Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1307

Takafumi Murai

A Real Variable Method for the Cauchy Transform, and Analytic Capacity

Springer-Verlag

Berlin Heidelberg New York London Paris Tokyo

Author

Takafumi Murai
Department of Mathematics, College of General Education
Nagoya University
Nagoya, 464, Japan

Mathematics Subject Classification (1980): Primary 30C85; secondary 42A50

ISBN 3-540-19091-0 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-19091-0 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

PREFACE

The purpose of this lecture note is to study the Cauchy transform on curves and analytic capacity. For a compact set Γ in the complex plane $\mathbb{C}$, $H^\infty(\Gamma^c)$ denotes the Banach space of bounded analytic functions in $\mathbb{C}\cup\{\infty\}-\Gamma$ $(=\Gamma^c)$ with supremum norm $\|\cdot\|_{H^\infty}$. The analytic capacity of Γ is defined by

$$\gamma(\Gamma) = \sup\{|f'(\infty)|;\ \|f\|_{H^\infty} \leqq 1,\quad f\in H^\infty(\Gamma^c)\},$$

where $f'(\infty) = \lim_{z\to\infty} z(f(z)-f(\infty))$. We also define

$$\gamma_+(\Gamma) = \sup\{(1/2\pi)\int d\mu;\ \|C\mu\|_{H^\infty} \leqq 1,\quad C\mu\in H^\infty(\Gamma^c),\quad \mu \geqq 0\},$$

where

$$C\mu(z) = (1/2\pi i)\int 1/(\zeta-z)\, d\mu(\zeta) \qquad (z \notin (\text{the support of } \mu)).$$

We are concerned with estimating $\gamma(\cdot)$ and $\gamma_+(\cdot)$. To do this, compact sets having finite 1-dimension Hausdorff measure are critical. Hence we assume that Γ is a finite union of mutually disjoint smooth arcs. Let $|\cdot|$ denote the 1-dimension Hausdorff measure (the generalized length). Let $L^p(\Gamma)$ $(1\leqq p\leqq\infty)$ denote the L^p space of functions on Γ with respect to the length element $|dz|$, and let $L^1_w(\Gamma)$ denote the weak L^1 space of functions on Γ. Put

$$\rho(\Gamma) = \inf \gamma(E)/|E|,\quad \rho_+(\Gamma) = \inf \gamma_+(E)/|E|,$$

where the infimums are taken over all compact sets E in Γ. The Cauchy(-Hilbert) transform on Γ is defined by

$$H_\Gamma f(z) = (1/\pi)\ \text{p.v.} \int_\Gamma f(\zeta)/(\zeta-z)\ |d\zeta| \qquad (z\in\Gamma).$$

Then we see that

$$\rho_+(\Gamma) \leqq \rho(\Gamma) \leqq \text{Const}\ \rho_+(\Gamma)^{1/3},\quad \text{Const}\ \rho_+(\Gamma) \leqq 1/\|H_\Gamma\|_{L^1(\Gamma),L^1_w(\Gamma)} \leqq \text{Const}\ \rho_+(\Gamma),$$

where $\|H_\Gamma\|_{L^1(\Gamma),L^1_w(\Gamma)}$ is the norm of H_Γ as an operator from $L^1(\Gamma)$ to $L^1_w(\Gamma)$ (Theorem D). Hence the study of $\gamma(\Gamma)$ is closely related to the study of H_Γ.

Here is a history of the study of the Cauchy transform on Lipschitz graphs. According to Professor Igari, the L^2 boundedness of the Cauchy transform on Lipschitz graphs was first conjectured by Professor Zygmund in his lecture at Orsay in 1960's. Let $\Gamma = \{(x,A(x));\ x\in\mathbb{R}\}$, $a(x) = A'(x)$, where $\mathbb{R}$ is the real line. Let $C[a]$ denote the singular integral operator defined by a kernel $1/\{(x-y)+i(A(x)-A(y))\}$. Then the above conjecture means the following assertion: $C[a]$ is bounded (from $L^2(\mathbb{R})$ to itself) if $a\in L^\infty(\mathbb{R})$. The operator $C[a]$ is formally expanded in the following form: $(-\pi)H + \Sigma_{n=0}^\infty(-i)^n\, T_n[a]$, where H is the Hilbert transform and $T_n[a]$ is the singular integral operator defined by a kernel $(A(x)-A(x))^n/(x-y)^{n+1}$. In 1965, Calderón [3] showed that $T_1[a]$ is bounded if $a \in L^\infty(\mathbb{R})$ (Theorem A). This theorem is very important and closely related to the BMO($\mathbb{R}$) theory, where BMO($\mathbb{R}$) is the Banach space, modulo constants, of functions on $\mathbb{R}$ of bounded mean oscillation. Coifman-Meyer [8], [9] studied $T_n[a]$, Calderón [4] showed that $C[a]$ is bounded if $\|a\|_{L^\infty(\mathbb{R})}$ is sufficiently small, and consequently Coifman-McIntosh-Meyer [7] solved the above conjecture in the affirmative (Theorem B). David [17] studied H_Γ for continuous curves Γ. It is already known [44] that $\|C[a]\|_{L^2(\mathbb{R}),L^2(\mathbb{R})} \leqq \mathrm{Const}(1 + \sqrt{\|a\|_{BMO(\mathbb{R})}})$ (Theorem C) and that the square root is best possible [18]. Jones-Semmes gives a simple proof of Theorem B by complex variable methods. (See Appendix II.)

As a first step of the study of H_Γ for discontinuous curves Γ, we begin with a review of the study of $C[a]$. In CHAP. I, 8 proofs of Theorem A will be given. Once this theorem is known, we can easily deduce Theorem B (cf. CHAP. II), and hence Theorem A is very important in the study of $C[a]$. As is easily seen, if $f, g \in L^2(\mathbb{R})$ have analytic extensions $f(z)$, $g(z)$ to the upper half plane (such that $\lim_{y\to\infty} f(iy) = \lim_{y\to\infty} g(iy) = 0$), then the Poisson extension of $(fg)(x)$ to the upper half plane is identical with $f(z)g(z)$. This simple property of analytic functions is essential in a proof of Theorem A by complex variable methods. We shall give, in CHAP. I, various interpretations of this property from the point of view of real analysis (cf. Coifman-Meyer-Stein [13]). These proofs are, of course, mutually very close, but each proof has proper applications and is interesting in itself.

In CHAP. II, we shall give the proofs of Theorems B and C by perturbation. Our method is an improvement of Calderón's perturbation [4] and David's perturbation [17]. Put

$$\sigma(C[a]) = \sup(1/|I|)\int_I |C[a](\chi_I f)(x)|\, dx,$$

where χ_I is the characteristic function of I and the supremum is taken over all intervals I and all real-valued functions f with $\|f\|_{L^\infty(\mathbb{R})} \leqq 1$. This quantity is comparable to $\|C[a]\|_{L^\infty(\mathbb{R}),BMO(\mathbb{R})}$ and convenient for our perturbation. Considering a suitable Calderón-Zygmund decomposition of a primitive $A(x)$ of $a(x)$ on I, we obtain an a-priori estimate of $(1/|I|)\int_I |C[a](\chi_I f)(x)|\, dx$ by moderate graphs. (See the figure in § 2.2.) Repeating this argument infinitely many times and estimating infinitely many error terms, we see that the boundedness of $C[a]$ is consequently reduced to the boundedness of H. For the proof, Theorem A is necessary. We shall also give a proof of Theorem A by perturbation [45]. Tools which we use are only the Calderón-Zygmund decomposition and the covering lemma. For the proof of Theorem C, we put

$$\hat{\sigma}(C[a]) = \sup(1/|I|)\int_I |C[a](\chi_I f)(x)|^2\, f(x)\, dx,$$

where the supremum is taken over all intervals I and all real-valued functions f with $0 \leq f \leq 1$. Then $\sigma(C[a])^2 \leq$ Const $\hat{\sigma}(C[a])$. Since $\int_I C[a](\chi_I f)(x)f(x)\,dx = 0$, this quantity behaves like a linear functional of $a(x)$, and this gives an a-priori estimate better than $\sigma(C[a])$. Our method is not short but very simple, and this is applicable to various kernels.

In CHAP. III, we shall study H_Γ for discontinuous graphs Γ and shall compare $\gamma(\cdot)$ with integralgeometric quantities. We first give the proof of Theorem D. As is well-known, planar Cantor sets are useful to construct various examples (cf. Denjoy [23], Vitushkin [52]). Let $Q_0 = [0,1]\times[0,1]$ and let Q_n $(n \geq 1)$ be the union of 4^n closed squares with sides of length 4^{-n} obtained from Q_{n-1} with each component of Q_{n-1} replaced by four squares in the four corners of the component. Put $Q_\infty = \bigcap_{n=0}^\infty Q_n$. Then $\gamma(Q_\infty) = 0$ and $|Q_\infty| > 0$ (Garnett [28]). This shows that two classes of null sets of $\gamma(\cdot)$ and $|\cdot|$ are different. We shall try to give grounds to this example. We may consider that Q_n is a graph. (See the figure in § 3.3.) Let $T_{s_1,..,s_n}$ $(s_1,..,s_n \in \mathbb{R})$ be the singular integral operator defined by a kernel

$$1/\{(x-y)+i(A_{s_1,..,s_n}(x) - A_{s_1,..,s_n}(y))\},$$

where $A_{s_1,..,s_n}(x) = s_k$ $((k-1)/n \leq x < k/n,\ 1 \leq k \leq n)$ and $A_{s_1,..,s_n}(x) = 0$ $(x \in [0,1))$. Then we see that

$$\max\{\sigma(T_{s_1,..,s_n});\ s_1,..,s_n \in \mathbb{R}\}$$

is comparable to $\sqrt{\log(n+1)}$ (Theorem G), and, if we neglect constant multiples, an n-tuple $(s_1^0,..,s_n^0)$ obtained from a graph $\{(x, A_{s_1^0,..,s_n^0}(x));\ x \in [0,1)\}$ similar to Q_m $(m =$ (the integral part of $(\log n)/4))$ is a solution of this extremal problem. Hence planar Cantor sets are worst curves in a sense. We shall also generalize Q_n. A segument $[0,1)$ is called a (thick) crank of degree 0 and a finite union Γ_n of segments parallel to the x-axis is called a (thick) crank of degree n, if Γ_n is obtained from a crank Γ_{n-1} of degree $n-1$ with each component J of Γ_{n-1} replaced by a finite number of segments $J_1,..,J_{2^p}$ $(p=p(J))$ parallel to the x-axis such that $|J_k| = 2^{-p}|J|$, the distance between J_k and J is less than or equal to $2^{-p}|J|$ $(1 \leq k \leq 2^p)$ and the projections of these segments to $\mathbb{R}$ are mutually disjoint and contained in the projection of J. We shall show that, for any crank Γ of degree n, $\|H_\Gamma\|_{L^2(\Gamma),L^2(\Gamma)} \leq$ Const $\sqrt{n}$ and that this estimate is best possible (Theorem E). To prove this, we define $n+1$ singular integral operators $\{T_k\}_{k=0}^n$ such that $T_0 = (-\pi)H$, $\|\Sigma_{k=0}^n T_k\|_{L^2(\mathbb{R}),L^2(\mathbb{R})} = \|H_\Gamma\|_{L^2(\Gamma),L^2(\Gamma)}$ and $\{T_k\}_{k=0}^n$ are mutually almost orthogonal. Hence we see that the meaning of $\sqrt{n}$ is the central limit theorem.

We define integralgeometric quantities $Cr_\alpha(\cdot)$ $(0<\alpha<1)$ as follows. Let $D(z,r)$ be the open disk of center z and radius r. For a compact set E, $N_E(r,\theta)$ $(r>0, |\theta| \leq \pi)$ denotes the (cardinal) number of elements of $E \cap L(r,\theta)$, where $L(r,\theta)$ is the straight line defined by the equation $x\cos\theta + y\sin\theta = r$. We put

$$Cr_\alpha(E) = \lim_{\varepsilon\to 0} Cr_\alpha^{(\varepsilon)}(E),$$

$$Cr_\alpha^{(\varepsilon)}(E) = \inf \int_{-\pi}^{\pi}\{\int_0^\infty N_{\partial\{\bigcup_{k=1}^n D(z_k,r_k)\}}(r,\theta)^\alpha\, dr\}\, d\theta \quad (\varepsilon>0),$$

where $\partial\{\bigcup_{k=1}^n D(z_k,r_k)\}$ is the boundary of $\bigcup_{k=1}^n D(z_k,r_k)$ and the infimum is taken over all finite coverings $\{D(z_k,r_k)\}_{k=1}^n$ of E with radii less than ε. Since $\gamma(E) \leqq \text{Const } Cr_1(E)$, it is interesting to compare $\gamma(\cdot)$ with $Cr_\alpha(\cdot)$ (cf. Marshall [37]). As an application of Theorem E, we shall show that, for $0<\alpha<1/2$, there exists a compact set E_α such that $\gamma(E_\alpha) = 1$ and $Cr_\alpha(E_\alpha) = 0$ (Theorem F). For the proof, we use a branching process. Let $\{X_n\}_{n=1}^\infty$ be a sequence of independent random variables on the standard probability space $([0,1),\mathcal{B},\text{Prob})$ such that $\text{Prob}(X_n = \pm 1) = 1/2$ $(n\geqq 1)$, and let $S_0 = 0$, $S_n = \Sigma_{k=1}^n X_k$ $(n\geqq 1)$. We define a Galton-Watson process $\{y_n\}_{n=0}^\infty$ by $y_0(x) = 1$, $y_n(x) = y_{n-1}(x) + S_{y_{n-1}(x)}(x)$ $(n\geqq 1)$. Then we see that, for $n\geqq 1$, there exists a crank Γ_n of degree n such that $Cr_\alpha(\Gamma_n)$ is comparable to $\Sigma_{k=0}^\infty k^\alpha\, \text{Prob}(y_n=k)$. This quantity is comparable to $1/n^{1-\alpha}$. Using the difference of order between $1/\sqrt{n}$ (the central limit theorem) and $1/n^{1-\alpha}$ (the Galton-Watson process), we construct the required set E_α.

I express my hearty thanks to Professors M.Ohtsuka, R.R.Coifman, P.W.Jones who gave me the chance to lecture during the academic year 1986-1987, and I am grateful to Professors S.Kakutani, T.Tamagawa, J.Garnett, S.Semmes, T.Steger, G.David, C.Bishop for their variable comments and suggestions. I especially express my appreciation to Professor W.H.J.Fuchs for his encouragement. I also thank to Mrs. Mel D. for typing the manuscript. This note is dedicated to the memory of my mother who died while I was staying at Yale University.

New Haven, July, 1987

CONTENTS

CHAPTER I. The Calderón commutator (8 proofs of its boundedness) 1

1.1. Calderón's theorem 1

1.2. Proof of (1.3) 1

1.3. Area integral 2

1.4. Good λ inequalities 4

1.5. BMO 6

1.6. The Coifman-Meyer expression 9

1.7. A tent space 11

1.8. The McIntosh expression 13

1.9. Almost orthogonality 15

1.10. Interpolation 21

1.11. Successive compositions of kernels 24

CHAPTER II. A real variable method for the Cauchy transform on graphs 31

2.1. Coifman-McIntosh-Meyer's theorem 31

2.2. Two basic principles 32

2.3. σ-function 35

2.4. A-priori estimates 39

2.5. Proof of Theorem A by perturbation 47

2.6. Proof of Theorem B by perturbation 50

2.7. Estimates of norms of $E[\cdot]$ and $,C[\cdot]$ 53

2.8. Proof of (2.38) 55

2.9. Proof of (2.39) 61

2.10. Application of (2.38) 68

CHAPTER III. Analytic capacities of cranks 71

3.1. Relation between $\gamma(\cdot)$ and H 71

3.2. Vitushkin's example, Garnett's example, Calderón's problem and extremal problems 79

3.3. The Cauchy transform on cranks 83

3.4. Proof of the latter half of Theorem E 91

3.5. Analytic capacities of fat cranks 99

3.6. Analytic capacity and integralgeometric quantities 105

3.7. Proof of Theorem F 112

APPENDIX I. An extremal problem 117

APPENDIX II. Proof of Theorem B by P.W.Jones-S.Semmes 126

REFERENCES 129

SUBJECT INDEX 132

CHAPTER I. THE CALDERÓN COMMUTATOR
(8 PROOFS OF ITS BOUNDEDNESS)

§1.1. Calderón's Theorem (Calderón [3])

Let L^p $(1 \leq p \leq \infty)$ denote the L^p space on the real line $\mathbb{R}$ with respect to the 1-dimension Lebesgue measure $|\cdot|$. Its norm is denoted by $\|\cdot\|_p$. Let BMO denote the Banach space, modulo constants, of functions f on $\mathbb{R}$ such that $\|f\|_{BMO} = \sup(1/|I|) \int_I |f(x)-(f)_I| dx$ is finite, where the supremum is taken over all (finite) intervals I and $(f)_I$ is the mean of f over I. For $a \in L^\infty$, we define a kernel

$$(1.1) \quad T[a](x,y) = \{A(x) - A(y)\} /(x-y)^2,$$

where A is a primitive of a. We write simply by $T[a]$ the operator from L^2 to itself defined by the above kernel, i.e.,

$$(1.2) \quad T[a]f(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} T[a](x,y)f(y)dy.$$

Calderón showed

<u>Theorem A ([3])</u>. For any $f \in L^2$, $T[a]f(x)$ exists a.e.

$$(1.3) \quad \|a\|_\infty \leq \text{Const } \|T[a]\|_{2,2}$$

and

$$(1.4) \quad \|T[a]\|_{2,2} \leq \text{Const } \|a\|_\infty ,$$

where $\|T[a]\|_{2,2}$ is the norm of $T[a]$ (as an operator from L^2 to itself).

In §1.2, we show (1.3). In §1.3-1.11, we show various proofs of (1.4).

§1.2. Proof of (1.3) (Coifman-Rochberg-Weiss [15])

For a set $E \subset \mathbb{R}$, χ_E denotes the characteristic function of E. We put

$$\rho_\varepsilon(x) = \left|\int_{I_{+\varepsilon}} \{\int_{I_{-\varepsilon}} (A(s)-A(t))dt\}ds\right| \qquad (x \in \mathbb{R}, \quad \varepsilon > 0),$$

where $I_{+\varepsilon} = (x, x+\varepsilon)$, $I_{-\varepsilon} = (x-\varepsilon, x)$. Then $\lim_{\varepsilon \to 0} \rho_\varepsilon/\varepsilon^3 = \text{Const } |a|$ a.e. We have, for almost all x,

$$\rho_\varepsilon(x) = |\int_{I_{+\varepsilon}} [\int_{I_{-\varepsilon}} T[a](s,t)\{(s-x)^2 + 2(s-x)(x-t) + (x-t)^2\}\,dt\}\,ds|$$

$$\le \int_{I_{+\varepsilon}} [(s-x)^2 |T[a]\chi_{I_{-\varepsilon}}(s)| + 2|s-x|\,|T[a]\{(x-\cdot)\chi_{I_{-\varepsilon}}\}(s)|$$

$$+ |T[a]\{(x-\cdot)^2\chi_{I_{-\varepsilon}}\}(s)|]\,ds$$

$$\le \text{Const}[\varepsilon^{5/2}\|T[a]\chi_{I_{-\varepsilon}}\|_2 + \varepsilon^{3/2}\|T[a]\{(x-\cdot)\chi_{I_{-\varepsilon}}\}\|_2$$

$$+ \varepsilon^{1/2}\|T[a]\{(x-\cdot)^2\chi_{I_{-\varepsilon}}\}\|_2]$$

$$\le \text{Const}\,\|T[a]\|_{2,2}\{\varepsilon^{5/2}\|\chi_{I_{-\varepsilon}}\|_2 + \varepsilon^{3/2}\|(x-\cdot)\chi_{I_{-\varepsilon}}\|_2$$

$$+ \varepsilon^{1/2}\|(x-\cdot)^2\chi_{I_{-\varepsilon}}\|_2\} \le \text{Const}\|T[a]\|_{2,2}\,\varepsilon^3 ,$$

and hence

$$|a| = \text{Const}\lim_{\varepsilon\to 0}\rho_\varepsilon/\varepsilon^3 \le \text{Const}\|T[a]\|_{2,2} \qquad \text{a.e.}$$

Thus we obtain (1.3).

§1.3. Area integral ([3])

In this section we show the proof of (1.4) by Calderón. Let C_0^∞ denote the totality of infinitely differentiable functions with compact support, $(\cdot,\cdot)$ denote the inner product and $\Upsilon_\varepsilon = \chi_{(-\varepsilon,\varepsilon)^c}$ $(\varepsilon > 0)$. Given real-valued functions a,f,g in C_0^∞ and $\varepsilon > 0$, we estimate

$$(T^\varepsilon[a]g,f) = \int_{-\infty}^{\infty} T^\varepsilon[a]g(x)f(x)\,dx,$$

where $T^\varepsilon[a]$ is an operator defined by a kernel $\Upsilon_\varepsilon(x-y)\,T[a](x,y)$. We may assume that $A(x) = \int_{-\infty}^{x} a(s)\,ds$. Then $A(x) = \int_{-\infty}^{\infty} e(x-s)a(s)\,ds$, where $e = \chi_{[0,\infty)}$. We have

$$(T^\varepsilon[a]g,f) = \int_{-\infty}^{\infty} a(s)[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\Upsilon_\varepsilon(x-y)}{(x-y)^2}\{e(x-s)-e(y-s)\}g(y)f(x)\,dy\,dx]\,ds.$$

Set

$$f_\pm(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(x)}{x-z}\,dx \qquad \operatorname{Im} z \begin{Bmatrix} > 0 \\ < 0 \end{Bmatrix} .$$

We denote also by $f_\pm(x)$ $(x \in \mathbb{R})$ the non-tangential limit of $f_\pm(z)$, respectively. We define analogously $g_\pm(z)$, $g_\pm(x)$. Then $f = f_+ - f_-$, $g = g_+ - g_-$, $\|f_\pm\|_2 \le \|f\|_2$ and $\|g_\pm\|_2 \le \|g\|_2$. Let

$$K_0(x,y,s) = \Upsilon_\varepsilon(x-y)\ \{e(x-s)-e(y-s)\}\ /(x-y)^2,$$

$$K_1^{\pm}\ (x,y,s) = \{e(x-s)-e(y-s)\}\ /(x-y \pm i\varepsilon)^2,$$

$$K_2(x,y,s) = \varepsilon/\{(x-s)^2 + (y-s)^2 + \varepsilon^2\}^{3/2}\ .$$

Then $|K_0(x,y,s) - K_1^{\pm}(x,y,s)| \leqq \text{Const}\ K_2(x,y,s)$. We have

$$|(T^\varepsilon[a]g,f)| = |\int_{-\infty}^{\infty} a(s)[\int_{-\infty}^{\infty} K_0(x,y,s)\{g_+(y) - g_-(y)\}\ f(x)dydx]\ ds|$$

$$\leqq |\int_{-\infty}^{\infty} a(s)[\int_{-\infty}^{\infty} K_1^+(x,y,s)g_+(y)f(x)dydx]\ ds|$$

$$+ |\int_{-\infty}^{\infty} a(s)[\int_{-\infty}^{\infty} K_1^-(x,y,s)g_-(y)f(x)dydx]\ ds|$$

$$+ \text{Const} \int_{-\infty}^{\infty} |a(s)|\ [\int_{-\infty}^{\infty} K_2(x,y,s)\ \{|g_+(y)| + |g_-(y)|\}\ |f(x)|\ dydx]ds$$

$$(= |\int_{-\infty}^{\infty} a(s)\ k_1^+(s)ds| + |\int_{-\infty}^{\infty} a(s)k_1^-(s)ds| + \text{Const} \int_{-\infty}^{\infty} |a(s)|k_2(s)ds, \text{say}).$$

We now estimate $k_1^{\pm}(s)$, $k_2(s)$. We have

$$k_1^+(s) = \int_{-\infty}^{\infty} f(x)\ \{ \int_{-\infty}^{\infty} K_1^+(x,y,s)g_+(y)dy\}\ dx$$

$$= \int_{-\infty}^{\infty} f(x)\ \{e(x-s) \int_{-\infty}^{\infty} \frac{g_+(y)}{(x-y-i\varepsilon)^2}\ dy - \int_s^{\infty} \frac{g_+(y)}{(x-y-i\varepsilon)^2}\ dy\ \}\ dx$$

$$= -i \int_{-\infty}^{\infty} f(x)\ [\int_0^{\infty} g_+(s+it)/\{(x-i\varepsilon)-(s+it)\}^2\ dt]\ dx$$

$$= -i \int_0^{\infty} g_+(s+it)\ [\int_{-\infty}^{\infty} f(x)/\{(x-i\varepsilon)-(s+it)\}^2\ dx]\ dt$$

$$= 2\pi \int_0^{\infty} f_+'(s+i(t+\varepsilon))g_+(s+it)\ dt.$$

Let

$$F(z) = -i \int_0^{\infty} f_+'(z+i(t+\varepsilon))\ g_+(z+it)\ dt \qquad (z \in U),$$

where $U = \{(x,y);\ x \in \mathbb{R},\ y > 0\}$. Then F is analytic in U and the non-tangential limit $F(s)$ equals $(1/2\pi i)\ k_1^+(s)$. Here is a main lemma necessary for the proof of (1.4). Let $P_y(x)$ be the Poisson kernel, i.e., $P_y(x) = y/\{\pi(x^2+y^2)\}$. For a differentiable function $v(x,y)$ in U, we write
$|\nabla v(x,y)| = \{|\partial v/\partial x|^2 + |\partial v/\partial y|^2\}^{1/2}$.

Lemma 1.1 ([3]). For $v \in L^1$, we define

$$A(v)(x) = \{\iint_{\Delta(x)} |\nabla v(\xi,\eta)|^2\ d\xi\ d\eta\}^{1/2} \qquad (x \in \mathbb{R}),$$

where $v(\xi,\eta) = P_\eta * v(\xi)$ and $\Delta(x) = \{(\xi,\eta);\ |\xi - x| < \eta\}$. Then $\|v\|_1 \leqq \mathrm{Const}\|A(v)\|_1$.

Once this lemma is known, (1.4) is deduced as follows. Since $F'(z) = f_+'(z + i\varepsilon)g_+(z)$, we have $A(F)(s) \leqq A(f_+)(s)m(g_+)(s) \leqq \mathrm{Const}\ A(f_+)(s)\ \mathcal{M} g_+(s)$, where $m(g_+)(s) = \sup\{|g_+(\xi,\eta)|;(\xi,\eta) \in \Delta(x)\}$ and $\mathcal{M}$ is the non-centered maximal operator (Journé [35, p.6]). (See Lemma 2.3.) We have $\|\mathcal{M}g_+\|_2 \leqq \mathrm{Const}\|g_+\|_2$. Green's formula shows that $\|A(f_+)\|_2 = \mathrm{Const}\|f_+\|_2$. Thus we have, by Lemma 1.1,

$$\begin{aligned}|\int_{-\infty}^{\infty} a(s)k_1^+(s)ds| &\leqq 2\pi\|a\|_\infty \|F\|_1 \leqq \mathrm{Const}\|a\|_\infty \|A(F)\|_1 \\ &\leqq \mathrm{Const}\ \|a\|_\infty \|A(f_+)\|_2 \|m(g_+)\|_2 \leqq \mathrm{Const}\ \|a\|_\infty \|f_+\|_2 \|g_+\|_2 \\ &\leqq \mathrm{Const}\ \|a\|_\infty \|f\|_2 \|g\|_2 .\end{aligned}$$

In the same manner, we have $|\int_{-\infty}^{\infty} a(s)k_1^+(s)ds| \leqq \mathrm{Const}\ \|a\|_\infty \|f\|_2 \|g\|_2$.
We have

$$\begin{aligned}k_2(s) &\leqq \int_{-\infty}^{\infty} \frac{\varepsilon}{(x-s)^2+\varepsilon^2} |f(x)| \left[\int_{-\infty}^{\infty} \frac{\sqrt{(x-s)^2+\varepsilon^2}}{(x-s)^2+(y-s)^2+\varepsilon^2}\{|g_+(y)|+|g_-(y)|\}dy\right]dx \\ &\leqq \mathrm{Const}\ \mathcal{M} f(s)\ \{\mathcal{M}g_+(s) + \mathcal{M}g_-(s)\} ,\end{aligned}$$

and hence

$$\int_{-\infty}^{\infty} |a(s)|k_2(s)\, ds \leqq \mathrm{Const}\ \|a\|_\infty \|f\|_2 \|g\|_2 .$$

Consequently $|(T^\varepsilon[a]g,f)| \leqq \mathrm{Const}\ \|a\|_\infty \|f\|_2 \|g\|_2$. Since $f,g \in C_0^\infty$, $\varepsilon > 0$ are arbitrary, we have (1.4) for $a \in C_0^\infty$. In the general case, we can deduce (1.4) from the boundedness of maximal operators $T^*[b]$ $(b \in C_0^\infty)$ and Fatou's lemma. (See Lemma 2.5.)

§1.4. Good λ inequalities ([2], [26], [48])

In this section we give the proof of Lemma 1.1 by the so-called "good λ inequalities". We put $\tilde{m}(v)(x) = \sup\{|v(x,y)|;\ y > 0\}$. Fixing a sufficiently large τ, we prove

$$\begin{aligned}(1.5)\qquad &|x;\ \tilde{m}(v)(x) > \tau\lambda\ ,\ A(v)(x) \leqq \lambda/\tau\ | \\ &\leqq (\mathrm{Const}/\tau^2)\ |x;\ \tilde{m}(x) > \lambda| \qquad (\lambda > 0).\end{aligned}$$

Let $W(\lambda) = \{x;\ \tilde{m}(x) > \lambda\}$, $\delta(\lambda) = |W(\lambda)|$. Then we can write $W(\lambda) = \cup_{k=1}^\infty I_k$ with a sequence $\mathcal{M}_\lambda = \{I_k\}$ of mutually disjoint open intervals. It is sufficient to show that, for each $I \in \mathcal{M}_\lambda$,

$$(1.6)\qquad |E| \leqq (\mathrm{Const}/\tau^2)\ |I|,$$

where $E = \{x \in I; \tilde{m}(v)(x) > \tau\lambda , A(v)(x) \leqq \lambda/\tau\}$. To do this we may assume that $A(v)(\xi) \leqq \lambda/\tau$ for some $\xi \in I$; otherwise $E = \emptyset$. Since $A(v)(\xi) \leqq \lambda/\tau$, we have, for any $x \in I$, $y \geqq 2|I|$,

$$(1.7) \quad |v(\alpha,y) - v(x,y)| \leqq \text{Const } A(v)(\xi) \leqq \text{Const } \lambda/\tau ,$$

where α is the left endpoint of I. We choose τ large enough so that the last quantity in (1.7) is less than λ. Since $\tilde{m}(v)(\alpha) \leqq \lambda$, we have $|v(x,y)| \leqq 2\lambda$ $(x \in I,\ y \geqq 2|I|)$. Hence, for any $x \in E$, there exists $0 < y_x < 2|I|$ such that $|v(x,y_x)| = \sup\{|v(x,y)|;\ y > y_x\} = \tau\lambda$. Let $J(x) = (x - (y_x/5),\ x + (y_x/5))$, $\tilde{J}(x) = \{(\xi,y_x);\ |\xi - x| < y_x/10\}$ $(x \in E)$. Then, for any $(\xi,y_x) \in \tilde{J}(x)$, we have $|v(\xi,y_x)| \geqq |v(x,y_x)| - \text{Const } A(v)(x) \geqq \tau\lambda - \text{Const } \lambda/\tau \geqq \tau\lambda/2$. There exist a finite number of mutually disjoint intervals $\{J(x_\mu)\}$ such that $|E| \leqq 5\,\Sigma\,|J(x_\mu)|$. (See §2.2.) Let $R = Q_0 \cap \cup\, \tilde{\Delta}(x_\mu)$, where $Q_0 = \{(\xi,\eta);\ \xi \in I,\ 0 < \eta < 2|I|\}$, $\tilde{\Delta}(x_\mu) = \{(\xi,\eta);\ |\xi - x_\mu| < \eta/10,\ \eta > y_{x_\mu}\}$. Green's formula shows that

$$(1.8) \quad \int_{\partial R} \left\{ \frac{\partial\eta}{\partial n}|v|^2 - \eta\frac{\partial|v|^2}{\partial n} \right\} ds = \text{Const} \iint_R \eta\,|\nabla v|^2\, d\xi\, d\eta,$$

where $\partial/\partial n$ is the inner normal derivative and ds is the length element. Let $A_R(v)(x) = \{\iint_{\Delta^*(x) \cap R} |\nabla v|^2\, d\xi\, d\eta\}^{1/2}$, where $\Delta^*(x) = \{(\xi,\eta);\ |\xi - x| < \eta/10\}$. Then a geometric observation shows that $A_R(v)(x) \leqq A(v)(x_\nu) \leqq \lambda/\tau$, where x_ν is a point which is nearest to x in $\{x_\mu\}$. Hence the right-hand side of (1.8) is dominated by:

$$\text{Const} \int_I A_R(v)(x)^2 dx \leqq \text{Const}(\lambda/\tau)^2 |I| \leqq \text{Const } \lambda^2\, |I|.$$

We divide ∂R into the following three parts: $\partial R_0 = \partial R \cap \cup\, \tilde{J}(x_\mu)$, $\partial R_1 = \{(\xi,\eta);\ \xi \in I,\ \eta = 2|I|\}$, $\partial R_2 = \partial R - (\partial R_0 \cup \partial R_1)$. Note that $\eta|\nabla v(\xi,\eta)| \leqq \text{Const } \lambda/\tau$ on ∂R. By the definition of y_x $(x \in E)$, we have, for any $(\xi,\eta) \in \partial R$, $|v(\xi,\eta)| \leqq \tau\lambda + \text{Const } \lambda/\tau \leqq \text{Const } \tau\lambda$. Thus

$$\left|\int_{\partial R} \eta \frac{\partial|v|^2}{\partial n} ds\right| \leqq \text{Const} \int_{\partial R} \eta|\nabla v|\,|v|\, ds$$

$$\leqq \text{Const } (\lambda/\tau)\, \tau\lambda \int_{\partial R} ds \leqq \text{Const } \lambda^2\, |I|.$$

Since $|v(\xi,\eta)| \leqq \text{Const } \lambda$ on ∂R_1, we have $\left|\int_{\partial R_1} \frac{\partial\eta}{\partial n} |v|^2 ds\right| \leqq \text{Const } \lambda^2\, |I|$. These estimates yield that $\int_{\partial R_0 \cup \partial R_2} \frac{\partial\eta}{\partial n} |v|^2 ds \leqq \text{Const } \lambda^2\, |I|$. Since $\partial\eta/\partial n \geqq 0$ on ∂R_2, $\partial\eta/\partial n = 1$ on ∂R_0 and $|v(\xi,\eta)| \geqq \tau\lambda/2$ on ∂R_0, we have

$$\tau^2 \lambda^2 |E| \leqq \mathrm{Const} \int_{\partial R_0} \frac{\partial \eta}{\partial n} |v|^2 ds \leqq \mathrm{Const} \int_{\partial R_0 \cup \partial R_2} \frac{\partial \eta}{\partial n} |v|^2 ds \leqq \mathrm{Const}\, \lambda^2 |I|,$$

which shows (1.6). Consequently (1.5) holds.

By (1.5), we have, with a constant C_0,

$$(1.9) \qquad \delta(\tau\lambda) \leqq \tilde{\delta}(\lambda/\tau) + (C_0/\tau^2)\, \delta(\lambda),$$

where $\tilde{\delta}(\lambda) = |x;\ A(v)(x) > \lambda|$. We now choose $\tau = 2\, C_0$ and integrate each quantity in (1.9) by $d\lambda$ from 0 to infinity. Then we obtain $\|\tilde{m}(v)\|_1 \leqq \mathrm{Const}\ \|A(v)\|_1$, which gives $\|v\|_1 \leqq \mathrm{Const}\ \|A(v)\|_1$. This completes the proof of Lemma 1.1.

§1.5. BMO (Fefferman-Stein [27])

Theorem A is closely related to the theory of BMO [27]. In this section, we show the proof of Theorem A by Fefferman-Stein. We say that a non-negative measure $d\mu(x,y)$ in U is a Carleson measure with constant B if

$$\iint_{I \times (0,|I|)} d\mu(x,y) \leqq B\, |I|$$

for any interval $I \subset R$. The following two facts are elementary.

Lemma 1.2 ([27]). Let $a \in \mathrm{BMO}$. Then $y|\nabla a(x,y)|^2\, dx\, dy$ is a Carleson measure with constant $\mathrm{Const}\ \|a\|^2_{\mathrm{BMO}}$, where $a(x,y) = P_y * a(x)$.

Proof. Given an interval I, we put

$$a^{(1)}(x) = (a(x) - (a)_I)\, \chi_{I^*}(x), \quad a^{(2)}(x) = (a(x) - (a)_I)\, \chi_{I^{*c}}(x),$$

where $(a)_I = (1/|I|) \int_I a(y)dy$ and I^* is the double of I, i.e., the (open) interval of the same midpoint as I and of length $2|I|$. Then

$$a(x,y) = P_y * a^{(1)}(x) + P_y * a^{(2)}(x) + (a)_I$$
$$(= a^{(1)}(x,y) + a^{(2)}(x,y) + (a)_I, \text{ say}).$$

John-Nirenberg's inequality [32] shows that $\|a^{(1)}\|_2 \leqq \mathrm{Const}\ \|a\|_{\mathrm{BMO}} \sqrt{|I|}$. (See Lemma 2.5.) Hence we have, with $\hat{I} = I \times (0,|I|)$,

$$\iint_{\hat{I}} y\, |\nabla a^{(1)}(x,y)|^2\, dx\, dy \leqq \iint_U y\, |\nabla a^{(1)}(x,y)|^2\, dx\, dy$$
$$= \mathrm{Const}\ \|a^{(1)}\|_2^2 \leqq \mathrm{Const}\ \|a\|^2_{\mathrm{BMO}}\, |I|.$$

Note that $|(a)_{I_j} - (a)_I| \leqq \mathrm{Const}\ j\|a\|_{\mathrm{BMO}}$ $(j \geqq 1)$, where I_j is the interval of the same midpoint as I and of length $2^j|I|$. We have, for $(x,y) \in \hat{I}$

$$|\nabla a^{(2)}(x,y)| \leqq \text{Const} \int_{I^{*c}} \frac{1}{(x-s)^2} |a^{(2)}(s)|\, ds$$

$$\leqq \text{Const} \sum_{j=1}^{\infty} |I_j|^{-2} \int_{I_{j+1}-I_j} |a(y)-(a)_I|\, ds$$

$$\leqq \text{Const} \sum_{j=1}^{\infty} |I_j|^{-2} |I_{j+1}| \{\|a\|_{BMO} + |(a)_{I_j} - (a)_I|\}$$

$$\leqq \text{Const}\left(\sum_{j=1}^{\infty} j\, 2^{-j}\right) \|a\|_{BMO}/|I| ,$$

and hence

$$\iint_{\hat{I}} y\, |\nabla a^{(2)}(x,y)|^2\, dx\, dy \leqq \text{Const}(\|a\|_{BMO}/|I|)^2 \iint_{\hat{I}} y\, dx\, dy$$

$$\leqq \text{Const}\ \|a\|_{BMO}^2\, |I| .$$

Thus

$$\iint_{\hat{I}} y\ |\nabla a(x,y)|^2\, dx\, dy \leqq \text{Const}\ \{\iint_{\hat{I}} y\ |\nabla a^{(1)}(x,y)|^2\, dx\, dy$$

$$+ \iint_{\hat{I}} y\ |\nabla a^{(2)}(x,y)|^2\, dx\, dy\} \leqq \text{Const}\ \|a\|_{BMO}^2\, |I| . \qquad \text{Q.E.D.}$$

Lemma 1.3 ([35, p. 85]). Let $d\mu(x,y)$ be a Carleson measure with constant B. Then, for any $f \in L^2$,

$$\iint_U |f(x,y)|^2\, d\mu(x,y) \leqq \text{Const}\ B\|f\|_2^2 \qquad (f(x,y) = P_y * f(x)).$$

Proof. Let $W(\lambda) = \{(x,y) \in U;\ |f(x,y)| > \lambda\}$, $\delta(\lambda) = \iint_{W(\lambda)} d\mu(x,y)$ ($\lambda > 0$). Then the left-hand side of our lemma is dominated by

Const $\int_0^\infty \lambda\delta(\lambda)d\lambda$. If $(x,y) \in W(\lambda)$, then $\lambda \leqq \sup\{|f(\xi,\eta)|;\ |x-\xi| < \eta\} \leqq C\mathcal{M}f(x)$ for some constant C. Hence $W(\lambda)$ is contained in $W_0(\lambda) = \cup\, I \times (0,|I|)$, where the union is taken over all components I of $\{x;\ \mathcal{M}f(x) > C\lambda\}$. Thus

$$\delta(\lambda) \leqq \iint_{W_0(\lambda)} d\mu(x,y) \leqq B|x;\ \mathcal{M}f(x) > C\lambda| ,$$

which gives

$$\int_0^\infty \lambda\, \delta(\lambda)d\lambda \leqq B \int_0^\infty \lambda|x;\ \mathcal{M}f(x) > C\lambda|\, d\lambda$$

$$\leqq \text{Const}\ B\ \|\mathcal{M}f\|_2^2 \leqq \text{Const}\ B\ \|f\|_2^2 . \qquad \text{Q.E.D.}$$

We now prove Theorem A. The Hilbert transform H is defined by

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|s-x| > \varepsilon} \frac{f(s)}{s-x}\, ds .$$

For $a, f \in C_0^\infty$, we have

$$(1.10) \quad T[a]f(x) = -\pi\ H(af)(x) + \pi [A,H]f'(x),$$

where $[A,H]f' = A(Hf') - H(Af')$. Since $\|H(af)\|_2 \leq \|a\|_\infty \|f\|_2$, it is sufficient to show that $\|[A,H]f'\|_2 \leq \text{Const}\ \|a\|_\infty \|f\|_2$; we will prove a better inequality.

$$(1.11) \quad \|[A,H]f'\|_2 \leq \text{Const}\ \|a\|_{BMO}\ \|f\|_2 .$$

Without loss of generality we may assume that a, f are real-valued. We have, for any real-valued function $g \in C_0^\infty$,

$$([A,H]f',g) = \int_{-\infty}^{\infty} [A,H]f'(x)g(x)dx = (A, Hf'\cdot g + f'Hg)$$

$$= 4\ \text{Im}(A, f_+'\ g_+) = 4\ \text{Im}(A,F') = -4\ \text{Im}(a,F),$$

where

$$(1.12) \quad F(x) = \int_{-\infty}^{x} f_+'(s)g_+(s)ds = -i \int_0^\infty f_+'(x+is)g_+(x+is)ds.$$

Let $a(x,y) = P_y * a(x,y)$, $F(x,y) = P_y * F(x)$. Since $f_+'(z)$, $g_+(z)$ are analytic in U, we have $\frac{\partial F}{\partial x}(x,y) = f_+'(x+iy)g_+(x+iy)$. Thus Lemmas 1.2, 1.3 and Parseval's formula yield that

$$|(a,F)| = \text{Const}\ \left|\iint_U y\ \frac{\partial a}{\partial x}(x,y)\ \frac{\partial F}{\partial x}(x,y)\ dx\ dy\right|$$

$$= \text{Const}\ \left|\iint_U y\ \frac{\partial a}{\partial x}(x,y) f_+'(x+iy)g_+(x+iy)\ dx\ dy\right|$$

$$\leq \text{Const}\ \{\iint_U y|f_+'(x+iy)|^2\ dx\ dy\}^{1/2}\ \{\iint_U y\ |\nabla a(x,y)|^2 |g_+(x+iy)|^2 dx\ dy\}^{1/2}$$

$$\leq \text{Const}\ \|f_+\|_2\ \|a\|_{BMO}\ \|g_+\|_2 \leq \text{Const}\ \|a\|_{BMO}\ \|f\|_2\ \|g\|_2 .$$

This completes the proof of Theorem A.

Fefferman-Stein [27] showed also the following inequality, which is essentially same as (1.11).

Lemma 1.4 ([27]). Let $a \in BMO$. Then $\|[a,H]\|_{2,2} \leq \text{Const}\ \|a\|_{BMO}$.

Proof. Without loss of generality we may assume that a is real-valued. We have, for any real-valued functions $f,g \in C_0^\infty$,

$$([a,H]f,g) = (a,\ Hf\cdot g + fHg) = -4\ \text{Im}\ (a, f_+ g_+).$$

Let $G(x) = f_+(x)g_+(x)$. Then Parseval's formula shows that, with $G(x,y) = P_y * G(x)$, $a(x,y) = P_y * a(x)$,

$$|(a,f_+g_+)| = |(a,G)| = \text{Const}\ |\iint_U y\ \frac{\partial a}{\partial x}(x,y)\ \frac{\partial G}{\partial x}(x,y)\ dx\ dy\ |$$

$$\leq \text{Const}\ \{\iint_U y\ |\nabla a|^2\ |G|\ dx\ dy\}^{1/2}\ \{\iint_U y\ |\nabla G|^2\ |G|^{-1}\ dx\ dy\ \}^{1/2}.$$

Since $\log|G(x,y)|$ is subharmonic in U,

$$\Delta \log|G| = (\Delta|G| - \frac{|\nabla|G||^2}{|G|})\ \frac{1}{|G|} \geq 0,$$

and hence

$$\frac{|\nabla G|^2}{|G|} = \Delta|G| + \frac{|\nabla|G||^2}{|G|} \leq 2\ \Delta|G|.$$

This shows that

$$\iint_U y\ |\nabla G|^2\ |G|^{-1}\ dx\ dy \leq 2\ \iint_U y\ \ \Delta|G|\ dx\ dy = \text{Const}\ \|G\|_1.$$

Since $|G(x,y)|^{1/2}$ is subharmonic in U, we have $|G(x,y)| \leq P_y * (|G|^{1/2})(x)^2$. Hence Lemmas 1.2 and 1.3 yield that

$$\iint_U y\ |\nabla a(x,y)|^2 |G(x,y)|\ dx\ dy \leq \iint_U y\ |\nabla a(x,y)|^2\ P_y * (|G|^{1/2})(x)^2 dx\ dy$$

$$\leq \text{Const}\ \|a\|^2_{BMO}\ \|G\|_1.$$

Consequently, we have

$$|([a,H]f,g)| \leq \text{Const}\ \|a\|_{BMO}\ \|G\|_1 \leq \text{Const}\ \|a\|_{BMO}\ \|f\|_2\ \|g\|_2. \qquad \text{Q.E.D.}$$

§1.6. The Coifman-Meyer expression (Coifman-Meyer [8])

It is important to understand Theorem A from the point of view of real analysis. Coifman-Rochberg-Weiss [15] showed Lemma 1.4 without using analytic functions. Coifman-Meyer gave the following expression.

Lemma 1.5 ([8]). $[A,H]f'(x)$

$$= -\ \text{Const} \int_{-\infty}^{\infty} [a_{-s},H]f_s(x)/(1+s^2)ds \qquad (a \in BMO,\ f \in C_0^\infty),$$

where $a_s = k_s * a$, $f_s = k_s * f$, $k_s(x) = \Xi_s/|x|^{1+is}$ and $\Xi_s = \Gamma((1+is)/2)/\ \{\Gamma(-is/2)\ \pi^{is}\}$.

Proof. We have, for $a, f \in C_0^\infty$,

$$[A,H]f'(x) = \text{Const}\ i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\xi+\eta)x}\ \{\text{sign}\,\eta - \text{sign}(\xi+\eta)\}\ \frac{\hat{a}(\xi)}{i\xi}\ i\eta\hat{f}(\eta)d\xi\ d\eta\ ,$$

where $\hat{a}$, $\hat{f}$ are the Fourier transform of a, f, respectively. Note that

$\{\mathrm{sign}\,\eta - \mathrm{sign}(\xi+\eta)\}\,(\eta/\xi) = -\{\mathrm{sign}\,\eta - \mathrm{sign}(\xi+\eta)\}\,\chi_{(0,1)}(|\eta/\xi|)$. Since

$$|\eta/\xi| = \mathrm{Const}\int_{-\infty}^{\infty} |\eta/\xi|^{is}/(1+s^2)ds \qquad (|\eta/\xi| \leqq 1),$$

$\hat{a}_{-s}(\xi) = \hat{k}_{-s}(\xi)\hat{a}(\xi) = |\xi|^{-is}\,\hat{a}(\xi)$ and $\hat{f}_s(\eta) = |\eta|^{is}\,\hat{f}(\eta)$, we have

$$[A,H]f'(x) = -\mathrm{Const}\int_{-\infty}^{\infty} [i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\xi+\eta)x}\{\mathrm{sign}\,\eta - \mathrm{sign}(\xi+\eta)\}$$
$$\hat{a}_{-s}(\xi)\,\hat{f}_s(\eta)\,d\xi\,d\eta]/(1+s^2)\,ds = -\mathrm{Const}\int_{-\infty}^{\infty}[a_{-s},H]f_s(x)/(1+s^2)ds.$$

(In the case of $a \in BMO$, $f \in C_0^\infty$, it is necessary to show the convergence of the quantity in the right-hand side of Lemma 1.5. This will be shown later in the proof of Theorem A.) Q.E.D.

Here is another lemma necessary for the proof of Theorem A.

Lemma 1.6 ([8]). $\|a_s\|_{BMO} \leqq \mathrm{Const}(1 + |s|^{3/4})\,\|a\|_{BMO}$.

Proof. Without loss of generality we may assume that $s > 0$. We put $a^{(1)} = (a-(a)_I)\,\chi_{I^*}$, $a^{(2)} = (a-(a)_I)\,\chi_{I^{*c}}$. (See Lemma 1.2.) Then $a_s = a_s^{(1)} + a_s^{(2)}$, where $a_s^{(j)} = k_s * a^{(j)}$ $(j = 1,2)$. John-Nirenberg's inequality shows that $\|a^{(1)}\|_2 \leqq \mathrm{Const}\|a\|_{BMO}\sqrt{|I|}$, and hence

$$\int_I |a_s^{(1)}(x)|dx \leqq \|a_s^{(1)}\|_2\sqrt{|I|} = \|a^{(1)}\|_2\sqrt{|I|} \leqq \mathrm{Const}\,\|a\|_{BMO}\,|I|.$$

Note that $|\Xi_s| \leqq \mathrm{Const}(1+\sqrt{s})$. In the same manner as in Lemma 1.2, we have, with x_0 = (the midpoint of I),

$$\int_I |a^{(2)}(x) - a^{(2)}(x_0)|\,dx$$
$$= |\Xi_s|\int_I \left|\int_{-\infty}^{\infty}\left\{\frac{1}{|x-y|^{1+is}} - \frac{1}{|x_0-y|^{1+is}}\right\} a^{(2)}(y)dy\right| dx$$
$$\leqq \mathrm{Const}\,\{|\Xi_s|\,(1+s^{1/4})\}\int_I |x-x_0|^{1/4}\left\{\int_{I^{*c}} \frac{1}{|x_0-y|^{5/2}}\,|a(y)-(a)_I|dy\right\} dx$$
$$\leqq \mathrm{Const}\,(1+s^{3/4})\,\|a\|_{BMO}\,|I|.$$

Thus we obtain

$$(|a_s - (a_s)_I|)_I \leqq 2(|a_s - a_s^{(2)}(x_0)|)_I \leqq \mathrm{Const}\,(1+s^{3/4})\,\|a\|_{BMO},$$

which gives the required inequality. Q.E.D.

Theorem A is deduced from Lemmas 1.4-1.6 as follows. Inequality (1.10) shows that it is sufficient to show that $\|[A,H]f'\|_2 \leqq \mathrm{Const}\,\|a\|_\infty\|f\|_2$. Lemmas 1.4-1.6