Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Asymptotic Analysis of Soliton Problems

An Inverse Scattering Approach

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How, in frames at rest,
the tail goes west,
while the east is won
by the soliton

PREFACE

A few years ago, when I started reading about solitons, I was fascinated by the beauty of the theory but at the same time astonished by the amount of assertions without even a shadow of a proof. No doubt, this lack of rigour is connected with the fact that many great discoveries were made during a relatively short period of time. In particular, in the early seventies it went more or less like this: A discovery was done and the proof was sketched. Immediately another discovery followed and the process repeated itself.

Of course, in this way a lot of questions remained unanswered. To mention only two of them: (i) How do KdV solitons emerge from arbitrary initial conditions? (ii) What are the phase shifts of these solitons as they interact both with the other solitons and with the dispersive wave-train?

The purpose of this volume is to provide answers to these and similar questions. Specifically, we give a complete, rigorous and explicit description of the emergence of solitons from various classes of nonlinear partial differential equations solvable by the inverse scattering technique. To this end we present an almost uniform method to obtain the asymptotic behaviour for large time of solutions of soliton problems in those coordinate regions where the nonsoliton component can be considered as a perturbation of the soliton component. The conditions under which our method works are remarkably general. For instance in the KdV analysis of Chapter 2 a mild algebraic decay of the initial function, so as to ensure that the associated reflection coefficient has a second derivative decaying at infinity as the inverse of its argument, is already sufficient.

The chapters in this volume are essentially self-contained with the exception of Chapter 1, which uses some concepts that are discussed in more detail in Chapter 2. Therefore the reader not particularly familiar with soliton theory is advised to read Chapter 2 before Chapter 1.

It is an honour to focus my thanks on two outstanding mathematicians, Wiktor Eckhaus and Aart van Harten, for their continuous interest in my work and for their pleasant way of combining moral support with valuable criticism.

To Wilma van Nieuwamerongen I am much indebted for skilfully typing the manuscript.

August 1986 Peter Schuur

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INTRODUCTION

For centuries nonlinearity formed a dark mystery. Nowadays, though things still look rather black, there are a few bright spots where we may confidently expect steady progress. This volume deals with one of these sparkles of hope: *the inverse scattering transformation*.

1. Historical remarks.

Many physical phenomena are nonlinear in nature. More often than not they can be modelled by nonlinear partial differential equations offering a wide range of complexity. Until the late sixties of this century the analyst had, roughly speaking, the choice: approximate or apologize. In the past two decades this situation changed, since various powerful nonperturbative mathematical techniques made their entrance. One of these is the inverse scattering technique (IST), also called inverse scattering transformation or spectral transform.

Its discovery is due to Gardner, Greene, Kruskal and Miura (GGKM for short) and was first reported in 1967 in their famous two-paged signal paper [9]. In this paper GGKM showed how to obtain the solution $u(x,t)$ of the Korteweg-de Vries (KdV) initial value problem

(1.1a) $u_t - 6uu_x + u_{xxx} = 0, \quad -\infty < x < +\infty, \quad t > 0$

(1.1b) $u(x,0) = u_0(x)$.

Here and in the sequel a subscript variable indicates partial differentiation, e.g. $u_x = \frac{\partial u}{\partial x}$. Equation (1.1a) was first derived by Korteweg and de Vries [13] in 1895 in the context of free-surface gravity waves propagating in shallow water (see [4] for its historical background).

Below we shall discuss the GGKM method in some detail. Here we only mention its amazing starting point, namely the introduction of the solution u(x,t) of (1.1) as a potential in the Schrödinger scattering problem.

In 1968 Lax [16] put the GGKM method into a framework that clearly indicated its generality and had a substantial influence on future developments. In particular, Lax showed that (1.1a) is a member of an infinite family of nonlinear partial differential equations that can all be analysed in a similar fashion.

Guided by Lax' generalization of the pioneering work of GGKM, Zakharov and Shabat [26] were able to solve the initial value problem for another nonlinear equation of physical importance, the nonlinear Schrödinger equation (NLS)

(1.2) $iu_t = u_{xx} + 2|u|^2u$.

To this end they associated (1.2) with a spectral problem based on a system of two coupled first order ordinary differential equations. Incidentally, the NLS shows up in the description of plasma waves and models plane self-focusing and one-dimensional self-modulation.

Subsequently, Tanaka [20], [22] extended and rigorized the direct and inverse scattering theory for the Zakharov-Shabat system, motivated by the surprising discovery of Wadati [23] that another interesting nonlinear evolution equation could be solved by this system, namely the modified Korteweg-de Vries equation (mKdV)

(1.3) $u_t + 6u^2u_x + u_{xxx} = 0$

which appears in the continuum limit of a one-dimensional lattice with quartic anharmonicity [5].

Ablowitz, Kaup, Newell and Segur [1], [2] then showed that NLS and mKdV belong to a large class of nonlinear partial differential equations that can be solved via a generalized version of the Zakharov-Shabat scattering problem. Among these newly found integrable equations were several of physical importance, such as the sine-Gordon equation

$$(1.4) \qquad u_t = \tfrac{1}{2} \sin\left[2 \int_{-\infty}^{x} u(x',t)\,dx'\right]$$

which arises as an equation for the electric field in quantum optics [15], though the related forms

$$(1.4)' \qquad \sigma_{xt} = \sin\sigma \quad \text{and}$$

$$(1.4)'' \qquad \sigma_{xx} - \sigma_{tt} = \sin\sigma$$

appear more frequently in the literature (cf. [12]).

Herewith the triumphal march of the inverse scattering technique began. We shall not follow it further but refer to the survey articles [5], [10], [15], [17], [18] as well as the many textbooks on solitons [3], [6], [7], [8], [14], [25] currently available. We only mention that several other classes of physically relevant equations were found to be solvable by inverse scattering methods. In fact the process of finding new integrable nonlinear evolution equations has continued until this very day and has grown out into a major industry. Moreover, IST had its spin-off's to other areas of mathematics, like algebraic and differential geometry, functional and numerical analysis, etc. Nowadays - as stated in [6] - its applications range from nonlinear optics to hydrodynamics, from plasma to elementary particle physics, from lattice dynamics to electrical networks, from superconductivity to cosmology and geophysics. Moreover, IST is developing into an interdisciplinary subject, since it has recently penetrated in epidemiology and neurodynamics.

An essential reason for this wide applicability has not been mentioned so far: a dominant feature of nonlinear evolution equations of physical importance solvable via IST is that they admit exact solutions that describe the propagation and interaction of *solitons*.

At the moment there is no generally accepted mathematical definition of a soliton. As a working definition of a soliton we might take (cf. [5]) that it is a "localized" wave (in the sense of sufficiently rapidly decaying) which asymptotically preserves its shape and speed upon interaction with any other such localized wave. However, the concept of a soliton has a great intuitive appeal and is a good illustration of the fact that a happily chosen terminology is half of the success of a theory. The soliton was discovered in 1965 by Zabusky and Kruskal [24] while performing a numerical study of the KdV. Actually, the name "soliton" was suggested by Zabusky, who originally used the term "solitron" instead (see [6], pp. 176, 177).

Let us discuss their discovery in some detail. Already Korteweg and de Vries theirselves knew [13] that the KdV had a special travelling wave solution, the solitary wave

$$(1.5) \qquad u(x,t) = -2k_0^2 \operatorname{sech}^2[k_0(x - x_0 - 4k_0^2 t)], \qquad (\operatorname{sech} z = \frac{2}{e^z + e^{-z}})$$

where k_0 and x_0 are constants. Observe, that the velocity of this wave, $4k_0^2$, is proportional to its amplitude, $2k_0^2$. Now, in [24] Zabusky and Kruskal considered two waves such as (1.5), with the smallest to the right, as initial condition to the KdV. They discovered that after a certain time the waves overlap (the bigger one catches up), but that next the bigger one separates from the smaller and gradually the waves regain their initial shape and speed. The only permanent effect of the interaction is a phase shift, i.e. the center of each wave is at a different position than where it would have been if it had been travelling alone. Specifically, the bigger one is shifted to the right, the smaller to the left. The name soliton was chosen so as to stress this remarkable particle-like behaviour.

To conclude these introductory remarks, let us not forget to mention that, although in the past few years soliton interaction has been observed in various physical systems (see [3]), the first physical observation of what is now known as the single soliton solution (1.5) of the KdV already took place in the month of August 1834 by John Scott Russell, during his celebrated chase on horseback of a huge wave in the Union Canal, which from Edinburgh, joins with the Forth-Clyde canal and thence to the two coasts of Scotland. His own report of this experience, though classical by now, cannot be missed in any true soliton story.

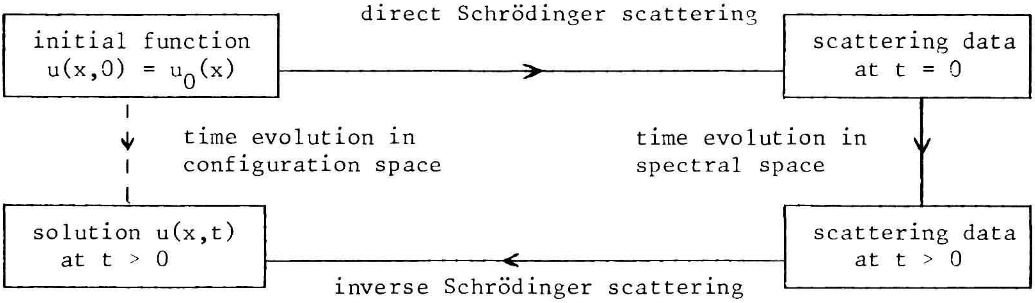
It reads as follows [19]:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon ...".

2. IST for KdV: the gist of the method.

To comfort the reader who is completely new to the subject, let us at least give a rough sketch of how IST works, referring to [8] for the many intricate mathematical details. To this end we indicate here very briefly the basic features of the GGKM method, which is the first and undoubtedly the most fundamental example of an inverse scattering method.

Let us consider the KdV initial value problem (1.1) with $u_0(x)$ an arbitrary real function, sufficiently smooth and rapidly decaying for $x \to \pm\infty$. The surprising discovery of GGKM is now, that the nonlinear problem (1.1) can be solved in a series of linear steps, schematically representable in the following diagram



The manipulations suggested by this diagram are the following: For each $t \geqq 0$, introduce the real function $u(x,t)$ as a potential in the Schrödinger scattering problem

$$(2.1) \qquad \psi_{xx} + (k^2 - u(x,t))\psi = 0, \qquad -\infty < x < +\infty.$$

For $t = 0$, compute the associated bound states $-\kappa_1^2 < -\kappa_2^2 < \dots < -\kappa_N^2$, $\kappa_j > 0$, right normalization coefficients c_j^r and right reflection coefficient $b_r(k)$ (see Chapter 2 for their definition and properties), in other words, compute the right scattering data $\{b_r(k), \kappa_j, c_j^r\}$ associated with $u_0(x)$. Then, as $u(x,t)$ evolves according to the KdV, its right scattering data evolve in the following simple way:

$$(2.2a) \qquad \kappa_j(t) = \kappa_j$$

$$(2.2b) \qquad c_j^r(t) = c_j^r \exp\{4\kappa_j^3 t\}, \qquad j = 1,2,\dots,N$$

$$(2.2c) \qquad b_r(k,t) = b_r(k) \exp\{8ik^3 t\}, \qquad -\infty < k < +\infty.$$

To recover $u(x,t)$ from these data, one applies the inverse scattering procedure for the Schrödinger equation found by Gel'fand and Levitan [11], and defines

$$(2.3) \qquad \Omega(\xi;t) = 2 \sum_{j=1}^{N} [c_j^r(t)]^2 e^{-2\kappa_j \xi} + \frac{1}{\pi} \int_{-\infty}^{\infty} b_r(k,t) e^{2ik\xi} dk.$$

Next, one solves the Gel'fand-Levitan equation

$$(2.4) \qquad \beta(y;x,t) + \Omega(x+y;t) + \int_0^{\infty} \Omega(x+y+z;t)\beta(z;x,t)dz = 0$$

with $y > 0$, $x \in \mathbb{R}$, $t > 0$. The solution $\beta(y;x,t)$ has the important property

$$\beta(0^+;x,t) = \int_x^\infty u(x',t)dx', \quad x \in \mathbb{R}, \quad t > 0, \tag{2.5}$$

and so we find that the solution of the KdV problem (1.1) is given by

$$u(x,t) = -\frac{\partial}{\partial x}\beta(0^+;x,t), \quad x \in \mathbb{R}, \quad t > 0. \tag{2.6}$$

Notice that the original problem for the nonlinear partial differential equation (1.1) is essentially reduced in this way to the problem of solving a one-dimensional linear integral equation.

Explicit solutions of (2.4) have only been obtained for $b_r \equiv 0$. The solution $u_d(x,t)$ of the KdV with scattering data $\{0,\kappa_j,c_j^r(t)\}$ is called the pure N-soliton solution associated with $u_0(x)$, on account of its asymptotic behaviour displayed in the following remarkable result due to Tanaka [21]

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \left| u_d(x,t) - \sum_{p=1}^{N} \left(-2\kappa_p^2 \operatorname{sech}^2[\kappa_p(x-x_p^+-4\kappa_p^2 t)]\right)\right| = 0 \tag{2.7a}$$

$$x_p^+ = \frac{1}{2\kappa_p}\log\left\{\frac{[c_p^r]^2}{2\kappa_p}\prod_{\ell=1}^{p-1}\left(\frac{\kappa_\ell-\kappa_p}{\kappa_\ell+\kappa_p}\right)^2\right\}. \tag{2.7b}$$

Thus for large positive time $u_d(x,t)$ arranges itself into a parade of N solitons with the largest one in front and this happens uniformly with respect to x on $\mathbb{R}$.

3. Asymptotics for nonzero reflection coefficient: main purpose of the book.

As illustrated by the previous section, the inverse scattering method enables us to obtain rather explicit exact solutions to nonlinear wave equations and to determine their asymptotic behaviour, which generally corresponds to a decomposition into solitons. Evidently, the problem of the asymptotic behaviour evolving from an arbitrary initial condition is in this way far from exhausted. It is still necessary to determine the asymptotic properties of the "nonsoliton part" of the solution whose presence is connected with the reflection coefficient being nonzero. In this volume we concern ourselves with this problem.

Rather than to give an elaborate general discussion, let us illustrate the ideas involved by considering again the KdV problem (1.1).

Suppose $u_0(x)$ is not a reflectionless potential. Then, in view of the fact that the linearized version of (1.1a) is a dispersive equation with associated group velocity $v_g = -3k^2 \leqq 0$, one expects that for large time the soliton part and the dispersive component will separate out, the dispersive wavetrain moving leftward and the solitons nicely arranging theirselves into a parade moving to the right similar to that described by (2.7). However, this is only heuristic reasoning. In fact it is dangerous reasoning too, since for nonlinear equations there is no such thing as a superposition principle.

The circumstance that at the time the question of validity of the above "plausible" conjecture had not been answered in a mathematically satisfactory way, formed the impetus for the research laid down in the present volume .

The main purpose of this book is therefore to give a complete and rigorous description of the emergence of solitons from various (classes of) nonlinear partial differential equations solvable by the inverse scattering technique.

Throughout the book we focus our attention on coordinate regions where the dispersive component is sufficiently small, e.g. $x \geqq -t^{1/3}$ for the (m)KdV problem. The behaviour of the solution in other regions, where the dispersive waves interact, is not discussed, since entirely different techniques are needed. For recent results in those regions we refer to [3].

4. Brief description of the contents.

The chapters in this volume are largely self-explanatory. Only Chapter 1 forms an exception. We therefore advise the reader new to the field to start with Chapter 2. In fact, both chapters deal with the KdV. However, in Chapter 1 the central ideas of our asymptotic method are exposed in the simplest nontrivial setting, whereas Chapter 2 serves to extend the results of Chapter 1, as well as to supply the details of the inverse scattering machinery. Also, the discussion of existence and uniqueness for the KdV initial value problem is postponed to Chapter 2.

In Chapter 1 we present a rigorous demonstration of the emergence of solitons from the KdV initial value problem with arbitrary real initial function. We show that for any choice of the constants $v > 0$ and $M \geq 0$ there exists a function $\sigma(t)$ tending to zero as $t \to \infty$, such that

$$(4.1) \qquad \sup_{x \geq -M+vt} |u(x,t) - u_d(x,t)| = O(\sigma(t)), \quad \text{as } t \to \infty.$$

The exact behaviour of $\sigma(t)$ depends on properties of u_0. If u_0 decays exponentially for $x \to \pm\infty$, then so does $\sigma(t)$ for $t \to \infty$. If the decay of u_0 is only algebraic then also the decay of $\sigma(t)$ is algebraic.

In Chapter 2 we extend the asymptotic analysis given in Chapter 1. In fact, we no longer restrict our investigation to right half lines linearly moving rightward, but allow the right half lines to move slowly leftward. It is shown that in the absence of solitons the solution of (1.1) satisfies

$$(4.2) \qquad \sup_{x \geq -t^{1/3}} |u(x,t)| = O(t^{-2/3}), \quad \text{as } t \to \infty,$$

whereas in the general case

$$(4.3) \qquad \sup_{x \geq -t^{1/3}} |u(x,t) - u_d(x,t)| = O(t^{-1/3}), \quad \text{as } t \to \infty.$$

The emergence of solitons is clearly displayed by the remarkable convergence result

$$(4.4) \qquad \lim_{t\to\infty} \sup_{x \geq -t^{1/3}} \left| u(x,t) - \sum_{p=1}^{N} \left(-2\kappa_p^2 \operatorname{sech}^2[\kappa_p(x - x_p^+ - 4\kappa_p^2 t)] \right) \right| = 0$$

with x_p^+ as in (2.7b). In addition, we construct explicit x and t dependent bounds for the nonsoliton component of the solution and establish some interesting momentum and energy decomposition formulae. To support the analysis we only need to require - apart from the obvious assumption that IST works at all - that the right reflection coefficient b_r is of class $C^2(\mathbb{R})$ such that the derivatives $b_r^{(j)}(k)$, $j = 0,1,2$ behave as $O(|k|^{-1})$ for $k \to \pm\infty$. This condition is extremely weak. Hence our results apply to a large class of KdV initial value problems.