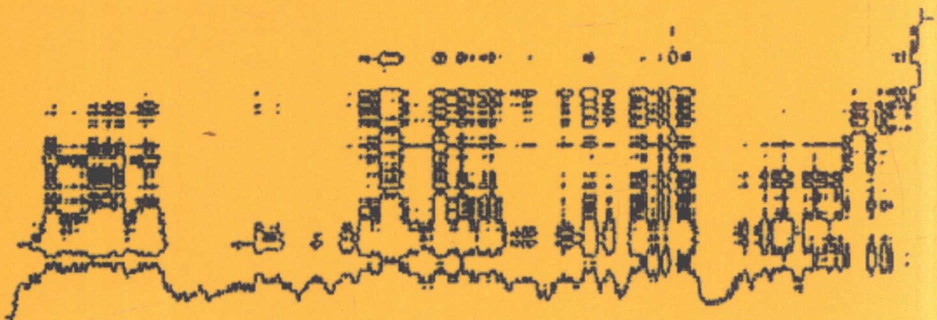
Lecture Notes in Mathematics

Robert Dalang
Davar Khoshnevisan
Carl Mueller
David Nualart
Yimin Xiao

A Minicourse on Stochastic Partial Differential Equations

962

Editors: Davar Khoshnevisan
Firas Rassoul-Agha



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Authors and Editors

Robert C. Dalang
Institut de Mathématiques
Ecole Polytechnique Fédérale de Lausanne
Station 8
1015 Lausanne
Switzerland
robert.dalang@epfl.ch

Carl Mueller
Department of Mathematics
University of Rochester
Rochester, NY 14627
USA
cmlr@math.rochester.edu

Firas Rassoul-Agha
Department of Mathematics
University of Utah
Salt Lake City, UT 84112-0090
USA
firas@math.utah.edu

Davar Khoshnevisan
Department of Mathematics
University of Utah
Salt Lake City, UT 84112-0090
USA
davar@math.utah.edu

David Nualart
Department of Mathematics
University of Kansas
Lawrence, KS 66045
USA
nualart@math.ku.edu

Yimin Xiao
Department of Statistics and Probability
Michigan State University
A-413 Wells Hall
East Lansing, MI 48824
USA
xiao@stt.msu.edu

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Preface

From May 8 to May 19th of 2006, the Department of Mathematics at the University of Utah hosted a minicourse on some modern topics in stochastic partial differential equations [SPDEs]. The participants included graduate students and recent PhDs from across North America, as well as research mathematicians at diverse stages of their careers. Intensive courses were given by Robert C. Dalang, Davar Khoshnevisan, An Le, Carl Mueller, David Nualart, Boris Rozovsky, and Yimin Xiao. The present book is comprised of most of those lectures.

For nearly three decades, the topic of SPDEs has been an area of active research in pure and applied mathematics, fluid mechanics, geophysics, and theoretical physics. The theory of SPDEs has a similar flavor as PDEs and interacting particle systems in the sense that most of the interesting developments generally evolve in two directions: There is the general theory; and then there are specific problem-areas that arise from concrete questions in applied science. As such, it is unlikely that there ever will be a cohesive all-encompassing theory of stochastic partial differential equations. With that in mind, the present volume follows the style of the Utah minicourse in SPDEs and attempts to present a selection of interesting themes within this interesting area. The presentation, as well as the choice of the topics, were motivated primarily by our desire to bring together a combination of methods and deep ideas from SPDEs (Chapters 1, 2, and 4) and Gaussian analysis (Chapters 3 and 5), as well as potential theory and geometric measure theory (Chapter 5). Ours is a quite novel viewpoint, and we believe that the interface of the mentioned theories is fertile ground that shows excellent potential for continued future research.

We are aware of at least four books on SPDEs that have appeared since we began to collect the material for this project [4; 8; 12; 14]. Although there is little overlap between those books and the present volume, the rapidly-growing number of books on different aspects of SPDEs represents continued, as well as a growing, interest in both the theory as well as the applications of the subject. The reader is encouraged to consult the references for examples in: (i) Random media [2; 4; 18] and filtering theory [15]; (ii) applications in fluid dynamics and turbulence [1; 2; 17]; and (iii) in statistical physics of disordered media [2; 6; 7; 10]. Further references are scattered throughout the lectures that follow. The reader is invited to consult the references to this preface, together with their voluminous bibliographies, for some of the other viewpoints on this exciting topic.

The Utah Minicourse on SPDEs was funded by a generous VIGRE grant by the National Science Foundation, to whom we are grateful. We thank also the lecturers and participants of the minicourse for their efforts. Finally, we extend our wholehearted thanks to the anonymous referee; their careful reading and thoughtful remarks have led to a more effective book.

Salt Lake City, Utah
July 1, 2008

Davar Khoshnevisan
Firas Rassoul-Agha

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Contributors

Davar Khoshnevisan
Department of Mathematics
The University of Utah
Salt Lake City
UT 84112–0090, USA
davar@math.utah.edu
http://www.math.utah.edu/~davar

Robert C. Dalang
Institut de Mathématiques
Ecole Polytechnique Fédérale de Lausanne
Station 8, CH-1015
Lausanne, Switzerland
robert.dalang@epfl.ch
http://ima.epfl.ch/~rdalang/

David Nualart
Department of Mathematics
University of Kansas
Lawrence, KS 66045, USA
nualart@math.ku.edu
http://www.math.ku.edu/~nualart/

Carl Mueller
Department of Mathematics
University of Rochester, Rochester
NY 14627, USA
cmlr@math.rochester.edu
http://www.math.rochester.edu/~cmlr

Yimin Xiao
Department of Statistics and Probability
A-413 Wells Hall
Michigan State University
East Lansing
MI 48824, USA
xiao@stt.msu.edu
http://www.stt.msu.edu/~xiaoyimi

Contents

A Primer on Stochastic Partial Differential Equations

Davar Khoshnevisan

Summary. These notes form a brief introductory tutorial to elements of Gaussian noise analysis and basic stochastic partial differential equations (SPDEs) in general, and the stochastic heat equation, in particular. The chief aim here is to get to the heart of the matter quickly. We achieve this by studying a few concrete equations only. This chapter provides sufficient preparation for learning more advanced theory from the remainder of this volume.

1 What is an SPDE?

Let us consider a perfectly even, infinitesimally-thin wire of length L. We lay it down flat, so that we can identify the wire with the interval $[0, L]$. Now we apply pressure to the wire in order to make it vibrate.

Let $F(t, x)$ denote the amount of pressure per unit length applied in the direction of the y-axis at place $x \in [0, L]$: $F < 0$ means we are pressing down toward $y = -\infty$; and $F > 0$ means the opposite is true. Classical physics tells us that the position $u(t, x)$ of the wire solves the partial differential equation,

$$\frac{\partial^2 u(t\,,x)}{\partial t^2} = \kappa \frac{\partial^2 u(t\,,x)}{\partial x^2} + F(t\,,x) \qquad (t \geq 0,\ 0 \leq x \leq L), \tag{1}$$

where κ is a physical constant that depends only on the linear mass density and the tension of the wire.

Equation (1) is the so-called *one-dimensional wave equation*. Its solution—via separation of variables and superposition—is a central part of the classical theory of partial differential equations.

D. Khoshnevisan and F. Rassoul-Agha (eds.) *A Minicourse on Stochastic Partial Differential Equations*.
Lecture Notes in Mathematics 1962.

We are interested in addressing the question, "*What if F is random noise*"? There is an amusing interpretation, due to Walsh [30], of (1) for random noise F: If a guitar string is bombarded by particles of sand, then the induced vibrations of the string are determined by a suitable version of (1).

It turns out that in most cases of interest to us, when F is random noise, Equation (1) does not have a classical meaning. But it can be interpreted as an infinite-dimensional integral equation. These notes are a way to get you started thinking in this direction. They are based mostly on the Saint-Flour lecture notes of Walsh from 1986 [30, Chapters 1–3]. Walsh's lecture notes remain as one of the exciting introductions to this subject to date.

2 Gaussian Random Vectors

Let $\boldsymbol{g} := (g_1, \ldots, g_n)$ be an n-dimensional random vector. We say that the distribution of $\boldsymbol{g}$ is *Gaussian* if $\boldsymbol{t}\cdot\boldsymbol{g} := \sum_{j=1}^n t_j g_j$ is a Gaussian random variable for all $\boldsymbol{t} := (t_1, \ldots, t_n) \in \mathbf{R}^n$. It turns out that $\boldsymbol{g}$ is Gaussian if and only if there exist $\boldsymbol{\mu} \in \mathbf{R}^n$ and an $n \times n$, symmetric nonnegative-definite matrix $\boldsymbol{C}$ such that

$$\mathrm{E}\left[\exp\left(i\boldsymbol{t}\cdot\boldsymbol{g}\right)\right] = \exp\left(i\boldsymbol{t}\cdot\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{t}\cdot\boldsymbol{C}\boldsymbol{t}\right). \tag{2}$$

Exercise 2.1. Prove this assertion. It might help to recall that $\boldsymbol{C}$ is *nonnegative definite* if and only if $\boldsymbol{t}\cdot\boldsymbol{C}\boldsymbol{t} \geq 0$ for all $\boldsymbol{t} \in \mathbf{R}^n$. That is, all eigenvalues of $\boldsymbol{C}$ are nonnegative.

3 Gaussian Processes

Let T be a set, and $G = \{G(t)\}_{t\in T}$ a collection of random variables indexed by T. We might refer to G as either a *random field*, or a [*stochastic*] *process indexed by* T.

We say that G is a *Gaussian process*, or a *Gaussian random field*, if $(G(t_1), \ldots, G(t_k))$ is a k-dimensional Gaussian random vector for every $t_1, \ldots, t_k \in T$. The *finite-dimensional distributions* of the process G are the collection of all probabilities obtained as follows:

$$\mu_{t_1,\ldots,t_k}(A_1, \ldots, A_k) := \mathrm{P}\left\{G(t_1) \in A_1, \ldots, G(t_k) \in A_k\right\}, \tag{3}$$

as $A_1, \ldots, A_k$ range over Borel subsets of $\mathbf{R}$ and k ranges over all positive integers. In principle, these are the only pieces of information that one has about the random process G. All properties of G are supposed to follow from properties of these distributions.

The consistency theorem of Kolmogorov [19] implies that the finite-dimensional distributions of G are uniquely determined by two functions:

1. The *mean function* $\mu(t) := \mathrm{E}[G(t)]$; and
2. the covariance function

$$C(s\,,t) := \mathrm{Cov}(G(s)\,,G(t)).$$

Of course, μ is a real-valued function on T, whereas C is a real-valued function on $T \times T$.

Exercise 3.1. Prove that if G is a Gaussian process with mean function μ and covariance function C then $\{G(t) - \mu(t)\}_{t \in T}$ is a Gaussian process with mean function zero and covariance function C.

Exercise 3.2. Prove that C is *nonnegative definite*. That is, prove that for all $t_1, \ldots, t_k \in T$ and all $z_1, \ldots, z_k \in \mathbf{C}$,

$$\sum_{j=1}^{k} \sum_{l=1}^{k} C(t_j\,,t_l) z_j \overline{z_l} \geq 0. \tag{4}$$

Exercise 3.3. Prove that whenever $C : T \times T \to \mathbf{R}$ is nonnegative definite and symmetric,

$$|C(s\,,t)|^2 \leq C(s\,,s) \cdot C(t\,,t) \qquad \text{for all } s, t \in T. \tag{5}$$

This is the *Cauchy–Schwarz inequality*. In particular, $C(t\,,t) \geq 0$ for all $t \in T$.

Exercise 3.4. Suppose there exist $E, F \subset T$ such that $C(s\,,t) = 0$ for all $s \in E$ and $t \in F$. Then prove that $\{G(s)\}_{s \in E}$ and $\{G(t)\}_{t \in F}$ are *independent* Gaussian processes. That is, prove that for all $s_1, \ldots, s_n \in E$ and all $t_1, \ldots, t_m \in F$, $(G(s_1)\,, \ldots, G(s_n))$ and $(G(t_1)\,, \ldots, G(t_m))$ are independent Gaussian random vectors.

A classical theorem—due in various degrees of generality to Herglotz, Bochner, Minlos, etc.—states that the collection of all nonnegative definite functions f on $T \times T$ matches all covariance functions, as long as f is symmetric. [*Symmetry* means that $f(s\,,t) = f(t\,,s)$.] This, and the aforementioned theorem of Kolmogorov, together imply that given a function $\mu : T \to \mathbf{R}$ and a nonnegative-definite function $C : T \times T \to \mathbf{R}$ there exists a Gaussian process $\{G(t)\}_{t \in T}$ whose mean function is μ and covariance function is C.

Example 3.5 (Brownian motion). Let $T = \mathbf{R}_+ := [0, \infty)$, $\mu(t) := 0$, and $C(s, t) := \min(s, t)$ for all $s, t \in \mathbf{R}_+$. I claim that C is nonnegative definite. Indeed, for all $z_1, \ldots, z_k \in \mathbf{C}$ and $t_1, \ldots, t_k \geq 0$,

$$\begin{aligned}\sum_{j=1}^{k}\sum_{l=1}^{k} \min(t_j, t_l) z_j \overline{z_l} &= \sum_{j=1}^{k}\sum_{l=1}^{k} z_j \overline{z_l} \int_0^\infty \mathbf{1}_{[0,t_j]}(x)\mathbf{1}_{[0,t_l]}(x)\, dx \\ &= \int_0^\infty \left| \sum_{j=1}^{k} \mathbf{1}_{[0,t_j]}(x) z_j \right|^2 dx,\end{aligned} \tag{6}$$

which is greater than or equal to zero. Because C is also symmetric, it must be the covariance function of *some* mean-zero Gaussian process $B := \{B(t)\}_{t \geq 0}$. That process B is called *Brownian motion*; it was first invented by Bachelier [1].

Brownian motion has the following additional property. *Let $s > 0$ be fixed. Then the process $\{B(t+s) - B(s)\}_{t \geq 0}$ is independent of $\{B(u)\}_{0 \leq u \leq s}$.* This is the socalled *Markov property* of Brownian motion, and is not hard to derive. Indeed, thanks to Exercise 3.4 it suffices to prove that for all $t \geq 0$ and $0 \leq u \leq s$,

$$\mathrm{E}[(B(t+s) - B(s))B(u)] = 0. \tag{7}$$

But this is easy to see because

$$\begin{aligned}\mathrm{E}[(B(t+s) - B(s))B(u)] &= \mathrm{Cov}(B(t+s), B(u)) - \mathrm{Cov}(B(s), B(u)) \\ &= \min(t+s, u) - \min(s, u) \\ &= u - u \\ &= 0.\end{aligned} \tag{8}$$

By d-dimensional Brownian motion we mean the d-dimensional Gaussian process $B := \{(B_1(t), \ldots, B_d(t))\}_{t \geq 0}$, where $B_1, \ldots, B_d$ are independent [one-dimensional] Brownian motions.

Exercise 3.6. Prove that if $s > 0$ is fixed and B is Brownian motion, then the process $\{B(t+s) - B(s)\}_{t \geq 0}$ is a *Brownian motion* independent of $\{B(u)\}_{0 \leq u \leq s}$. This and the independent-increment property of B [Example 3.5] together prove that B is a *Markov process.*

Example 3.7 (Brownian bridge). The *Brownian bridge* is a mean-zero Gaussian process $\{b(x)\}_{0 \leq x \leq 1}$ with covariance,

$$\mathrm{Cov}(b(x), b(y)) := \min(x, y) - xy \qquad \text{for all } 0 \leq x, y \leq 1. \tag{9}$$

The next exercise shows that the process b looks locally like a Brownian motion. Note also that $b(0) = b(1) = 0$; this follows because $\mathrm{Var}(b(0)) = \mathrm{Var}(b(1)) = 0$, and motivates the ascription "bridge." The next exercise explains why b is "brownian."

Exercise 3.8. Prove that if B is Brownian motion, then b is Brownian bridge, where

$$b(x) := B(x) - xB(1) \qquad \text{for all } 0 \le x \le 1. \tag{10}$$

Also prove that the process b is independent of $B(1)$.

Example 3.9 (OU process). Let $B := \{B(t)\}_{t\ge 0}$ denote a d-dimensional Brownian motion, and define

$$X(t) := \frac{B(e^t)}{e^{t/2}} \qquad \text{for all } t \ge 0. \tag{11}$$

The coordinate processes $X_1, \ldots, X_d$ are i.i.d. Gaussian processes with mean function $\mu(t) := 0$ and covariance function

$$\begin{aligned} C(s\,,t) &:= \mathrm{E}\left[\frac{B_1\left(e^s\right) B_1\left(e^t\right)}{e^{(s+t)/2}}\right] \\ &= \exp\left(-\tfrac{1}{2}|s-t|\right). \end{aligned} \tag{12}$$

Note that $C(s\,,t)$ depends on s and t only through $|s-t|$. Such processes are called *stationary Gaussian processes.* This particular stationary Gaussian process was predicted in the works of Dutch physicists Leonard S. Ornstein and George E. Uhlenbeck [29], and bears their name as a result. The existence of the Ornstein–Uhlenbeck process was proved rigorously in a landmark paper of Doob [10].

Example 3.10 (Brownian sheet). Let $T := \mathbf{R}_+^N := [0\,,\infty)^N$, $\mu(\boldsymbol{t}) := 0$ for all $\boldsymbol{t} \in \mathbf{R}_+^N$, and define

$$C(\boldsymbol{s}\,,\boldsymbol{t}) := \prod_{j=1}^{N} \min(s_j\,,t_j) \qquad \text{for all } \boldsymbol{s}, \boldsymbol{t} \in \mathbf{R}_+^N. \tag{13}$$

Then C is a nonnegative-definite, symmetric function on $\mathbf{R}_+^N \times \mathbf{R}_+^N$, and the resulting mean-zero Gaussian process $B = \{B(\boldsymbol{t})\}_{\boldsymbol{t}\in\mathbf{R}_+^N}$ is the N-parameter *Brownian sheet.* This generalizes Brownian motion to an N-parameter random field. One can also introduce d-dimensional, N-parameter Brownian sheet as the d-dimensional process whose coordinates are independent, [one-dimensional] N-parameter Brownian sheets.

Example 3.11 (OU sheet). Let $\{B(\boldsymbol{t})\}_{\boldsymbol{t}\in\mathbf{R}_+^N}$ denote N-parameter Brownian sheet, and define a new N-parameter stochastic process X as follows:

$$X(\boldsymbol{t}) := \frac{B\left(e^{t_1}, \ldots, e^{t_N}\right)}{e^{(t_1+\cdots+t_N)/2}} \qquad \text{for all } \boldsymbol{t} := (t_1\,, \ldots, t_N) \in \mathbf{R}_+^N. \tag{14}$$

This is called the N-paramerter *Ornstein–Uhlenbeck sheet,* and generalizes the Ornstein–Uhlenbeck process of Example 3.9.

Exercise 3.12. Prove that the Ornstein–Uhlenbeck sheet is a mean-zero, N-parameter Gaussian process and its covariance function $C(\boldsymbol{s}\,,\boldsymbol{t})$ depends on $(\boldsymbol{s}\,,\boldsymbol{t})$ only through $|\boldsymbol{s}-\boldsymbol{t}| := \sum_{i=1}^N |s_i - t_i|$.

Example 3.13 (White noise). Let $T := \mathscr{B}(\mathbf{R}^N)$ denote the collection of all Borel-measurable subsets of $\mathbf{R}^N$, and $\mu(A) := 0$ for all $A \in \mathscr{B}(\mathbf{R}^N)$. Define $C(A\,,B) := \lambda^N(A \cap B)$, where λ^N denotes the N-dimensional Lebesgue measure. Clearly, C is symmetric. It turns out that C is also nonnegative definite (Exercise 3.14 on page 6). The resulting Gaussian process $\dot{W} := \{\dot{W}(A)\}_{A\in\mathscr{B}(\mathbf{R}^N)}$ is called *white noise* on $\mathbf{R}^N$.

Exercise 3.14. Complete the previous example by proving that the covariance of white noise is indeed a nonnegative-definite function on $\mathscr{B}(\mathbf{R}^N) \times \mathscr{B}(\mathbf{R}^N)$.

Exercise 3.15. Prove that if $A, B \in \mathscr{B}(\mathbf{R}^N)$ are disjoint then $\dot{W}(A)$ and $\dot{W}(B)$ are independent random variables. Use this to prove that if $A, B \in \mathscr{B}(\mathbf{R}^N)$ are nonrandom, then with probability one,

$$\dot{W}(A \cup B) = \dot{W}(A) + \dot{W}(B) - \dot{W}(A \cap B). \tag{15}$$

Exercise 3.16. Despite what the preceding may seem to imply, $\dot{W}$ is not a random signed measure in the obvious sense. Let $N = 1$ for simplicity. Then, prove that with probability one,

$$\lim_{n\to\infty} \sum_{j=0}^{2^n-1} \left| \dot{W}\left(\left[\frac{j-1}{2^n}\,,\,\frac{j}{2^n}\right]\right)\right|^2 = 1. \tag{16}$$

Use this to prove that with probability one,

$$\lim_{n\to\infty} \sum_{j=0}^{2^n-1} \left| \dot{W}\left(\left[\frac{j-1}{2^n}\,,\,\frac{j}{2^n}\right]\right)\right| = \infty. \tag{17}$$

Conclude that if $\dot{W}$ *were* a random measure then with probability one $\dot{W}$ is not sigma-finite. Nevertheless, the following example shows that one can integrate some things against $\dot{W}$.

Example 3.17 (The isonormal process). Let $\dot{W}$ denote white noise on $\mathbf{R}^N$. We wish to define $\dot{W}(h)$ where h is a nice function. First, we identify $\dot{W}(A)$ with $\dot{W}(\mathbf{1}_A)$. More generally, we define for all disjoint $A_1, \ldots, A_k \in \mathscr{B}(\mathbf{R}^N)$ and $c_1, \ldots, c_k \in \mathbf{R}$,

$$\dot{W}\left(\sum_{j=1}^k c_j \mathbf{1}_{A_j}\right) := \sum_{j=1}^k c_j \dot{W}(A_j). \tag{18}$$

The random variables $\dot{W}(A_1), \ldots, \dot{W}(A_k)$ are independent, thanks to Exercise 3.15. Therefore,

$$\left\| \dot{W}\left(\sum_{j=1}^{k} c_j \mathbf{1}_{A_j} \right) \right\|_{L^2(\mathrm{P})}^2 = \sum_{j=1}^{k} c_j^2 |A_j| = \left\| \sum_{j=1}^{k} c_j \mathbf{1}_{A_j} \right\|_{L^2(\mathbf{R}^N)}^2. \tag{19}$$

Classical integration theory tells us that for all $h \in L^2(\mathbf{R}^N)$ we can find h_n of the form $\sum_{j=1}^{k(n)} c_{jn} \mathbf{1}_{A_{j,n}}$ such that $A_{1,n}, \ldots, A_{k(n),n} \in \mathscr{B}(\mathbf{R}^N)$ are disjoint and $\|h - h_n\|_{L^2(\mathbf{R}^N)} \to 0$ as $n \to \infty$. This, and (19) tell us that $\{\dot{W}(h_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathrm{P})$. Denote their limit by $\dot{W}(h)$. This is the *Wiener integral of* $h \in L^2(\mathbf{R}^N)$, and is sometimes written as $\int h\, dW$ [no dot!]. Its key feature is that

$$\left\| \dot{W}(h) \right\|_{L^2(\mathrm{P})} = \|h\|_{L^2(\mathbf{R}^N)}. \tag{20}$$

That is, $\dot{W} : L^2(\mathbf{R}^N) \to L^2(\mathrm{P})$ is an isometry; (20) is called *Wiener's isometry* [32]. [Note that we now know how to construct the stochastic integral $\int h\, dW$ only if $h \in L^2(\mathbf{R}^N)$ is *nonrandom*.] The process $\{\dot{W}(h)\}_{h \in L^2(\mathbf{R}^N)}$ is called the *isonormal process* [11]. It is a Gaussian process; its mean function is zero; and its covariance function is $C(h\,, g) = \int_{\mathbf{R}^N} h(x) g(x)\, dx$—the $L^2(\mathbf{R}^N)$ inner product—for all $h, g \in L^2(\mathbf{R}^N)$.

Exercise 3.18. Prove that for all [nonrandom] $h, g \in L^2(\mathbf{R}^N)$ and $a, b \in \mathbf{R}$,

$$\int (ah + bg)\, dW = a \int h\, dW + b \int h\, dW, \tag{21}$$

almost surely.

Exercise 3.19. Let $\{h_j\}_{j=1}^{\infty}$ be a complete orthonormal system [c.o.n.s.] in $L^2(\mathbf{R}^N)$. Then prove that $\{\dot{W}(h_j)\}_{j=1}^{\infty}$ is a complete orthonormal system in $L^2(\mathrm{P})$. In particular, for all Gaussian random variables $Z \in L^2(\mathrm{P})$ that are measurable with respect to the white noise,

$$Z = \sum_{j=1}^{\infty} a_j \dot{W}(h_j) \quad \text{almost surely,} \quad \text{with} \quad a_j := \mathrm{Cov}\left(Z\,, \dot{W}(h_j) \right), \tag{22}$$

and the infinite sum converges in $L^2(\mathrm{P})$. This permits one possible entry into the "Malliavin calculus." For this, and much more, see the course by D. Nualart in this volume.