show that there is no need to maintain any distinction between different $\mathcal{E}^\bullet$-Markov properties, provided the filtration is sufficiently rich.

(1.13) EXERCISE. *Let* (P_t) *preserve each of the* σ*-algebras* $\mathcal{E}^\bullet$, $\mathcal{E}'$, *with* $\mathcal{E} \subset \mathcal{E}^\bullet \subset \mathcal{E}' \subset \mathcal{E}^u$. *Let* $(X_t)_{t\geq 0}$ *be defined on* $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbf{P})$ *with* X *satisfying (1.8) for all* $t, s \geq 0$, $f \in \mathrm{b}\mathcal{E}^\bullet$. *Assume that* X *is* $\mathcal{E}'$*-adapted to* $(\mathcal{G}_t)$. *Prove that (1.8) holds then for* $f \in \mathrm{b}\mathcal{E}'$. *(Hint: choose* $f_1 \leq f \leq f_2$ *with* $f_1, f_2 \in \mathcal{E}^\bullet$ *and* $f_2 - f_1$ *null for the measure* $g \to \mathbf{P}g(X_{t+s}) = \mathbf{P}\, P_s g(X_t)$ $(g \in \mathrm{b}\mathcal{E}^\bullet)$. *Remember that a conditional expectation is an equivalence class of random variables.)*

(1.14) EXERCISE. *Let* $\mathcal{E}^\bullet \subset \mathcal{E}'$ *be* σ*-algebras preserved by* (P_t), *and assume that* X *satisfies (1.8) for every* $f \in \mathrm{b}\mathcal{E}^\bullet$. *Prove that for all* $f \in \mathrm{b}\mathcal{E}'$,

$$\mathbf{P}\{f(X_{t+s}) \mid \mathcal{F}'_t\} = P_s f(X_t).$$

(Hint: by (1.13), one may reduce to the case $f \in \mathrm{b}\mathcal{E}^\bullet$. *Show using monotone classes that, for every* $H \in \mathrm{b}\mathcal{F}'_t$, *there exist* $H_1 \leq H \leq H_2$ *with* $H_1, H_2 \in \mathrm{b}\mathcal{F}^\bullet_t$ *and* $H_2 - H_1$ *null for the measure* $G \to \mathbf{P}G$ $(G \in \mathrm{b}\mathcal{F}^\bullet_t)$.*)*

(1.15) EXERCISE. *Let* $(X_t)_{t\geq 0}$ *be Markov with transition function* $(P_{s,t})$. *Suppose also that* $(P_{s,t})$ *satisfies the measurability condition*

$$(s,t,x) \to P_{s,t}(x,B)1_{\{s\leq t\}} \quad \text{is in } \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{E}^\bullet \quad \forall\, B \in \mathcal{E}^\bullet.$$

Show that, with ϵ_u *denoting unit mass at* $u \in E$,

$$\tilde{P}_t\big((r,x),(ds,dy)\big) := \epsilon_{r+t}(ds) P_{r,r+t}(x,dy)$$

defines a Markov transition semigroup on $(\mathbf{R} \times E, \mathcal{B}(\mathbf{R}) \otimes \mathcal{E}^\bullet)$ *and the* **space-time** *process* $\tilde{X}_t := (t, X_t)$ *has the Markov property relative to* $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbf{P})$, *with transition semigroup* $(\tilde{P}_t)$.

(1.16) EXERCISE. *Verify, using the Kolmogorov existence theorem, that if* (P_t) *is a Markov transition semigroup on the Radon space* E, *and if* $(\mu_t)_{t>0}$ *is an arbitrary probability entrance law for* (P_t), *then there exists a unique probability measure* $\mathbf{P}$ *on the product space* $\Omega = E^{]0,\infty[}$ *with product* σ*-algebra* $\mathcal{G}$ *so that the coordinate maps* X_t *form a Markov process with transition function* (P_t) *and entrance law* (μ_t). *Formulate and check the temporally inhomogeneous version of this result.*

(1.17) EXERCISE. *Let* X_t *be a process with (not necessarily stationary) independent increments in* $\mathbf{R}^d$. *Show that* X *is Markovian and satisfies (1.6) for some transition function* $(P_{s,t})$.