SINGLE-VARIABLE CALCULUS

ROBERT A. ADAMS

Second Edition

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PREFACE

Like the first edition which preceded it, the second edition of *Single-Variable Calculus* has been designed for general calculus courses, as well as for courses for science and engineering students. As such it provides a complete introduction to the calculus of functions of a single, real variable and treats all those topics normally found in a two-semester course on the differentiation and integration of these functions. In addition to the usual "core material," it also covers a selection of optional and enrichment topics from which an instructor can select those appropriate for his or her class.

Much of the material of the first edition has been rewritten to make it more accessible to the average student with a reasonable background in high-school algebra and some previous exposure to analytic geometry. However, some optional material is more subtle and/or theoretical, and is intended mainly for stronger students. Throughout the book I have taken pains to make correct statements of results; I have tried to make the presentation as simple as possible, but no simpler.

The exercises vary greatly in both difficulty and subtlety. Numerous drill-type exercises are provided to help the student master core concepts, and many more thought-provoking ones, some theoretical, some computational, are also included to challenge the student to apply the concepts. More difficult and/or theoretical exercises are marked with an asterisk (*).

Because differential equations are used extensively to model phenomena in the sciences, this book introduces the terminology of differential equations and initial-value problems early as exercises to develop differentiation and integration skills, and to familiarize the students with the most important types. Exercises involving differential equations are marked with a dagger (†).

Principal Features of the Text

- There is an emphasis on geometry. Frequently applications of calculus are based on underlying geometric relationships among the variables involved.
- Trigonometric and exponential functions and their inverses are introduced early, and their major properties are developed before applications of differentiation are discussed. Thus these functions can be *freely* used in the applications. Students who have encountered the trigonometric functions earlier will find Section 3.1 a good review of their basic properties. Exponential functions are introduced prior to their inverses, the logarithms, but an optional section provides the alternate approach (used in the first edition) of introducing the natural logarithm first, as

the "area" under a curve. This can provide some advance motivation for the later development of the Fundamental Theorem.

- There is increased emphasis on numerical approximation of values of functions, roots of equations, and definite integrals. In particular, there is now a separate (optional) chapter on numerical integration. Topics such as the Romberg method and a discussion of the pitfalls of numerical methods help to make calculus more relevant in this age of computers and calculators. Where appropriate, students are encouraged to program calculators or computers to obtain numerical results efficiently.
- Precise statements are given for theorems. Proofs of most theorems are given or suggested immediately, but some proofs are postponed to the appendices. The three appendices develop, respectively, the technique of proof by mathematical induction, the properties of continuous functions defined on a closed, finite interval, and the properties of the Riemann integral. They provide suitable enrichment for particularly interested students and honours classes.
- The Mean-Value Theorem and its applications have been given a higher profile in a separate chapter.
- The definition of the definite integral in Chapter 6 now allows partitions with subintervals of unequal length, and a new example illustrates the added power of this improved definition.
- A chapter on plane curves now provides a (classical) introduction to the conic sections as well as the development of polar coordinates, plane parametric curves, and plane vector functions. This chapter follows that on applications of integration, so that lengths of polar and parametric curves, and areas bounded by them, can be done in proper sequence.
- The chapter on Taylor's formula and Taylor series has been reorganized so that its material can be treated either with or without having first covered most of the material in the previous chapter on numerical series.
- Like the revised first edition, this book contains a final chapter on partial differentiation, included for use in those courses which must cover a few weeks of multivariable calculus at the end of the second semester. (Of course, the author's companion volume, *Calculus of Several Variables*, takes up where *Single-Variable Calculus* leaves off, presenting a full treatment of partial differentiation, multiple integration and vector calculus.)

Core and Optional Material

Any division of material into "core" and "optional" is necessarily somewhat arbitrary. I regard most of the material of Chapters 1–6 as core, with the exception of Sections 3.6 (the alternate presentation of ln and exp), 3.8 (the hyperbolic functions), and the approximation methods of Sections 5.4 and 5.5. I also consider Sections 8.1–8.3 (basic geometric applications of integration), 9.2 (polar coordinates), 9.3 (parametric curves), 10.1 and 10.2 (the basics of infinite sequences and series), and most of Chapter 11 as core. These days, scientists and engineers require more training in the proper and effective use of numerical techniques, so Sections 5.5 (root finding), and much of Chapter 7 (numerical integration) are assuming more importance and might be regarded as "core" for some classes.

The remaining material is optional in the sense that (with minor exceptions) its prior coverage is not necessary for any of what follows. It is up to individual instructors to decide what is most appropriate for their classes.

Acknowledgments

The first edition of *Single-Variable Calculus* has been used, since its publication, for classes of general science, engineering, and mathematics majors and honours students at the University of British Columbia. I am grateful to colleagues and students at UBC, and at many other institutions where the book has been used, for their encouragement and useful comments and criticisms. Many of the changes in this edition are a result of that feedback. I am also grateful to several reviewers and proofreaders for their helpful suggestions during the preparation of this edition. Reviews of specific chapters were done by Professors T. Bisztriczky (University of Calgary), Ken Dunn (Dalhousie University), Tom Holens, (University of Manitoba), David J. Leeming, (University of Victoria), Richard Nowakowsky, (Dalhousie University), David Ryeburn (Simon Fraser University), Cedric Schubert (Queens University), and R. Grant Woods, (University of Manitoba). Painstaking reviews of the finished typescript were done by David Ryeburn and Ken Dunn, and by student Joanna Kwan (UBC). A final, thorough editorial proofreading was done by production editor Valerie Adams at Addison-Wesley.

I typeset this volume using TeX and PostScript on an AT microcomputer. I also generated most of the figures in PostScript using software developed by myself and my colleague, Professor Robert Israel. Some of the three dimensional air-brush art was prepared by Iris Ward. Prior to my starting the revision, the unrevised text of the first edition was committed to computer files in TeX format by Valerie Adams. I am very grateful to all these people for their excellent work.

I also wish to thank several people at Addison-Wesley for their assistance and encouragement. These include Sponsoring Editor Jim Grant, who guided the project and arranged for the reviews, Vice-President Andy Yull, with whom I enjoyed frequent stimulating discussions on matters of design and on numerous problems involving the TeX–PostScript interface, Executive Vice-President Joe Swan, and Editorial Director Ron Doleman, who supervised the publication of the first edition of *Single-Variable Calculus* and of *Calculus of Several Variables*, who first introduced the author to TeX and PostScript, and who assumed responsibility, along with Shirley Tessier, for the final stages of the publication of this edition.

Despite all the excellent help I have received, I am not so naïve as to believe that the text is now free of errors and obscurities, and I accept full responsibility for any that remain. Any comments, corrections, and suggestions for future revisions from readers will be much appreciated.

R.A.A.
Vancouver, Canada
September, 1989

CONTENTS

PREFACE

CHAPTER 1 **FUNCTIONS, LIMITS, AND CONTINUITY** 1

1.1 What is Calculus? 2

1.2 The Real Line and the Cartesian Plane 3

Intervals 5
The Absolute Value 7
Coordinates and Graphs 9

1.3 Functions 14

Operations on Functions 18
Odd and Even Functions 20

1.4 Inverse Functions 22

Inverting Non-One-to-One Functions 25

1.5 Limits 26

1.6 Extensions of the Limit Concept 33

One-Sided Limits 33
Limits at Infinity 35
Infinite Limits 37

1.7 Continuity 40

CHAPTER 2 **DIFFERENTIATION** 47

2.1 Tangent Lines and Their Slopes 48

Normals 52

2.2 The Derivative 53

Leibniz Notation 58
Differentials 60

2.3 Differentiation Rules (Sums, Products, Quotients) 62

Sums and Constant Multiples 63

The Product Rule 64
The Reciprocal Rule 65
The Quotient Rule 66

2.4 The Chain Rule 68

Proof of the Chain Rule 71

2.5 Implicit Differentiation 73

Derivatives of Inverse Functions 75

2.6 Interpretations of the Derivative 77

Velocity and Acceleration 79
Marginals 81

2.7 Higher-Order Derivatives 83

2.8 Antiderivatives and Indefinite Integrals 86

Differential Equations and Initial-Value Problems 89

CHAPTER 3 **THE ELEMENTARY TRANSCENDENTAL FUNCTIONS** 93

3.1 The Circular (Trigonometric) Functions 94

Cosine and Sine of Special Angles 97
The Addition Formulas 99
Other Trigonometric Functions 101
Some Trigonometry 102

3.2 Derivatives of the Trigonometric Functions 105

Simple Harmonic Motion 109

3.3 The Inverse Trigonometric Functions 114

The Inverse Sine Function 114
The Inverse Tangent Function 118
Other Inverse Trigonometric Functions 120

3.4 Exponential and Logarithmic Functions 122

3.5 Derivatives of Exponentials and Logarithms 125

Logarithmic Differentiation 130
The Growth of Exponentials and Logarithms 131

3.6 An Alternate Approach to Exp and Ln 133

3.7 Growth and Decay Problems 138

Interest on Investments 140
Logistic Growth 142

3.8 The Hyperbolic Functions and Their Inverses 144

3.9 Second Order Constant Coefficient Differential Equations 149

CHAPTER 4 **THE MEAN-VALUE THEOREM AND CURVE SKETCHING** 155

4.1 The Mean-Value Theorem 156

Some Consequences of the Mean-Value Theorem 158
Proof of the Mean-Value Theorem 161

4.2 Critical Points and Extreme Values 162

Maximum and Minimum Values 163
Endpoints, Critical Points and Singular Points 164
The First Derivative Test 165

4.3 Concavity and Inflections 169

The Second Derivative Test 172

4.4 Sketching the Graph of a Function 174

Asymptotes 176
Examples of Formal Curve Sketching 179

CHAPTER 5 **APPLICATIONS OF DIFFERENTIATION** 187

5.1 Optimization Problems 188

Procedure for Solving Optimization Problems 191

5.2 Related Rates 198

Procedures for Related Rates Problems 200

5.3 The Tangent-Line Approximation 206

The Error Estimate 208
Errors in Measurement 210
The Generalized Mean-Value Theorem 211
Proof of the Error Estimate (Theorem 5.3.4) 212

5.4 Higher Order Approximations – Taylor's Formula 213

Taylor's Formula 215

5.5 Finding Roots 218

The Bisection Method 218
Newton's Method 220
Fixed Point Iteration 223

5.6 Indeterminate Forms and l'Hôpital's Rules 227

l'Hôpital's Rules 228

CHAPTER 6 **INTEGRATION** 235

6.1 Areas Under Curves 236

Summation Notation 239
Some Area Calculations 241

6.2 The Definite Integral 245
Properties of the Definite Integral 250
The Mean-Value Theorem for Integrals 252
Definite Integrals of Piecewise Continuous Functions. 253
6.3 The Fundamental Theorem of Calculus 255
6.4 The Method of Substitution 261
Trigonometric Integrals 266
6.5 Inverse Substitutions 270
The Inverse Trigonometric Substitutions 270
Completing the Square 274
Other Inverse Substitutions 275
The tan(=2) Substitution 275
6.6 Integration By Parts 277
Reduction Formulas 281
6.7 The Method of Partial Fractions 284
Summary of Techniques of Integration 293
Review Exercises on Techniques of Integration 294
6.8 Improper Integrals 295
Estimating Convergence and Divergence 301

CHAPTER 7 **NUMERICAL INTEGRATION** 305
7.1 The Trapezoid and Midpoint Rules 306
The Midpoint Rule 309
Error Estimates 310
7.2 Simpson's Rule 313
7.3 Romberg Integration 318
Using the Romberg Method 319
Proof of Theorem 7.3.1 322
7.4 Other Aspects of Approximate Integration 324
Approximating Improper Integrals 325
Other Methods 326

CHAPTER 8 **APPLICATIONS OF INTEGRATION** 329
8.1 Areas of Plane Regions 330
Areas Between Two Curves 331
8.2 Volumes 335
Volumes by Slicing 336
Solids of Revolution 341
Cylindrical Shells 344

8.3 Arc Length and Surface Area 348
Arc Length 348
Areas of Surfaces of Revolution 353
8.4 Mass, Moments, and Centre of Mass 355
Moments and Centres of Mass 358
Two- and Three-Dimensional Examples 360
8.5 Centroids 363
The Pappus Theorem 367
8.6 Other Physical Applications 369
Hydrostatic Pressure 369
Work 371
Potential and Kinetic Energy 373
8.7 Probability 376
Expectation, Mean, Variance, and Standard Deviation 379
The Normal distribution 383
8.8 First-Order Separable and Linear Differential Equations 387
Separable Equations 387
First-Order Linear Equations 391

CHAPTER 9 **PLANE CURVES** 395
9.1 Conics 396
Circles 397
Parabolas 397
The Focal Property of a Parabola 399
Ellipses 401
The Focal Property of an Ellipse 402
Directrices 403
Hyperbolas 403
The Focal Property of the Hyperbola 405
Classifying General Conics 406
9.2 Polar Coordinates and Polar Curves 410
Some Polar Curves 412
Polar Conics 414
The Slope of a Polar Curve 416
Areas Bounded by Polar Curves 417
Arc Length of Polar Curves 418
9.3 Parametric Curves 420
General Plane Curves: Parametrizations 423
Some Interesting Plane Curves 424
9.4 Smooth Curves 427
The Slope of a Parametric Curve 428

Sketching Parametric Curves 429
Arc Lengths of Parametric Curves 430
Areas Bounded by Parametric Curves 432

9.5 Velocity, Acceleration and Plane Vectors 435

Plane Vectors 435
Position, Velocity, and Acceleration as Vectors 438
The Projectile Problem 441

CHAPTER 10 **SEQUENCES AND SERIES** 445

10.1 Sequences and Convergence 446

10.2 Infinite Series 453

Geometric Series 455
Telescoping Series and Harmonic Series 457
Some Theorems About Series 458

10.3 Convergence Tests for Positive Series 460

Comparison Tests 460
The Integral Test 464
The Ratio Test 466

10.4 Absolute and Conditional Convergence 469

The Alternating Series Test 471

10.5 Estimating the Sum of a Series 476

Integral Bounds 476
Geometric Bounds 479
Alternating Series Bounds 481

CHAPTER 11 **REPRESENTING FUNCTIONS BY POWER SERIES** 483

11.1 Taylor Polynomials and Taylor's Formula 484

Taylor and Maclaurin Series 489
Taylor's Theorem with Integral Remainder 489

11.2 Power Series 491

Algebraic Operations on Power Series 494
Differentiation and Integration of Power Series 496

11.3 Taylor and Maclaurin Series 502

Maclaurin Series for Some Elementary Functions 503
Other Maclaurin and Taylor Series 506
The Binomial Series 508

11.4 Applications of Taylor and Maclaurin Series 512

Approximating the Values of Functions 512
Functions Defined by Integrals 514
Indeterminate Forms 515

CHAPTER 12 **PARTIAL DIFFERENTIATION** 517

12.1 Functions of Several Variables 518

Graphical Representations 519
Limits and Continuity 521

12.2 Partial Derivatives 525

Tangent Planes 527
Higher Order Derivatives 529

12.3 The Chain Rule 534

Homogeneous Functions 538
Higher Order Derivatives 539
Approximations and Differentiability 540
Proof of the Chain Rule 542

12.4 Gradients and Directional Derivatives 545

Directional Derivatives 546
Tangent Lines to Level Curves 549
Higher Dimensional Vectors 550
The Gradient in Higher Dimensions 551
Tangents to the Graphs of Functions 553

12.5 Extreme Values 554

Classifying Critical Points 557
Extreme Values of Functions Defined on Closed, Bounded Sets 558
Extreme Value Problems with Constraints 560

APPENDIX I **MATHEMATICAL INDUCTION** 565

APPENDIX II **THE THEORETICAL FOUNDATIONS OF CALCULUS** 568

Limits of Functions 569
Continuous Functions 574
Completeness and Sequential Limits 575
Continuous Functions on a Closed, Finite Interval 577

APPENDIX III **THE RIEMANN INTEGRAL** 579

Uniform Continuity 583

ANSWERS TO ODD-NUMBERED EXERCISES 587

INDEX 615

CHAPTER 1

Functions, Limits, and Continuity

1.1 WHAT IS CALCULUS?

Much of our understanding of the world in which we live depends on our ability to describe how things change. Whether we are concerned with the motion of a pitched baseball or the path of a planet, whether the temperatures and currents of the oceans or the fluctuations of the stock market, whether the propagation of radio waves or the power produced by a chemical reaction, we are constantly forced to analyze relationships among quantities which change with time.

Algebra and geometry are useful tools for describing relationships among *static* quantities, but they do not involve concepts appropriate for describing how a quantity changes. For this we need new mathematical operations which go beyond the algebraic operations of addition, subtraction, multiplication, division and the taking of powers and roots. We require operations which measure the way related quantities change.

Calculus provides the tools for describing motion quantitatively. It introduces two new operations called *differentiation* and *integration* which, like addition and subtraction, are opposites of one another; what differentiation does, integration undoes.

For example, consider the motion of a falling rock. The height (in metres) of the rock t seconds after it is dropped from a height h_0 is a function $h(t)$ given by

$$h(t) = h_0 - 4.9t^2.$$

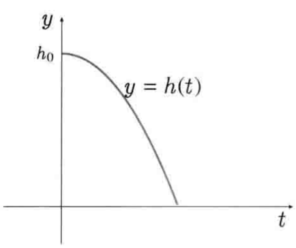

FIGURE 1.1.1

The graph of $y = h(t)$ is shown in Fig. 1.1.1. The process of differentiation enables us to find a new function, which we denote $h'(t)$ and call *the derivative* of h with respect to t:

$$h'(t) = -9.8t,$$

and which represents the *rate of change* of the height of the rock, that is, its *velocity* in metres/second.

Inversely, if we know the velocity of the rock as a function of time, integration enables us to find the height function $h(t)$.

Calculus was invented independently and in somewhat different ways by two 17th century mathematicians, Sir Isaac Newton and Gottfried Wilhelm Leibniz. Newton's motivation was a desire to analyze the motion of moving objects. Using his calculus he was able to formulate his laws of motion and gravitation, and to *calculate from them* that the planets must move around the sun in elliptical orbits, a fact that had been discovered half a century earlier by Johannes Kepler. Kepler's discovery was empirical, made from years of study of numerical data on the positions of planets.

Many of the most fundamental and important "laws of nature" are conveniently expressed as equations involving rates of change of quantities. Such equations are called *differential equations* and techniques for their study and solution are at the heart of calculus. In the falling rock example the appropriate law is Newton's second law of motion:

$$\text{Force} = \text{mass} \times \text{acceleration}.$$

The *acceleration*, -9.8 m/sec^2, is the rate of change (the *derivative*) of the velocity, which is in turn the rate of change (the *derivative*) of the height function.

Much of mathematics is related indirectly to the study of motion. We regard *lines* or *curves* as geometric objects, but the ancient Greeks thought of them as paths traced out by moving points. Nevertheless, the study of curves also involves geometric concepts such as tangency and area. The process of differentiation (Chapters 2–4) is closely tied to the geometric problem of finding tangent lines; similarly, integration (Chapters 6–8) is related to the geometric problem of finding areas of regions with curved boundaries.

Underpinning the study of calculus are the concepts of real number, coordinate system, and function. In the next three sections of this chapter we will review these concepts and set out the terminology and symbols we will use in referring to them throughout the book. The remaining sections introduce and explore the concept of **limit**, an operation on functions. The use of limits distinguishes calculus from other branches of mathematics (arithmetic, algebra, geometry) you have already encountered.

1.2 THE REAL LINE AND THE CARTESIAN PLANE

Elementary calculus depends heavily on properties of **real numbers**, that is, numbers expressible in decimal form such as

$$\begin{aligned} 5 &= 5.00000\ldots \\ -\tfrac{3}{4} &= -0.750000\ldots \\ \tfrac{1}{3} &= 0.3333\ldots \\ \sqrt{2} &= 1.4142\ldots \\ \pi &= 3.14159\ldots \end{aligned}$$

We expect that as a student of calculus you already have some familiarity with the real numbers and with the Cartesian coordinate system in the plane. Both are treated only briefly here to establish the terminology.

The real numbers can be represented geometrically as points on a number line, which we call the **real line**, shown in Fig. 1.2.1. The symbol $\mathbb{R}$ is used to denote either the real number system or, equivalently, the real line.

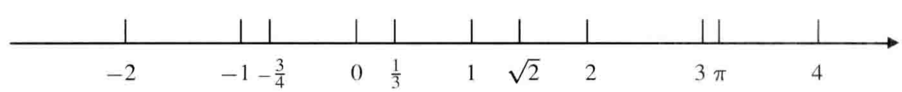

FIGURE 1.2.1

The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. The algebraic properties will already be familiar to you, and we will not dwell on them here; roughly speaking, they assert that real numbers may be added, subtracted, multiplied, and divided (except by zero) to produce more real numbers, and that the usual laws of arithmetic are satisfied.

The *order properties* refer to the order in which the numbers appear on the real line. If x lies to the left of y, then we say $x < y$ or $y > x$. Of course $x \leq y$ means that either $x < y$ or $x = y$. The order properties can be summarized as follows:

1.2.1
Order Properties of Real Numbers

i) If $x < y$ and z is any real number, then $x + z < y + z$.

ii) If $x < y$ and $z > 0$, then $xz < yz$.

iii) If $x < y$ and $z < 0$, then $xz > yz$; in particular, for $z = -1$, $-x > -y$.

iv) If $0 < x < y$ then $0 < \dfrac{1}{y} < \dfrac{1}{x}$.

Note especially the rules for multiplying an inequality by a number. If the number is positive, the inequality is preserved; if the number is negative, the inequality is reversed.

The *completeness* property of the real number system is more subtle and difficult to understand. One way to state it is as follows: If A is any set of real numbers having at least one number in it, and if there exists a real number y with the property that $x \leq y$ for every x in A, then there exists a *smallest* number y with the same property. Roughly speaking, this says that there can be no holes or gaps on the real line—every point corresponds to a real number. Certain important results in calculus require the completeness property for their proofs. Most of these results can be derived with no great difficulty from a few basic theorems, in particular Theorems 1.7.7 and 1.7.10 below. We do not prove these theorems in this chapter, but sketch their proofs in Appendix II, which is concerned with the theoretical foundations of calculus. The techniques for formal proofs involving limits in that appendix often are not studied in first courses in calculus but are deferred to subsequent courses in mathematical analysis. We will, however, make some direct use of completeness when we study infinite sequences and series in Chapter 11.

We distinguish three special subsets of the real numbers:

i) the **natural numbers**, namely the numbers 1, 2, 3, 4, ...

ii) the **integers**, namely, the numbers 0, ± 1, ± 2, ± 3, ...

iii) the **rational numbers**, that is, numbers that can be expressed in the form m/n, where m and n are integers, and $n \neq 0$.

The rational numbers are precisely those real numbers with decimal expansions that are either:

a) terminating, (that is, ending with an infinite string of zeros), or

b) repeating, (that is, ending with a string of digits that repeats over and over).