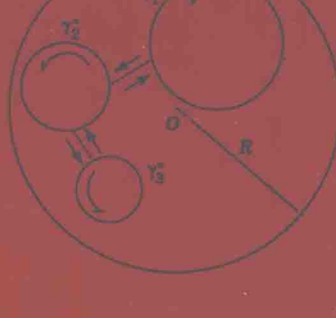
MODERN REAL and COMPLEX ANALYSIS



Bernard R. Gelbaum

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Preface

Real analysis and complex analysis are fundamental in modern mathematical education at the graduate and advanced undergraduate levels. The material covered in courses on those subjects has varied over the years. The tendency has been toward periodic revisions reflecting a changing consensus in the mathematical community. The aim of this book is to present a modern approach to the subjects as they are currently viewed.

The reader is assumed to be familiar with such terms as *continuity, power series, uniform convergence and uniform continuity, derivative, Riemann integral,* etc. On the other hand, a considerable effort has been made to provide, in the union of the text proper, the SYMBOL LIST, and the GLOSSARY/INDEX, complete definitions of all mathematical concepts introduced. The following notations obtain for assertions in formal logic:

$$\{A\} \Rightarrow \{B\} \text{ for } A \text{ implies } B$$
$$\{A\} \Leftrightarrow \{B\} \text{ for } A \text{ iff } B$$
$$A \wedge B : \text{ for } A \text{ and } B$$
$$A \vee B \text{ for } A \text{ or } B.$$

Among the novel and unique features of the text are the following.

In Chapter 1.
- a) Topology discussed three ways: via open sets, via nets, and via filters.
- b) Two proofs of Brouwer's Fixed Point Theorem.
- c) Uniform spaces.

In Chapter 2.
- a) Integration viewed as a Daniell functional.
- b) A detailed exploration of the connection between measure as derived from a Daniell functional and classical Lebesgue-Caratheodory measure.
- c) The Riesz Representation Theorem as a consequence of Daniell's approach.

In Chapter 3.
- a) Functional analysis and weak topologies.
- b) Banach algebras.
- c) Axiomatics of Hilbert space and linear operators.

In Chapter 4.
- a) The Fubini-Tonelli Theorems via Daniell's techniques.
- b) A unified approach to nonmeasurable sets.
- c) Differentiation by direct methods that avoid parameters of regularity, nicely shrinking sequences, etc.
- d) Haar measure by Daniell functionals.

In Chapter 5.
- a) Singular homology of the plane via the formulæ and theorems of Cauchy.
- b) Elementary exterior calculus as applied to complex function theory.

In Chapter 6.
- a) Subharmonic functions, barriers, and Perron's approach to Dirichlet's problem.
- b) Poisson's kernels and approximate identities in Banach algebras.

In Chapter 7.
- a) Runge's Theorem and its application to Mittag-Leffler's Theorem; the latter as a source of Weierestraß's product representation.
- b) Entire functions and their orders of growth.

In Chapter 8.
- a) Riemann's Mapping Theorem and its connection to Dirichlet's problem.
- b) Bergman's kernel functions and conformal mapping.
- c) Automorphic functions.
- d) Green's functions.

In Chapter 9.
- a) Picard/Montel Theorems and their consequences.

In Chapter 10.
- a) A thorough treatment of analytic continuation.
- b) The Riemann-Weierstraß-Weyl concepts of Riemann surfaces as well as Riemann surfaces defined as connected one-dimensional complex analytic manifolds.
- c) Covering spaces, sheaves, lifts.
- d) The General Uniformization Theorem derived via a sequence of carefully graded **Exercises**.

In Chapter 11.
- a) Thorin's Theorem.
- b) Applications to M. Riesz's Convexity Theorem and related parts of functional analysis.

In Chapter 12. An introduction to the theory of complex functions of more than one complex variable.

Within each section, all the numbered items save **Figure**s, e.g., THEOREMs, **Exercise**s, equations, are numbered consecutively as they appear. Thus in **Section 3.2**, the first item is **3.2.1** LEMMA, the second item is, **3.2.2** COROLLARY, the third item is (3.2.3), (the first) numbered equation, etc.

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REAL ANALYSIS

1
Fundamentals

1.1. Introduction

The text is addressed to readers with a standard background in undergraduate algebra, analysis, and elementary topology. Hence, when reference is made to concepts such as groups, maps, posets, topological spaces, etc., there is an underlying assumption that the reader is familiar with them. Nevertheless, all terms and notations essential for the understanding of the material are defined or explained in the **Chapter**s, the GLOSSARY/INDEX, or the SYMBOL LIST.

Real analysis deals with the study of functions defined on a set X and taking values, for some n in $\mathbb{N}$, in the set $\mathbb{R}^n$ of n-tuples of real numbers or occasionally in the set $\mathbb{C}^n$ of n-tuples of complex numbers.

On the other hand, *complex analysis* is confined to the study of *locally holomorphic* functions, i.e., for some nonempty connected open subset, i.e., *region*, Ω of $\mathbb{C}$, functions f in $\mathbb{C}^\Omega$ and differentiable throughout Ω.

Beginning with a system, e.g., that of Zermelo-Fraenkel, (ZF) [**Me**], of axioms for set theory, one can construct in turn the system $\mathbb{N} \stackrel{\text{def}}{=} \{1, 2, \ldots\}$ of *natural numbers*, the *ring* $\mathbb{Z}$ of *integers*, the *field* $\mathbb{Q}$ of *rational numbers*, and finally the field $\mathbb{R}$ of *real numbers* [**La**]. The result is a field endowed with an order $<$ (a transitive relation such that for any two (different) elements x and y, precisely one of $x < y$ and $y < x$ is true); $\mathbb{R}$ is complete with respect to $<$, i.e.,for every subset bounded above there is a unique supremum (least upper bound). Since any two complete ordered fields are field- and order-isomorphic [**O**], one may proceed directly as follows.

1.1.1 DEFINITION. THE SET $\mathbb{R}$ IS A COMPLETE ORDERED FIELD.

The multiplicative identity 1 of $\mathbb{R}$ gives rise to $1, 1+1, \ldots$, i.e., to the set $\mathbb{N}$ of natural numbers. The ring $\mathbb{Z}$ of integers is the set of *R-equivalence classes* of $\mathbb{N}^2$:

$$\{(m,n)R\,(m',n')\} \Leftrightarrow \{m + n' = m' + n\}.$$

The *field* $\mathbb{Q}$ of *rational numbers* is the set of all S-equivalence classes of $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$:

$$\{(p,q)S\,(p',q')\} \Leftrightarrow \{pq' = p'q\}.$$

The field $\mathbb{C}$ of *complex numbers* is the set

$$\mathbb{R}^2 \stackrel{\text{def}}{=} \{ (a,b) \ : \ \{a,b\} \subset \mathbb{R} \} \stackrel{\text{def}}{=} \mathbb{R} \times \mathbb{R}$$

in which the algebraic operations addition $(+)$ and multiplication $(\cdot)$ as well as the symbols 0 and i are defined according to:

$$\begin{gathered}(a,b)+(c,d) \stackrel{\text{def}}{=} (a+c, b+d),\\ (a,b)\cdot(c,d) \stackrel{\text{def}}{=} (a,b)(c,d) \stackrel{\text{def}}{=} (ac-bd, bc+ad),\\ 0 \stackrel{\text{def}}{=} (0,0), i \stackrel{\text{def}}{=} (0,1).\end{gathered}$$

Furthermore, $\mathbb{R}$ is identified with $\mathbb{R}\times\{0\}$ and then $(a,b) \stackrel{\text{def}}{=} a+ib$. When $z \stackrel{\text{def}}{=} a+ib \stackrel{\text{def}}{=} \Re(z)+i\Im(z) \in \mathbb{C}$, the *absolute value* of z is $|z| \stackrel{\text{def}}{=} \sqrt{a^2+b^2}$. (When $a\in\mathbb{R}$ and a is regarded as $a+i0$, an element of $\mathbb{C}$, the definition of $|a|$ as just given and the definition

$$|a| \stackrel{\text{def}}{=} \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

are equivalent.) If $z \stackrel{\text{def}}{=} a+ib \neq 0$ and $w \stackrel{\text{def}}{=} \dfrac{a-ib}{|z|^2}$, $wz = zw = 1$.

[**1.1.2 Note.** Below and throughout the book, to avoid imperatives, most **Exercise**s are phrased as assertions to be proved.]

1.1.3 Exercise. a) The set $\mathbb{N}$ is the intersection of all $\mathbb{R}$-subsets S containing 1 and such that if $x\in S$, $x+1\in S$. b) The order in $\mathbb{R}$ is *Archimedean*, i.e., if $\epsilon > 0$ and $M > 0$, then for some n in $\mathbb{N}$, $n\epsilon > M$.

[*Hint:* b) Otherwise, for some positive ϵ and M and each n in $\mathbb{N}$, $n\epsilon \leq M$, whence $\{ n\epsilon \ : \ n \in \mathbb{N} \}$ has a supremum.]

1.1.4 Exercise. If $a,b\in\mathbb{R}$, then $|a+b| \leq |a|+|b|$. Equality obtains iff for some nonnegative constants A, B, not both zero, $Aa = Bb$.

1.1.5 Exercise. a) Addition and multiplication in $\mathbb{C}$ are commutative operations. b) If $a+ib\in\mathbb{C}$, $|a+ib| \leq |a|+|b|$ and equality obtains iff $ab=0$.

The following special types of subsets of $\mathbb{R}$ and $\mathbb{C}$ appear frequently in the text:

- when $-\infty \leq a \leq b \leq \infty$, the *oriented real intervals*:

$$\begin{aligned}(a,b) &\stackrel{\text{def}}{=} \{ x \ : \ a < x < b \}, \text{ (open)},\\ [a,b) &\stackrel{\text{def}}{=} \{ x \ : \ a \leq x < b \}, \text{ (right-open)},\\ (a,b] &\stackrel{\text{def}}{=} \{ x \ : \ a < x \leq b \}, \text{ (left-open)},\\ [a,b] &\stackrel{\text{def}}{=} \{ x \ : \ a \leq x \leq b \}, \text{ (closed)};\end{aligned}$$

- when $S \subset \mathbb{R}$, $S^+ \stackrel{\text{def}}{=} S \cap [0, \infty)$;
- when $\{p, q\} \subset \mathbb{C}$ the *oriented complex intervals*:

$$\begin{aligned}
(p,q) &\stackrel{\text{def}}{=} \{\, z \;:\; z = (1-t)p + tq,\ 0 < t < 1 \,\},\ \text{(open)},\\
[p,q) &\stackrel{\text{def}}{=} \{\, z \;:\; z = (1-t)p + tq,\ 0 \le t < 1 \,\},\ \text{(right-open)},\\
(p,q] &\stackrel{\text{def}}{=} \{\, z \;:\; z = (1-t)p + tq,\ 0 < t \le 1 \,\},\ \text{(left-open)},\\
[p,q] &\stackrel{\text{def}}{=} \{\, z \;:\; z = (1-t)tp + tq,\ 0 \le t \le 1 \,\},\ \text{(closed)};
\end{aligned}$$

  ($[a, b)$, $(a, b]$, $[p, q)$, and $(p, q]$ are *half-open* complex intervals);
- the subgroup $\mathbb{T} \stackrel{\text{def}}{=} \{\, z \;:\; z \in \mathbb{C},\ |z| = 1 \,\}$ of the multiplicative group of nonzero elements of $\mathbb{C}$.

The (possibly empty) *interior* of any of the real intervals above is (a, b).

For a set $\{X_\gamma\}_{\gamma \in \Gamma}$, $\bigtimes_{\gamma\in\Gamma} X_\gamma$ is the *Cartesian product* of the sets X_γ, i.e., $\bigtimes_{\gamma\in\Gamma} X_\gamma \stackrel{\text{def}}{=} \{\, f \;:\; f : \Gamma \ni \gamma \mapsto f(\gamma) \in X_\gamma \,\}$. Since $f(\gamma) \in X_\gamma$, occasionally the notation x_γ is used for $f(\gamma)$ and an f is a *vector* $\{x_\gamma\}_{\gamma\in\Gamma}$ When, for some X, $X_\gamma \equiv X$, then $\bigtimes_{\gamma\in\Gamma} X_\gamma = X^\Gamma$, the set of all maps from Γ into X.

An n-dimensional *interval* I in $\mathbb{R}^n$ is either the empty set ($\emptyset$) or the Cartesian product of n one-dimensional intervals each of which has a nonempty interior. For n in $\mathbb{N}$, a *half-open n-dimensional interval* in

$$\mathbb{R}^n \stackrel{\text{def}}{=} \left\{ \mathbf{x} \stackrel{\text{def}}{=} (x_1, \ldots, x_n) \;:\; x_i \in \mathbb{R},\ 1 \le i \le n \right\}$$

is the Cartesian product $\bigtimes_{k=1}^n [a_k, b_k)$ of right-open intervals. If $b_k - a_k$ is k-free, the half-open n-dimensional interval is a *half-open n-dimensional cube*. When $n > 1$, elements of $\mathbb{R}^n$ or $\mathbb{C}^n$ are regarded as vectors and are denoted by **boldface** letters: $\mathbf{a}$, $\mathbf{x}$, … . The vector $(0, \ldots, 0)$ is denoted $\mathbf{O}$.

The *length* or *norm* of the vector $\mathbf{x} \stackrel{\text{def}}{=} (x_1, \ldots, x_n)$ is $\|\mathbf{x}\|_2 \stackrel{\text{def}}{=} \sqrt{\sum_{k=1}^n |x_k|^2}$.

The *cardinality* of X is denoted $\#(X)$, e.g., $\#(\mathbb{N}) \stackrel{\text{def}}{=} \aleph_0$, $\#(\mathbb{R}) \stackrel{\text{def}}{=} \mathfrak{c}$. The *ordinal number* of the *well-ordered* set of equivalence classes of well-ordered countable sets is Ω. (The previously introduced use of Ω—to denote a region—causes no difficulty since the two contexts—ordinal numbers and regions—do not occur together in the remainder of the book.)

When $\mathfrak{n}$ is a cardinal number the phrase—$\mathfrak{n}$ objects—means there is a set S consisting of pairwise different objects and $\#(S) = \mathfrak{n}$. Thus, the phrase—*two points x and y*—implies $x \neq y$.

On the other hand, the phrase—*the points x and y*—carries no such implication: both $x = y$ and $x \neq y$ are admissible.

1.2. Topology and Continuity

1.2.1 DEFINITION. A TOPOLOGICAL SPACE IS A SET X PAIRED WITH A SUBSET T OF $\mathfrak{P}(X)$, THE SET OF ALL SUBSETS OF X. THE SET T IS THE TOPOLOGY OF X AND THE ELEMENTS OF T ARE THE OPEN SETS OF X. THE AXIOMS GOVERNING T ARE:

a) $\emptyset \in \mathsf{T}$ AND $X \in \mathsf{T}$;
b) T IS CLOSED WITH RESPECT TO THE FORMATION OF ARBITRARY UNIONS AND FINITE INTERSECTIONS.

The set of open sets of X is also denoted $\mathsf{O}(X)$.
When $A \subset X$, a) and b) hold for the set

$$\mathsf{T}_A \stackrel{\text{def}}{=} \{ A \cap U \ : \ U \in \mathsf{T} \},$$

which endows A with the *relative topology induced by* T.

1.2.2 Exercise. For a topological space X, the set $\mathsf{F}(X)$ of complements of elements of $\mathsf{O}(X)$ is governed by:

a′) $\emptyset \in \mathsf{F}(X)$ and $X \in \mathsf{F}(X)$;
b′) $\mathsf{F}(X)$ is closed with respect to the formation of arbitrary intersections and finite unions.

When T and T' are topologies for X and $\mathsf{T} \subset \mathsf{T}'$, T' is *stronger* than T while T is *weaker* than T'.

1.2.3 Example. For any set X there are:

a) the *strongest* or *discrete topology* $\mathfrak{P}(X)$ consisting of all subsets of X;
b) the *weakest* or *trivial topology* consisting only of $\emptyset$ and X.

1.2.4 Example. For $\mathbb{R}$, the *customary topology* consists of all (arbitrary) unions of *open intervals*. Unless the contrary is stated, $\mathbb{R}$ is regarded as endowed with its customary topology.

On the other hand, the *Sorgenfrey topology* T_s for $\mathbb{R}$ consists of all (arbitrary) unions of left-closed intervals, i.e., unions of all sets of the form

$$[a, b) \stackrel{\text{def}}{=} \{ x \ : \ \mathbb{R} \ni a \le x < b \in \mathbb{R} \}.$$

For a topological space X a subset $\mathcal{B} \stackrel{\text{def}}{=} \{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ of T is a *base* for T iff every element (open set) in T is the union of (some) elements of $\mathcal{B}$.

1.2.5 Example. In both the usual and Sorgenfrey topologies for $\mathbb{R}$, $\mathbb{Q}$ meets every open set.

The countable set $\{ (a, b) \ : \ \mathbb{Q} \ni a < b \in \mathbb{Q} \}$ is a base for the customary topology. By contrast, if $\mathcal{B}$ is a base for T_s and $[a, b) \in \mathsf{T}_s$, a must belong to some base element B contained in $[a,b)$. Thus $\#(\mathcal{B}) \geq \#(\mathbb{R})$: there is no countable base for T_s.

If $\mathcal{B}$ is an arbitrary subset of $\mathfrak{P}(X)$, $\mathcal{B}$ is contained in the discrete topology $\mathfrak{P}(X)$: the set of all topologies containing $\mathcal{B}$ is nonempty. The intersection $\mathsf{T}_{\mathcal{B}}$ of all topologies containing $\mathcal{B}$ is the topology for which $\mathcal{B}$ is a base: $\mathcal{B}$ generates $\mathsf{T}_{\mathcal{B}}$.

When, for (X, T), there is a countable base for T, X is *second countable*. When X contains a countable subset meeting every element of $\mathsf{O}(X)$, X is *separable*.

1.2.6 Example. The set $\{ (a,b)^n : \{a,b\} \subset \mathbb{Q},\ a < b \}$ is a countable base for the customary topology for $\mathbb{R}^n$.

When X and Y are sets and $f \in Y^X$, f is:

- *injective* iff
$$\{f(a) = f(b)\} \Rightarrow \{a = b\};$$
- *surjective* iff $f(X) = Y$;
- *bijective* iff f is injective and surjective;
- *autojective* iff $X = Y$ and f is bijective.

Injective, surjective, ... maps are *injections*, *surjections*, ...

When (X_1, T_1) and (X_2, T_2) are topological spaces and $f \in X_2^{X_1}$ (the set of all maps from X_1 into X_2), f is:

- *continuous* iff $f^{-1}(\mathsf{T}_2) \subset \mathsf{T}_1$;
- *open* iff $f(\mathsf{T}_1) \subset \mathsf{T}_2$.
- a *homeomorphism* iff f is bijective and f is both continuous and open, i.e., iff f is bijective and both f and f^{-1} are continuous.

The set of continuous maps in $X_2^{X_1}$ is denoted $C(X_1, X_2)$.

1.2.7 Exercise. If $(f,g) \in C(X,Y) \times C(Y,Z)$, then $g \circ f \in C(X,Z)$.

When $A \subset X$ and $\mathsf{O}(A)$ is the set of open subsets of A,

$$A^\circ \stackrel{\text{def}}{=} \bigcup_{U \in \mathsf{O}(A)} U,$$

is the (possibly empty) *interior* of A. For a nonempty subset A of X, a *neighborhood* $N(A)$ is a set such that $A \subset N(A)^\circ$. For simplicity of notation, when $x \in X$, $N(x) \stackrel{\text{def}}{=} N(\{x\})$. The set of neighborhoods of A is $\mathcal{N}(A)$.

1.2.8 Exercise. If $A \neq \emptyset$, for $\mathcal{F} \stackrel{\text{def}}{=} \mathcal{N}(A)$, the following obtain:

a) $\mathcal{F} \neq \emptyset$, $\emptyset \notin \mathcal{F}$;
b) $\{F, F' \in \mathcal{F}\} \Rightarrow \{F \cap F' \in \mathcal{F}\}$;
c) $\{\{F \in \mathcal{F}\} \wedge \{F \subset G\}\} \Rightarrow \{G \in \mathcal{F}\}$.