Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1274

N. Christopher Phillips

Equivariant K-Theory and Freeness of Group Actions on C*-Algebras

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

Author

N. Christopher Phillips
Department of Mathematics, University of California
Los Angeles, CA 90024-1555, USA

Mathematics Subject Classification (1985): Primary: 46L55, 46L80, 46M20
Secondary: 19K33, 19K99, 19L47

ISBN 3-540-18277-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-18277-2 Springer-Verlag New York Berlin Heidelberg

Library of Congress Cataloging-in-Publication Data. Phillips, N. Christopher (Norman Christopher), 1956-. Equivariant K-theory and freeness of group actions on C*-algebras. (Lecture notes in mathematics ; 1274) Bibliography: p. Includes index. 1. K-theory. 2. C*-algebras. 3. Lie groups. I. Title. II. Series: Lecture notes in mathematics (Springer-Verlag) ; 1274. QA3.L28 no 1274 510 s 87-23345 [QA612.33] [512'.55]
ISBN 0-387-18277-2 (U.S.)

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

To

汪開祥

Wang Kai-Shyang

Acknowledgments

I would like to thank a number of people for their help and encouragement during the preparation of this book. Graeme Segal first called my attention to the paper from which I learned about the relation between equivariant K-theory and freeness of actions. I have had valuable discussions with, among others, Claude Schochet concerning Künneth theorems, I. M. Singer concerning pseudodifferential operators, Nigel Higson concerning extendible prequasihomomorphisms, and Jonathan Rosenberg concerning projective unitary representations. David Handelman suggested the main result of section 9.2 and its proof.

Most of all, however, I would like to thank Marc Rieffel. Many of the results in this book first appeared in my Ph.D. thesis written under his direction, and many others were first proved while I was working on my thesis. His patience, encouragement, and suggestions, both before and after I received my Ph.D., have been extremely valuable, and he has helped eliminate many obscurities in the exposition. He was also kind enough to show me some of his unpublished research notes, and to allow me to use some of the material from them in chapter 7.

This book is based on the author's Ph.D. thesis written at the University of California at Berkeley. However, it incorporates substantial revisions and many additional results. The research reported here has been partially supported by a National Science Foundation Graduate Fellowship, by the Mathematical Sciences Research Institute (NSF Grant 8120790), and by a National Science Foundation Postdoctoral Fellowship. All of this support is gratefully acknowledged. Thanks are also due to Marsha Colby, who converted an enormous number of sometimes rather illegible handwritten pages into the computer files from which this manuscript was produced, and who, with rare exceptions, was able to persuade a University of California computer to print many symbols that it had never been intended to make.

Preface

This book is about equivariant K-theory and K-theoretic conditions for freeness of actions of compact Lie groups on C^*-algebras. The introduction, explaining in detail the motivation for this work, is followed by two primarily expository chapters, one each on equivariant K-theory for C^*-algebras and equivariant KK-theory for C^*-algebras. The remaining six chapters contain the results of the author's research on K-theoretic conditions for freeness of actions on C^*-algebras. We assume throughout familiarity with the theory of C^*-algebras, including crossed product C^*-algebras and ordinary (that is, not equivariant) K-theory of C^*-algebras.

Our work is motivated by the observation that, given an action of a compact Lie group on a compact Hausdorff space, one can determine whether the action is free solely by examining the equivariant K-theory of the space for the given action. (Details, with proofs, are given in chapter 1.) Now the category of compact Hausdorff spaces is contravariantly equivalent to the category of commutative unital C^*-algebras via the functor assigning to each space the algebra of continuous complex-valued functions on the space. Therefore the relation between equivariant K-theory and freeness can be interpreted as an assertion about actions of compact Lie groups on commutative unital C^*-algebras. In view of the recent successes of noncommutative algebraic topology, in which general C^*-algebras are regarded as "noncommutative locally compact spaces," we are naturally led to try to generalize the results mentioned above to general C^*-algebras. However, no completely satisfactory notion of freeness of an action on a C^*-algebra is known. We therefore define and study conditions on the equivariant K-theory of a C^*-algebra which, in case the algebra is commutative, imply that the action on the underlying space is free.

Chapters 2 and 3 develop the necessary background material on C^*-algebraic equivariant K-theory and KK-theory respectively. Neither chapter depends significantly on the rest of the book. Chapter 2 consists mostly of material which has previously appeared only in somewhat condensed form, or which has never been published but straightforwardly generalizes ordinary K-theory. It is fairly comprehensive; however, there are some results purely about equivariant K-theory elsewhere, especially in sections 5.1 and 6.1-6.4. Previous knowledge of ordinary K-theory for C^*-algebras is assumed, but is used only at a few points, most notably for the six term exact sequence and Bott periodicity. Chapter 3 develops equivariant KK-theory from Cuntz' quasihomomorphism point of view. Most of the material has appeared previously but again only in very condensed form. We prove only the basic facts, up to the product and Bott periodicity, omitting however the construction of the six term exact sequences, for which we refer to a paper of Cuntz and Skandalis. Again, some additional material can be found elsewhere, particularly in sections 5.1 and 9.7.

In chapter 4, we define our K-theoretic notions of freeness. There and in the next two chapters we consider consistency theorems, the analogs of such facts as the freeness of the restriction of a free action to an invariant subspace or a closed subgroup. Most of the appropriate statements are either easily proved or, in bad cases, easily disproved. Two topics, namely actions of subgroups and actions on tensor products, present greater difficulties. Each of these topics gets a chapter to itself, and the gaps between our theorems and our counterexamples are larger than was generally the case in chapter 4.

Chapter 7 is devoted to the relation between our K-theoretic notions of freeness and previously known measures of freeness, especially Kishimoto's strong Connes spectrum. Our conditions are, unfortunately, trivially satisfied by a trivial action on a simple C^*-algebra whose K-groups are all zero. (This kind of difficulty cannot arise in the context of spaces.) However, if the K-theory of the C^*-algebra is sufficiently nontrivial, and if the group is sufficiently small, then K-theoretic freeness does imply other forms of freeness.

The remaining two chapters consider the implications of our K-theoretic conditions for freeness for actions on two special classes of C^*-algebras, namely type I and AF algebras. In both cases, we obtain analytic characterizations of several of our K-theoretic freeness conditions. In the type I case, one of our conditions is shown to be equivalent to freeness of the corresponding action on the primitive ideal space. For AF algebras, we obtain results of a somewhat different nature but which should not be surprising in view of known results about the ordinary K-theory of AF algebras. Our results here have led us to hope that there might be a good analytic notion of freeness which implies K-theoretic freeness in general and coincides with it on these special classes of C^*-algebras. However, we have made no attempt to investigate this question.

Table of Contents

Chapter 1

Introduction: The Commutative Case

1.1. Introduction

Recall that an action of a group G on a space X is said to be free if for $g \in G$, $g \neq e$, and $x \in X$, one has $gx \neq x$. Since the commutative C^*-algebras are exactly the algebras $C_0(X)$ of continuous complex valued functions vanishing at infinity on locally compact Hausdorff spaces X, we can transfer the definition of freeness to commutative C^*-algebras as follows: we say that a continuous action of a group G on $C_0(X)$ is free if the corresponding action on X is free. Viewing general C^*-algebras as "noncommutative topological spaces," we now seek generalizations of the concept of freeness to the case of actions of groups on noncommutative C^*-algebras.

The starting point of this work is a relation between freeness of a continuous action of a compact Lie group G on a compact space X and the equivariant K-theory of X. Before stating it, we introduce some notation. Recall ([98]) that equivariant K-theory is a generalized cohomology theory on locally compact G-spaces (that is, spaces carrying continuous actions of G), where G is a compact group. This theory is analogous to ordinary K-theory of locally compact spaces. The equivariant K-groups of a locally compact G-space X are written $K_G^0(X)$ and $K_G^1(X)$, and as usual we let $K_G^*(X)$ stand for the direct sum $K_G^0(X) \oplus K_G^1(X)$. These groups are in fact modules over the representation ring $R(G)$ of G, as defined in [97]. The ring $R(G)$ contains a distinguished ideal, namely the augmentation ideal $I(G)$. (It is defined in the example preceding proposition 3.8 of [97]. The definitions of $K_G^*(X)$, $R(G)$, and $I(G)$ will all be given in detail in chapter 2.)

We are now in a position to state the relations between freeness and equivariant K-theory. This result is essentially due to Atiyah and Segal ([6], proposition 4.3, and [98], proposition 4.1).

1.1.1 Theorem. Let G be a compact Lie group, and let X be a compact Hausdorff G-space. Then the following are equivalent:

(1) G acts freely on X.

(2) The natural map $K^*(X/G) \longrightarrow K_G^*(X)$ is an isomorphism.

(3) $K_G^*(X)$ is complete in the $I(G)$-adic topology.

(4) $K_G^*(X)$ is complete and Hausdorff in the $I(G)$-adic topology.

(5) $K_G^*(X)$ is discrete in the $I(G)$-adic topology.

(6) For all prime ideals $P \subset R(G)$ such that $P \not\supset I(G)$, the localization $K_G^*(X)_P$ is zero.

In condition (2), the space X/G is the space of orbits of the action of G on X. Since G is compact, X/G is a compact Hausdorff space with the quotient topology. The map sends the class $[E]$ of a vector bundle E over X/G to the class of the pull-back of E via the quotient map. More details will be given in the next section.

The methods used to generalize ordinary K-theory to C^*-algebras can also be used to generalize equivariant K-theory to C^*-algebras carrying continuous actions of compact groups. (The resulting theory is defined in, for example, [51]. See also [55]. We will develop its properties in detail in chapter 2.) Since equivariant K-theory is a covariant functor on C^*-algebras, we denote the equivariant K-theory of a C^*-algebra A carrying a continuous action of a compact group G by $K_*^G(A)$. We then have $K_*^G(C_0(X)) \simeq K_G^*(X)$ for locally compact G-spaces X.

The original motivation for the work reported here was an attempt to generalize theorem 1.1.1 to noncommutative C^*-algebras. Lacking an adequate notion of freeness of a group action on a C^*-algebra, we were led to define a concept of K-theoretic freeness in terms of one of the other conditions of this theorem. Our choice is condition (6). One reason for choosing this condition is that localization is an exact functor, so that exact sequences in equivariant K-theory yield exact sequences for the localized modules $K_*^G(A)_P$. In particular, if P is a prime ideal in $R(G)$, then $K_*^G(\cdot)_P$ is a generalized homology theory on C^*-algebras carrying continuous actions of the compact group G.

Theorem 1.1.1 fails for general locally compact G-spaces. Indeed, if X is a contractible space, such as $[0, 1)$, and the action of G on X is trivial, then conditions (2) through (6) are satisfied, since the K-groups which appear are all zero, but condition (1) fails. The situation can be saved by observing that a continuous action of a compact group G on a topological space X is free if and only if the restriction of the action to every G-invariant compact subset of X is free. Since compact subsets of Hausdorff spaces are closed, and since the C^* equivalent of a closed subset is the quotient by an ideal, we are led to the following definition: a continuous action of a compact Lie group G on a C^*-algebra A is called K-free if for every G-invariant ideal I in A and every prime ideal P in $R(G)$ not containing $I(G)$, we have $K_*^G(A/I)_P = 0$. This work, then, is devoted to the study of actions which are K-free or satisfy certain closely related conditions.

We have defined K-freeness only for compact Lie groups. We restrict ourselves to compact groups because equivariant K-theory is only defined for actions of compact groups. (Some work has been done toward a more general definition — see [56] and [8].) Furthermore, some of our work involves condition (2) of theorem 1.1.1 and its C^* analog. If G is not compact, then X/G need not be Hausdorff, and the C^* analog of X/G, namely the algebra A^G of fixed points under the action of G on the C^*-algebra A, is often so small as to be useless. The restriction that G be a compact Lie group is necessary to ensure that $R(G)$ is a Noetherian ring. (We do not require that Lie groups be connected. In particular, finite groups are not excluded.) The fact that $R(G)$ is Noetherian is used in the proof of theorem 1.1.1, and also in the proof of one of the basic lemmas on K-freeness, proposition 4.1.3. Finally, many of the most interesting results have actually been proved only for various classes of finite groups. Such restrictions are presumably not always necessary, but the generalizations of many of our results to infinite compact groups appear to be substantially more difficult.

We now outline the organization of this work. The remainder of this chapter is devoted to the proof of theorem 1.1.1 and several related results, assuming the basic properties of equivariant K-theory. This theorem is not merely motivation — the proofs of many of our results ultimately depend on it.

In chapter 2 we develop the properties of equivariant K-theory of C^*-algebras. The theorem of Julg ([51]), according to which there is a natural isomorphism $K_*^G(A) \simeq K_*(C^*(G,A))$, where $C^*(G,A)$ is the crossed product C^*-algebra, plays a central role in our development. Thus, we assume some knowledge of ordinary K-theory of Banach algebras. (See for example [26], [32], [46], and [105]; note that the hypothesis in [105] that all algebras are commutative is unnecessary.) We do not, however, assume any knowledge of equivariant K-theory of spaces. In particular, the results on equivariant K-theory used in the proof of theorem 1.1.1 are all proved in chapter 2. Little of this material, if any, is really new.

Chapter 3 is an introduction to the equivariant version of Kasparov's KK-theory [55], using the approach of Cuntz [25]. We use KK-theory in several ways: both as a useful technical device for proving things about K-theory, and because the analog of K-freeness using KK-theory is interesting in its own right. Again, the material is not new, although Cuntz does not deal with the equivariant theory in his account. This chapter is rather technical; we fill in many of the details omitted from [25].

In chapter 4, we formally define K-freeness and two related concepts, namely total K-freeness and KK-freeness. An action α of a compact Lie group G on a C^*-algebra A is called totally K-free if the restriction $\alpha|_H$ of α to every closed subgroup H is K-free. We find it necessary to introduce this concept because a simple example shows that K-freeness, unlike freeness, is not inherited by restrictions of actions. KK-freeness is the analog of K-freeness using KK-theory. We prove the basic properties of these concepts, showing, for example, that they behave well with respect to passage to G-invariant ideals and quotients by them. We also show that KK-freeness implies K-freeness. Perhaps the most significant results of chapter 4 are the following two results, showing that certain actions, which one would expect to be totally K-free or KK-free, in fact are. We prove that if α is a continuous action of a finite group G on a separable C^*-algebra A such that the induced action on the primitive ideal space of A is free, then α is totally K-free. We also prove that if α is an action of a compact Lie group on a separable unital C^*-algebra A such that the induced action on the maximal ideal space of the center of A is free, then α is KK-free. Unfortunately, this last result fails for nonunital algebras: there is a free action on a locally compact topological space X such that the corresponding action on $C_0(X)$ is not KK-free.

The next two chapters are devoted to two particularly difficult topics of the same general nature as those of chapter 4, namely restrictions of actions to subgroups and actions on tensor products. As mentioned before, the restriction of a K-free action to a closed subgroup need not be K-free. Nevertheless, in chapter 5 we prove that the restriction of a KK-free action of a finite group on a nuclear C^*-algebra to a subgroup *is* KK-free. The proof requires a technical lemma, the Ideal Decomposition Lemma, which is used to express an ideal invariant under the action of a subgroup in terms of ideals invariant under the action of the group. Going in the other direction, we are able to prove that if α is an action of a finite p-group G (a group whose order is a power of the prime number p) on a C^*-algebra A, and if $\alpha|_S$ is K-free for every cyclic subgroup S of G, then α is totally K-free. This is the best generalization we have of the fact that a continuous action of a compact Lie group on a space is free if the restriction of the action to every finite subgroup of prime order is free. An example given in chapter 9 shows that "cyclic" cannot be replaced by "prime order" in our theorem, but I do not know if the other hypotheses can be weakened.

In chapter 6, we turn to tensor products. Here the motivation is the fact that if X is a free G-space and Y is any G-space, then the diagonal action of G on $X \times Y$ is free. We prove that if G is a finite p-group which acts totally K-freely on a C^*-algebra A and arbitrarily on a C^*-algebra B, and if one of the actions satisfies certain technical conditions (including nuclearity of the algebra), then the diagonal action of G on $A \otimes B$ is totally K-free. The proof uses a Künneth theorem, similar to the one in [95], for the localized homology theories $K_*^H(\cdot)_P$ for certain groups H and certain prime ideals P in $R(H)$. The proof also uses the Ideal Decomposition Lemma from the previous chapter. I do not know if the hypothesis on G can be weakened, but examples show that total K-freeness cannot be replaced by K-freeness. Indeed, we produce an example in which the actions on *both* A and B are K-free, but the diagonal action is not K-free.

Chapter 7 examines the relations between K-freeness and other conditions for freeness of a group action on a C^*-algebra. One of the conditions considered is the C^* analog of condition (2) of theorem 1.1.1. Another type of condition, involving the Connes spectrum and its variants, has been investigated by Olesen and Pedersen in [71], [72], and [73], and by Kishimoto in [58], in connection with the problem of determining when crossed products of C^*-algebras by abelian groups are simple or prime. We obtain theorems along the following lines: if a finite abelian group G acts K-freely on a C^*-algebra A whose K-theory is sufficiently nontrivial in an appropriate sense, then the strong Connes spectrum ([58]) is the full dual group $\widehat{G}$. We are then able to conclude that the fixed point algebra A^G is strongly Morita equivalent ([87]) to the crossed product $C^*(G,A)$. If A is separable, it follows that there is a canonical isomorphism from $K_*(A^G)$ to $K_*^G(A)$. We then devote some space to showing that many C^*-algebras do indeed have sufficiently nontrivial K-theory. If $G = \mathbf{Z}/2\mathbf{Z}$, we can furthermore show that for a K-free action of G on an arbitrary C^*-algebra A, one has $K_*(A^G) \simeq K_*^G(A)$ up to 2-torsion.

The last two chapters examine the implications of K-freeness for two special classes of C^*-algebras. In chapter 8, we study type I algebras. We obtain a converse to one of the results of chapter 7: if α is an action of a compact abelian Lie group G on a separable type I algebra A, and if the strong Connes spectrum of α is the entire dual group $\hat{G}$, then the action is K-free. We also obtain a converse to a theorem in chapter 4: if a compact Lie group G acts totally K-freely on a type I algebra A, then the induced action on the primitive ideal space of A is actually free. If the group G is finite cyclic, then K-freeness of an action on a separable type I algebra actually implies total K-freeness, and we thus obtain a number of equivalent conditions for such an action to be K-free.

Finally, in chapter 9 we specialize to AF algebras. It is not known whether an action of a finite group on an AF algebra leaves invariant an increasing sequence of finite dimensional subalgebras whose union is dense in A; in order to obtain results about AF algebras we must assume that all actions considered do indeed satisfy this property. (The results in chapter 7 apply to AF algebras without any such assumption on the actions.) An action α of a finite abelian group G on an AF algebra A is shown to be K-free if $K_0^G(A)_P = 0$ for all primes $P \subset R(G)$ such that $P \not\supset I(G)$; thus one does not need to look at the equivariant K-theory of the quotients A/I. We furthermore show that if α is K-free, then in fact $I(G)K_*^G(A) = 0$, which is a strong version of condition (5) of theorem 1.1.1. These results simplify considerably the verification that an action is or is not K-free, and we use them to compute some of our most interesting examples. In particular, we construct a K-free action of $\mathbf{Z}/2\mathbf{Z}$ on a C^*-algebra A which induces the trivial action on the primitive ideal space of A, an action of $\mathbf{Z}/2\mathbf{Z}$ such that the strong Connes spectrum is the full dual group but which is not K-free, and an action of $\mathbf{Z}/4\mathbf{Z}$ which is not K-free but whose restriction to the two element subgroup is K-free. We then prove the main theorem on actions on AF algebras: we show that, for actions of finite abelian groups, KK-freeness, total K-freeness, and certain homotopy conditions are all equivalent. We are able to generalize part of this theorem to actions of arbitrary finite groups. The proof of the generalization uses a general method for computing $KK_G^*(A, B)$, where G is a finite group which acts on the AF algebras A and B in such a way that the assumption made above on invariant finite dimensional subalgebras is satisfied. We also examine the special case of locally representable actions.

It should be mentioned that the later chapters are not dependent on all of the preceding ones. They all, of course, depend on chapter 4, while chapter 6 also requires results from chapter 5. Chapter 7 depends only on chapter 4, except that it uses one lemma, not involving K-theory, from chapter 6. Chapter 8 uses material from chapter 7 but not from chapters 5 and 6. Chapter 9 depends only on chapters 4 and 5, except for one example which is related to chapter 7.

Some of the results of this work were announced in [81].

1.2. Proof of Theorem 1.1.1

In this section, we prove theorem 1.1.1 and an algebraic lemma, used in its proof, which will be needed later. We also prove, again for later use, the Localization Theorem ([98], proposition 4.1), which is a generalization of one of the implications in theorem 1.1.1.

Before proving theorem 1.1.1, we give a precise definition of the map appearing in condition (2) and recall a few facts about localization. If G is a compact group, and X is a compact G-space, then the map from $K^0(X/G)$ to $K_G^0(X)$ is defined as follows. Let π be the canonical identification map from X to X/G, and let E be a vector bundle over X/G with projection $p : E \longrightarrow X/G$. Then the image of the class $[E]$ of E in $K_G^0(X)$ is the class of the G-vector bundle

$$\pi^*E = \{(x,v) \in X \times E : \pi(x) = p(v)\},$$

where the G-action is $g \cdot (x,v) = (gx,v)$. It is clear that the resulting map from $K^0(X/G)$ to $K_G^0(X)$ is well defined. It is also easily seen to be a ring homomorphism. If X is not compact, then there is a canonical homeomorphism $X^+/G \simeq (X/G)^+$, where X^+ and $(X/G)^+$ are the one point compactifications of X and X/G. (The action of G on X^+ fixes the point at infinity.) It is clear that the image of $K^0(X/G)$ under the map from $K^0((X/G)^+)$ to $K_G^0(X^+)$ actually lies in $K_G^0(X)$, so that we have a homomorphism from $K^0(X/G)$ to $K_G^0(X)$ for arbitrary locally compact G-spaces X. By taking suspensions, we also obtain a homomorphism from $K^1(X/G)$ to $K_G^1(X)$.

We now recall the definition of localization. (See chapter 3 of [5] for details.) Let R be a commutative ring with identity, and let S be a multiplicative system in R, that is, a subset of R which contains 1, does not contain 0, and is closed under multiplication. Then there is a commutative ring $S^{-1}R$ consisting of all fractions a/s with $a \in R$ and $s \in S$. Two fractions a/s and b/t are equal if and only if there is $u \in S$ such that $u(ta - sb) = 0$. Similarly, if M is an R-module, then there is an $S^{-1}R$-module $S^{-1}M$ consisting of all fractions m/s with $m \in M$ and $s \in S$; the condition for equality of two such fractions is similar. The assignment $M \longrightarrow S^{-1}M$ is in fact a functor from R-modules to $S^{-1}R$-modules, and it can be easily shown ([5], proposition 3.3) that this functor preserves exactness. Now let $P \subset R$ be a prime ideal, that is, an ideal such that if $ab \in P$ then $a \in P$ or $b \in P$. Then $S = R - P$ is a multiplicative system. The ring $S^{-1}R$ is called the localization of R at P, and is written R_P. Similarly, $S^{-1}M$ is called the localization of M at P and written M_P.

We need one lemma before starting the proof of theorem 1.1.1.

1.2.1 Lemma. Let X be a locally compact G-space. Then there is a natural isomorphism of $R(G)$-modules $K_G^*(X) \simeq K_G^0(S^1 \times X)$, where S^1 is the unit circle, and the G-action on $S^1 \times X$ is given by $g(z,x) = (z,gx)$.

Proof. We identify S^1 with the one point compactification of $(0,1)$. We obtain a natural identification of the suspension $(0,1)\times X$ of X with an open subset U of $S^1\times X$. The complement of U is just a copy of X, and furthermore the inclusion of the complement has a left inverse, namely the projection on the second factor. We obtain an exact sequence in K-theory

$$K_G^1(S^1\times X) \longrightarrow K_G^1(X) \longrightarrow K_G^0(U) \longrightarrow K_G^0(S^1\times X) \longrightarrow K_G^0(X) \longrightarrow K_G^1(U).$$

The maps from $K_G^i(S^1\times X)$ to $K_G^i(X)$ have right inverses, so there is actually a split exact sequence

$$0 \longrightarrow K_G^0(U) \longrightarrow K_G^0(S^1\times X) \longrightarrow K_G^0(X) \longrightarrow 0.$$

Since $K_G^0(U)$ is naturally isomorphic to $K_G^1(X)$, this completes the proof. Q.E.D.

Proof of theorem 1.1.1. (1)$\Longrightarrow$(2). (This proof is taken from [98], proposition 2.1.) If G acts freely on X then G acts freely on $S^1\times X$. The previous lemma now implies that we can replace X by $S^1\times X$ and then only prove that $K^0(X/G) \longrightarrow K_G^0(X)$ is an isomorphism. Let E be a G-vector bundle over X. We claim E/G is a vector bundle over X/G, with the obvious projection map. The only part which is not obvious is showing that E/G is locally trivial. For this, let $x \in X$. Then by [14], theorem II.5.4, there is a subset Z of X with $x \in Z$ such that the map $(g,z) \longrightarrow gz : G\times Z \longrightarrow X$ defines a homeomorphism from $G\times_{G_x} Z$ onto a neighborhood $U = G\cdot Z$ of the orbit Gx. (Here G_x is the stabilizer group of the point x. The notation $G\times_{G_x} Z$ denotes the twisted product. See the discussion preceding definition 2.9.2 for details.) Since the action is free, $G\times_{G_x} Z = G\times Z$. Since E is locally trivial, we may assume, by choosing Z small, that the restricted bundle $E|_Z$ is trivial. Let $\pi : X \longrightarrow X/G$ be the identification map. Since $U \simeq G\times Z$, we obtain isomorphisms $(E/G)|_{\pi[U]} \simeq (E|_U)/G \simeq E|_Z$. Since $\pi[U]$ is a neighborhood of $\pi(x)$ in X/G, it follows that E/G is locally trivial, and hence a vector bundle.

We now claim that $[E] \longrightarrow [E/G]$ defines an inverse for $[F] \longrightarrow [\pi^*(F)]$. Let $p : E \longrightarrow X$ and $p_0 : E/G \longrightarrow X/G$ be the projections; then recall that

$$\pi^*(E/G) = \{(x,v) \in X\times E/G : \pi(x) = p_0(v)\}.$$

The function from E to $\pi^*(E/G)$ defined by $e \longrightarrow (p(e), \pi(e))$ is then obviously an isomorphism of G-vector bundles. In the other direction, if F is a vector bundle over X/G, then the map sending the class $[x,v] \in \pi^*(F)/G$ of $(x,v) \in \pi^*(F)$ to v is clearly an isomorphism. It follows that $[F] \longrightarrow [\pi^*(F)]$ is an isomorphism from $K^0(X/G)$ to $K_G^0(X)$ as desired.

(2)$\Longrightarrow$(5). (This is from [6], proof of proposition 4.3.) Again, by lemma 1.2.1, we can replace X by $S^1\times X$, and consider only K^0 and K_G^0. The ring $R(G)$ is noetherian ([97], corollary 3.3), hence $I(G)$ is finitely generated. Since $K^0(X/G) \simeq K_G^0(X)$, it follows that $K^0(X/G)$ is also an $R(G)$-module. (This module structure is not particularly easy to describe.) Since it is also a commutative ring with identity, the ideal $I(G)K^0(X/G)$ is also finitely generated. Each element in it has the form $[E]-[F]$, where E and F are vector bundles such that for $x \in X/G$, the fiber dimensions $dim(E_x)$ and $dim(F_x)$ are equal. (This follows from the fact that elements of $I(G)$ have the form $[V]-[W]$, where V and W are representation spaces of G with $dim V = dim W$.) By [2], proposition 3.1.6, all such elements are nilpotent. Since $I(G)K^0(X/G)$ is finitely generated, there is an integer n such that $(I(G)K^0(X/G))^n = 0$, and since $K^0(X/G)$ is unital it follows that $I(G)^n K^0(X/G) = 0$. Thus $I(G)^n K_G^0(X) = 0$, as desired.

(5)$\Longrightarrow$(4) $\Longrightarrow$(3): trivial.

(3)$\Longrightarrow$(1): (This is also from [6], although our argument below that $R(H)$ is not complete is different.) Suppose G does not act freely on X. Then there is $x \in X$ and a subgroup H of G of prime order p such that $H \subset G_x$. The composite ring homomorphism

$$K_G^*(X) \longrightarrow K_H^*(X) \longrightarrow K_H^*(\{x\}) \simeq R(H)$$

makes $R(H)$ into a topological $K_G^*(X)$-module, where everything has the $I(G)$-adic topology. Now $R(H)$ is finitely generated as a module over $K_G^*(X)$, since by [97], proposition 3.2, it is already finitely generated over $R(G)$ and $K_G^*(X)$ is a unital $R(G)$-algebra. By [5], theorem 10.13, $R(H)$ is complete in the $I(G)$-adic topology if $K_G^*(X)$ is. Then $R(H)$ would also be complete in the $I(H)$-adic topology, since by [97], corollary 3.9, it is the same as the $I(G)$-adic topology. We will show that this is not the case.

We have $R(G) \simeq \mathbf{Z}[x]/ < 1 - x^p >$ (the quotient of the polynomial ring in one variable over $\mathbf{Z}$ by the ideal generated by $1 - x^p$), and $I(H) = < 1 - x >$. Clearly $\bigcap_n I(H)^n = \{0\}$, whence $R(H)$ is Hausdorff. It is also metrizable. (Set $d(\eta,\lambda) = 2^{-n}$ where n is the largest integer such that $\eta - \lambda \in I(H)^n$.) One has $(1-x)^n \neq 0$ for any n, but $(1-x)^n \longrightarrow 0$ as $n \longrightarrow \infty$, so that the open set $R(H) - \{0\}$ is dense; therefore all sets $R(H) - \{\eta\}$ for $\eta \in R(H)$ are dense and open. We have

$$\emptyset = \bigcap_{\eta \in R(H)} R(H) - \{\eta\}.$$

Since $R(H)$ is countable, we have shown that the empty set is a countable intersection of dense open subsets. By the Baire Category Theorem, $R(H)$ is therefore not complete. So $K_G^*(X)$ cannot be complete.

(5)$\Longleftrightarrow$(6). $R(G)$ is a noetherian ring, $I(G)$ is a prime ideal in it, and $K_G^*(X)$ is a unital algebra over $R(G)$, since X is compact. The equivalence of condition (5) and (6) then follows immediately from the the following lemma, which for later use is stated in greater generality than needed here.

1.2.2 Lemma. Let R be a Noetherian ring, let I be a prime ideal in R, and let M be an R-module. Then the following are equivalent:

(1) $M_P = 0$ for all prime ideals $P \subset R$ such that $P \not\supset I$.

(2) For every $\eta \in M$ there is $n \in \mathbf{Z}$ such that $I^n \cdot \eta = 0$.

If M is either finitely generated or a unital algebra (not necessarily commutative) over R, then these are equivalent to:

(3) There is $n \in \mathbf{Z}$ such that $I^n \cdot M = 0$.

Proof. (1)$\Longrightarrow$(2): Let $\eta \in M$. Let $J = \{r \in I : r\eta = 0\}$ and let

$$rad(J) = \{a \in R : a^n \in J \text{ for some } n\}$$

be the radical of J. By [5], proposition 1.14, $rad(J)$ is the intersection of all prime ideals of R which contain J.

Let P be a prime ideal of R which does not contain I. Then by hypothesis, the image of η in M_P is zero, so there is $r \notin P$ such that $r\eta = 0$. Let s be any element of I not in P. Then $rs\eta = 0$ and $rs \in I$, that is, $rs \in J$. Also $rs \notin P$ because P is a prime ideal. So $J \not\subset P$. Thus, any prime ideal containing J also contains I, and hence $rad(J) \supset I$. Since $J \subset I$ and I is prime, we actually have $rad(J) = I$. Since R is noetherian, proposition 7.14 of [5] implies that there is n such that $I^n \subset J$, whence $I^n \cdot \eta = 0$.

(2)$\Longrightarrow$(1). Let $P \subset R$ be a prime ideal with $P \not\supset I$, and let $r \in I$, $r \notin P$. For every $\eta \in M$, there is n such that $I^n \cdot \eta = 0$, and in particular $r^n \eta = 0$. Since P is prime, $r^n \notin P$ for any n, and therefore the image of η in M_P is zero. Since this is true for all η, we obtain $M_P = 0$.

(3)$\Longrightarrow$(2) is obvious for any M.

(2)$\Longrightarrow$(3) for finitely generated modules: Let $\eta_1, \ldots, \eta_k$ generate M, and find n_i such that $I^{n_i} \cdot \eta_i = 0$. Then clearly $I^n M = 0$ with $n = \max_i(n_i)$.

(2)$\Longrightarrow$(3) for unital algebras: Choose n such that $I^n \cdot 1 = 0$, where 1 is the unit of M. Then for every $\eta \in M$, $I^n \cdot \eta = I^n \cdot 1 \cdot \eta = 0$, so that $I^n M = 0$.

This completes the proof of the lemma, and thus also of the theorem. Q.E.D.

1.2.3 Corollary (of theorem 1.1.1). Let G be a compact Lie group, and let X be a locally compact free G-space. Then the map $K^*(X/G) \longrightarrow K_G^*(X)$ is an isomorphism.

Proof. Let $(X_\alpha)_{\alpha \in I}$ be an increasing family, indexed by a directed set I, of open G-invariant subsets of X with compact closures $\bar{X}_\alpha$, such that $X = \bigcup_{\alpha \in I} X_\alpha$. Then the boundaries ∂X_α are also compact G-invariant subsets, and there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
K^{1-i}(\bar{X}_\alpha/G) & \longrightarrow & K^{1-i}(\partial X_\alpha/G) & \longrightarrow & K^i(X_\alpha/G) & \longrightarrow & K^i(\bar{X}_\alpha/G) & \longrightarrow & K^i(\partial X_\alpha/G) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_G^{1-i}(\bar{X}_\alpha) & \longrightarrow & K_G^{1-i}(\partial X_\alpha) & \longrightarrow & K_G^i(X_\alpha) & \longrightarrow & K_G^i(\bar{X}_\alpha) & \longrightarrow & K^i(\partial X_\alpha).
\end{array}$$

Since $\bar{X}_\alpha$ and ∂X_α are compact free G-spaces, the implication (1)$\Longrightarrow$(2) of theorem 1.1.1 implies that all the vertical arrows except the middle one are isomorphisms. By the Five Lemma, the middle vertical arrow is also an isomorphism. Now