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Edited by A. Dold and B. Eckmann

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Proceedings of the Second International Colloquium on Dynamical Systems, held in Mexico, July 1986

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Editors

Xavier Gomez-Mont
José A. Seade
Instituto de Matemáticas, Area de la Investigación Científica
Circuito Exterior, Ciudad Universitaria
México 04510, D.F., México

Alberto Verjovski
International Centre for Theoretical Physics
P.O. Box 586, 34100 Trieste, Italy

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Dedicated to Professors

Abdus Salam

Jacob Palis

and

Christopher Zeeman

in Recognition of the Influence that the International Centre for Theoretical Physics (ICTP), Trieste, Italy has on the Development of Mathematics in Mexico.

PREFACE

The Semester of Activities on Dynamical Systems was held at the Instituto de Matemáticas of the National University of Mexico from January to June 1986, and it concluded with the Second Colloquium of the Mexican Mathematical Society held in the same Institute and in the University of Guadalajara, in Chapala, Mexico, in July 1986. We would like to thank the participants for their mathematical interest and the following institutions for their financial support:

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The Editors

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First Integrals for Singular Holomorphic Foliations With Leaves of Bounded Volume

J. C. Alexander* and Alberto Verjovsky†

Department of Mathematics and Institute for Physical Science and Technology
University of Maryland
College Park, MD 20742 USA

Departamento de Matemáticas
Centro de Investigación del IPN
Apartado Postal 14-740
México 14, D. F., México

We consider the germ of k-dimensional holomorphic foliation in $\mathbb{C}^n$ with an isolated singularity at the origin. Under the assumption that the germs of the leaves have bounded k-volume, it is proved that all leaves are closed and that at least one separatrix exists. If the k-volume (or k-dimensional Hausdorff measure) of the separatrix set is also finite, the germ has a very regular structure. In particular, the leaf space is a complex analytic space. The problem is motivated by the study of singularities of complex differential equations. Illustrative examples and a partial converse are presented.

1. Introduction

The subjects of complex dynamics, and more generally, of holomorphic foliations, have characters different from their real counterparts, due to the rich structure of complex analysis. Many of the results of complex analytic geometry have important implications for holomorphic foliations. In this report we consider one such implication. Bishop [1] has shown that a bound on the volumes and Hausdorff measure of analytic sets has geometric consequences. We study the consequences for the structure of a holomorphic foliation in the neighborhood of an isolated singularity. The foliation has a very regular structure. It contains separatrices. Leaves which are not separatrices are closed. The leaf space has a complex analytic structure, so that the foliation has the maximal number of first integrals. In this report we develop such consequences of a bound on the volume of leaves.

A nonsingular foliation of a manifold is a decomposition of the manifold into disjoint immersed submanifolds, called *leaves*. Foliations with singularities correspond to integrable systems of forms. It is convenient to begin with the following [22, def. 3.1, ch. III, pp. 106–107]. Let $\Omega \subset \mathbb{C}^n$ be an open subset and let $0 < k < n$. An $(n-k)$*-dimensional holomorphic Frobenius structure* F on Ω is a collection of $n-k$ holomorphic one-forms $F = \{\omega_1, \dots, \omega_{n-k}\}$ on Ω such that for each $i = 1, \dots, n-k$, the integrability condition

$$d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_{n-k} = 0$$

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† Present address: International Centre for Theoretical Physics, Strada Costiera 11, Miramare, 34100 Trieste, Italy

is satisfied. For each $z \in \Omega$, let

$$K_z = \bigcap_{i=1}^{n-k} \ker \omega_i(z),$$

a subspace of the tangent space at z. The *singular locus* of F is the set

$$S(F) = \{z \in \Omega : \dim_{\mathbb{C}}(K_z) > k\}.$$

This is an analytic subset of Ω. The Frobenius system F is *regular* if $\dim_{\mathbb{C}}(S(F)) < k$. On the complement of $S(F)$ in Ω, the forms $\omega_i \in F$ are linearly independent and thus determine a nonsingular k-dimensional foliation $\mathcal{F}(F)$ of $\Omega - S(F)$.

More generally, a *holomorphic foliation of codimension* q with singularities in the complex manifold M is a nonsingular foliation of codimension q in $M - A$, where A is an analytic set of codimension bigger than 1. If A has codimension bigger than q, we say the foliation is *regular*. The forms that define the foliation in A may be taken to be those local 1-forms which are tangent to the foliation in $M - A$.

In particular, a foliation of codimension $n - 1$ in a manifold of dimension n may be given by the solutions of an ordinary complex differential equation

$$\frac{dz}{dT} = f(z), \quad T \in \mathbb{C}. \tag{1.1}$$

The orbits of (1.1) are the leaves and the stationary points constitute the singular set. However note that if $n > 2$, the resulting foliation is not in general an $(n - 1)$-dimensional holomorphic Frobenius structure.

A (holomorphic) *first integral* of a foliation defined on Ω is a (nontrivial) holomorphic function $\rho : \Omega \to \mathbb{C}$ which is constant on leaves. There are a number of adjectives ('strong', 'weak', 'formal') that can be put in front of the term, depending on the particular context, and a number of results concerning the existence and number of such integrals can be found in [7, 13, 16, 17, 18, 20, 21, 27, 28]. In the context of Frobenius structures, first integrals are related to the *integrability problem* [22]. A first integral is a function defined on the leaf space of the foliation, that is a map to a one-dimensional variety. If a foliation admits r first integrals, they form a map from the leaf space to an r dimensional variety. If the map does not factor through an $(r - 1)$-dimensional variety, the first integrals are *independent*. In this paper, we introduce a condition of a differential-geometric nature, essentially that the k-volumes of the leaves of a k-dimensional foliation are bounded, and under this condition, prove the existence of the maximal number $(n - k)$ of independent first integrals. Indeed we determine the structure of the leaf space of the germ of the foliation. Our results are somewhat analogous to those of Epstein [9] and Edwards-Millett-Sullivan [8].

We recall some terminology and results. A leaf of a non-singular foliation is (locally) an analytic variety if and only if it is (locally) closed [13]. A variety V may be the union of a finite number of irreducible components. The *dimension* of V is the maximum of the dimensions of its components. It is *purely k-dimensional* if all of its components are exactly k-dimensional. For k-dimensional V, let $\mathrm{Vol}_{2k}(V)$ denote the Euclidean $2k$-dimensional volume of V as a (possibly singular) submanifold of $\mathbb{C}^n$. Given any subset S of Ω, let $\mathcal{L}(S) = \mathcal{L}_\Omega(S)$, called the *saturation* of S in Ω, be the union of the leaves which intersect S. A subset of Ω is *saturated* if it is it own saturation. If F is a holomorphic foliation defined on $\Omega \subset \mathbb{C}^n$, nonsingular in $\Omega - A$, a *separatrix* of F is an analytic set $W \subset \Omega$ such that $A \cap W \neq \emptyset$ and $W - A$ is a leaf of $\mathcal{F}(F)$. Let $\Sigma(F)$ denote the union of all separatrices; $\Sigma(F)$ is called the *separatrix set* of F. An *orbifold* (or V-manifold) is the quotient of a finite group action on a complex manifold [6, 14, 24]. An orbifold is a normal space [6]. We also use Hausdorff measure for subsets of $\mathbb{C}^n$ and the Hausdorff metric on the set of closed subsets of $\mathbb{C}^n$, see e.g. [23]. We collect our results in an omnibus theorem.

Theorem. *Let $\mathcal{F}$ be a holomorphic foliation of codimension $n-k$ defined on a neighborhood $\mathcal{U}$ of the origin in $\mathbb{C}^n$, $0 < k < n$, nonsingular in $\mathcal{U} - \{0\}$. Suppose there exists a positive constant K such that for any leaf L of $\mathcal{F}$,*

$$\mathrm{Vol}_{2k}(L) \leq K. \tag{1.2}$$

Then every leaf is closed in $\mathcal{U} - \{0\}$ and is thus a k-dimensional variety. Those leaves which are not closed in $\mathcal{U}$ are precisely the separatrices. There exists at least one separatrix. Let $\Sigma = \Sigma(\mathcal{F})$ denote the union of the separatrices. If the $2k$-dimensional Hausdorff measure $\mathcal{H}_{2k}(\Sigma)$ is finite, then there exists a subneighborhood $\mathcal{V}$ of the origin such that in $\mathcal{V}$:

1. *Σ is a purely k-dimensional subvariety of $\mathcal{V}$ and in particular has a finite number of irreducible components;*
2. *if F_i is a sequence of closed leaves converging to any subset of Σ, it converges to all of Σ.*
3. *there are an $(n-k)$-dimensional singular space S, a point $p \in S$, and a holomorphic map $\pi: \mathcal{V} \to S$ such that $\pi^{-1}(p) = \Sigma$ and $\pi^{-1}(q)$ is a leaf of the foliation distinct from Σ, for $q \neq p$ in S.*

Thus we have $n-k$ first integrals in the map π. The proof occupies section 4.

The question of the existence of separatrices is very old. It was proposed by Briot-Bouquet [3] in 1856 for the case of holomorphic differential equations in $\mathbb{C}^2$ with an isolated singularity at the origin. The existence of a separatrix in this case was settled affirmatively in [5].

A partial converse of the theorem is valid. If there exists a map $\pi: \mathcal{U} \to V$, by Fubini's theorem the integral of the k volumes of the leaves (= fibers) is integrable over V. Several questions can be raised. Is the main theorem valid if the k-volumes are only integrable in some sense instead of uniformly bounded? Or does the existence of π ensure that the k-volumes are uniformly bounded? In particular, if all the leaves that are not separatrices are closed, are the volumes uniformly bounded? Also suppose $\mathcal{F} = \{\omega_1, \ldots, \omega_k\}$ is a regular holomorphic Frobenius system near the origin in $\mathbb{C}^n$, $0 < k < n$, and suppose that in some neighborhood of the origin,

$$\int_{\mathcal{U}} \log \|\omega_1(z) \wedge \omega_2(z) \wedge \cdots \wedge \omega_k(z)\| \, dz < \infty.$$

Does $\mathcal{F}$ have k independent holomorphic first integrals? The volume of the leaves is related to the integral.

The theorem implies that the foliation is transversally Riemannian off of the origin [22].

The authors would like to thank Xavier Gómez-Mont for helpful discussions and a careful reading of the paper.

2. Examples

We consider several examples, all differential equations, which illustrate some aspects of the theorem. These are derived from [4, 10]. Consider the complex differential system

$$\frac{dz}{dT} = Az, \qquad z \in \mathbb{C}^n, \quad A \in GL(n, \mathbb{C}). \tag{1.3}$$

For simplicity, suppose A is diagonal, with entries $\lambda_1, \ldots, \lambda_n$. The solution (leaf) through a point $(z_1, \ldots, z_n)$ is given by

$$\phi(z_1, \ldots, z_n, T) = (e^{\lambda_1 T}, \ldots, e^{\lambda_n T}). \tag{1.4}$$

1. If all the λ_i are equal, the leaves are all the punctured complex lines. Each leaf is a separatrix. Any continuous function constant on the leaves must be constant; there are no first integrals. Although the volumes of the leaves are bounded, the 2-dimensional Hausdorff measure of the separatrix set Σ is infinite.
2. Suppose A is hyperbolic and in the Poincaré domain (i.e., the convex hull of the eigenvalues of A does not contain the origin and the eigenvalues are independent over the reals). In this case, there is a nonzero λ_0 such that $\arg(\lambda - i/\lambda_0) < \pi/2$ for all $i = 1, \ldots, n$. For each $i = 1, \ldots, n$, let $T_N = -\lambda_0/N\lambda_i$ for $N = 1, \ldots, \infty$; we see that every nonsingular leaf contains at least one eigenspace in its closure. The separatrices are the eigenspaces. The closures of the other orbits are not analytic. It can be verified explicitly that the volumes of the leaves are not uniformly bounded near the origin.
3. Let $n = 3$ and suppose the convex hull of the eigenvalues contains the origin in its interior. The solution through a point (z_1, z_2, z_3) with all $z_i \neq 0$ is closed in $\mathbb{C}^n$. For suppose $a > 0$ and let $P(a) = \{(z_1, z_2, z_3) : |z_i| \leq a, i = 1, 2, 3\}$ be a polydisk, then $\{T \in \mathbb{C} : \phi(z_1, z_2, z_3, T) \in P(a)\}$ is a compact convex subset of $\mathbb{C}$. There are leaves $\phi(z_1, z_2, z_3, T)$ with some of the $z_i = 0$ which contain eigenspaces in their closures. The leaves are generically closed, but the volumes are not uniformly bounded.
4. Suppose $n = 2$. Consider the hyperbolic resonant case with $\lambda_1/\lambda_2 = -p/q$ for real positive integers p, q. Then the flow has the first integral $f: \mathbb{C}^2 \to \mathbb{C}$ given by $f(z_1, z_2) = z_1^q z_2^p$. It is easy to verify directly from the uniformization of the leaves given by the flow that the leaves have uniformly bounded volume. Note however that a transversal to the separatrix $\{z_2 = 0\}$ intersects each closed leaf p times, whereas a transversal to the separatrix $\{z_1 = 0\}$ intersects each closed leaf q times. The group Γ is cyclic of order q or p, depending on the separatrix. In this case, because the group is cyclic, the orbifold V is not singular, although the projection π is. It would be interesting to have an example with non-cyclic group.
5. Consider the elliptic resonant case for $n = 2$ with $\lambda_1/\lambda_2 = +p/q$ for real positive integers p, q (this case is related to example 1). Then there is a first integral on the complement of the origin, to wit $(z_1, z_2) \mapsto z_1^q z_2^{-p}$. However this integral does not extend across the origin. In this case all leaves are separatrices.

From these examples it is evident that having leaves with uniformly bounded volume is a highly non-generic situation. However having first integrals is also non-generic. We mention for example, the result of Mattei-Moussou [18] (which subsumes part of ours in the case of codimension one). Their result states that a codimension-one foliation admits a first integral if and only if the leaves are closed in $\mathcal{U} - \{0\}$ and if the set of leaves containing 0 in their closures in countable. There may be some kind of 'sliced' or 'fibered' version of our theorem: if there is some kind of r-codimensional 'slice' of $\mathbb{C}^n$ such that the leaves have finite $(k - r)$-volume, then are there $n - k - r$ first integrals? Making sense of the words is part of the question.

3. Bishop's results

We will need some results of Bishop relating k-volumes of subsets and analyticity. For details see [1, 25]. For convenience we collect them here. A sequence of subsets (in particular varieties) $\{V_i\}$, $i = 1, 2, \ldots$ in Ω has a (set) limit V_∞ if for each compact $C \subset \Omega$, the Hausdorff metric $d(V_i \cap C, V_\infty \cap C) \to 0$ as $i \to \infty$.

Bishop 1. *Let $\{V_i\}$ be a sequence of purely k-dimensional varieties in an open subset $\Omega \subset \mathbb{C}^n$ with* *uniformly bounded 2k-volumes; that is* $\mathrm{Vol}_{2k}(V_i) \le K$ *for all i. Suppose* $\lim_i V_i = V_\infty$. *Then* V_∞ *is also a purely k-dimensional variety in* Ω *and* $\mathrm{Vol}_{2k}(V_\infty) \le K$.

Bishop 2. *Let* V_1 *be a subvariety of an open set* $\Omega \subset \mathbb{C}^n$. *If V is a purely k-dimensional subvariety of* $\Omega - V_1$ *such that* $\overline{V} \cap V_1$ *has zero 2k-dimensional Hausdorff measure (*$\overline{V}$ *denotes closure in* Ω*), then* $\overline{V}$ *is a k-dimensional variety in* Ω.

Bishop 3. *Let* V_1 *be a subvariety of an open set* $\Omega \subset \mathbb{C}^n$. *If V is a purely k-dimensional subvariety of* $\Omega - V_1$ *with* $\mathrm{Vol}_{2k}(V) < \infty$, *then* $\overline{V}$ *is a purely k-dimensional variety in* Ω.

4. The proof

In this section we assume that $\mathcal{F}$ is a regular foliation of codimension $n - k$ defined in a neighborhood $\mathcal{U}$ of 0 in $\mathbb{C}^n$. Moreover we assume that (1.2) holds. For the first proposition, we do not need to assume that 0 is an isolated singularity. Let $S(\mathcal{F})$ be the singularity set in $\mathcal{U}$.

Proposition. *Under the above hypotheses, every leaf is closed in* $\mathcal{U} - S(\mathcal{F})$ *and hence is an analytic subvariety of* $\mathcal{U} - S(\mathcal{F})$. *For any* $z \in S(\mathcal{F})$, *there is at least one separatrix containing z in its closure. The separatrices are precisely the leaves* $L \subset \mathcal{U} - S(\mathcal{F})$ *such that the closure* $\overline{L}$ *of L in* $\mathcal{U}$ *intersect* $S(\mathcal{F})$.

Proof. Suppose there is a leaf L in $\mathcal{U} - S(\mathcal{F})$ which is not closed. Then there is a sequence $\{z_i\} \subset L$ which converges to $z \notin L$. Let $\mathcal{W}$ be a foliated chart of z in $\mathcal{U} - S(\mathcal{F})$. That is, $\mathcal{W}$ is holomorphically equivalent, say by f to a product $\mathcal{W}_1^k \times \mathcal{W}_2^{n-k}$, where $\mathcal{W}_i^r$ is open in $\mathbb{C}^r$ and the leaves of $\mathcal{F}|_{\mathcal{W}}$ are $f^{-1}(\mathcal{W}_1^k \times \{z_2\})$, called *plaques*. The k-volumes of subsets of $\mathcal{W}$ with the metric of $\mathbb{C}^n$ and with the metric of $\mathcal{W}_1^k \times \mathcal{W}_2^{n-k} \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$ are not the same. However because f is Lipschitz, each is bounded by some constant multiple of the other. In particular, the volumes of a sequence of sets is unbounded in one metric if and only if it is unbounded in the other. The set $L \cap \mathcal{W}$ consists of an infinite number of plaques converging to the plaque containing z. Hence the $2k$-volume of $L \cap \mathcal{W}$ is infinite, contradicting the assumption (1.2). Thus every leaf is closed in $\mathcal{U} - S(\mathcal{F})$. By the regularity of $\mathcal{F}$, the Hausdorff measure $\mathcal{H}_{2k}(S(\mathcal{F})) = 0$. Thus by Bishop 2, the closure $\overline{L}$ of L in $\mathcal{U}$ is a purely k-dimensional analytic subvariety of $\mathcal{U}$, so if $\overline{L} \cap S(\mathcal{F}) \neq \emptyset$, then L is a separatrix.

Let $z \in S(\mathcal{F})$. We show there is a separatrix containing z in its closure. Let $\{z_i\}$ be a sequence in $\mathcal{U} - S(\mathcal{F})$ which converges to z. Let $L(z_i)$ denote the leaf through z_i. Let $\mathcal{U} = \cup C_j$, $C_j \subset C_{j+1}$ be a description of $\mathcal{U}$ as an increasing sequence of compact sets containing all the z_i. Then for each j, $\overline{L}(z_i) \cap C_j$ is a sequence of compact subsets of C_j. The set of closed subsets of a compact set endowed with the Hausdorff metric is compact (Blaschke's selection lemma [2], see [15,§42.II, 23]). Hence there is a convergent subsequence of $\overline{L}(z_i) \cap C_j$. By Cantor's diagonal process, there is a convergent subsequence of $\overline{L}(z_i)$. Let $W(z)$ denote the limit. By Bishop 1, $W(z)$ is a purely k-dimensional variety containing z. Hence $W(z) - S(\mathcal{F})$ must be a finite union of leaves of $\mathcal{U} - S(\mathcal{F})$. At least one of them has to contain z in its closure. The result is proved.

Now suppose in addition that $S(\mathcal{F}) = \{0\}$ and that $\mathcal{H}_{2k}(\Sigma(\mathcal{F})) < \infty$, where Σ is the separatrix set. By Bishop 3, $\Sigma(\mathcal{F})$ is a purely k-dimensional variety and hence is the finite union of irreducible components $\Sigma_1 \cup \cdots \cup \Sigma_r$. Each Σ_i is an irreducible variety which is possibly singular only at the origin.

We recall the cone theorem of Milnor [19, thm. 2.10], which is also valid for analytic varieties [11]: Let $\Sigma^l \subset \mathbb{C}^n$ be an l-dimensional variety which is singular (possibly) only at the origin. Then there exists $\epsilon > 0$ such that every sphere $S_\eta^{2n-1} = \{z \in \mathbb{C}^n : |z| = \eta\}$ with $\eta \le \epsilon$ intersects Σ transversally in a real nonsingular analytic variety $\mathrm{Lk}_\eta(\Sigma)$, called the *link* of Σ^l. Furthermore, if D^{2n}_ϵ denotes the closed disk of radius ϵ, the pair $(D^{2n}_\epsilon, \Sigma^l \cap D^{2n}_\epsilon)$ is homeomorphic to the pair $(D^{2n}_\epsilon, \mathrm{Cone}\,\mathrm{Lk}_\epsilon(\Sigma))$. Actually more is proved. The homeomorphism is a real analytic equivalence on $D^{2n}_\epsilon - \{0\}$, so that for any $0 < \eta < \epsilon$, the intersection of Σ^l with the set $S^{2n}_{\eta,\epsilon} = \{z \in \mathbb{C}^n : \eta \leq |z| \leq \epsilon\}$ is real analytically the product $\mathrm{Lk}_\epsilon(\Sigma) \times [\eta, \epsilon]$.

Let ϵ be so small that $D^{2n}_\epsilon \subset \mathcal{U}$ and so that the conclusions of the cone theorem are valid for this ϵ for all the components Σ_i of the separatrix, $i = 1, \ldots, r$. Let $M_i = \mathrm{Lk}_\epsilon(\Sigma_i)$. Consider $\mathcal{F}|_{S^{2n-1}_\epsilon}$. It defines a foliation of S^{2n-1}_ϵ which is possibly singular. The leaves are the components of the intersections of the leaves of $\mathcal{F}$ with S^{2n-1}_ϵ. To distinguish them from the leaves in $\mathcal{U}$, we denote the leaf in S^{2n-1}_ϵ containing $x \in S^{2n-1}_\epsilon$ by $L_\epsilon(x)$. By transversality, the foliation of S^{2n-1}_ϵ is nonsingular in a closed tubular neighborhood $T_{\delta_i}(M_i)$ of radius δ_i of each M_i, $i = 1, \ldots, r$. In $T_{\delta_i}(M_i)$, each leaf is an irreducible real analytic variety and they are closed.

Lemma 1. *For each $i = 1, \ldots, r$, there exists $\delta_i > 0$ and $c_i > 0$ such that $\mathrm{Vol}_{2k-1}(L_\epsilon) < c_i$ for all leaves L_ϵ which intersect $T_{\delta_i}(M_i)$.*

Proof. Let $x_j \in T_{\delta_i}(M_i)$, $j = 1, 2, \ldots$ be a sequence converging to $x \in M_i$ such that $\mathrm{Vol}_{2k-1}(L_\epsilon(x_j)) \to \infty$. We claim eventually all the $L_\epsilon(x_j) \subset T_{\delta_i}(M_i)$. If not, choose a subsequence such that $L_\epsilon(x_i) \cap \partial T_{\delta_i}(M_i) \neq \emptyset$ for all j. By Blaschke's selection lemma, there is a further subsequence which converges in the Hausdorff-metric topology, say to $M \subset S^{2n-1}_\epsilon$. Cover the compact $T_{\delta_i}(M_i)$ with a finite number of foliated charts in $\mathbb{C}^n$. Denote the union of these by Y and let $L_Y(x_j) = \mathcal{L}_Y(\{x_j\})$ be the leaf in Y containing x_j. There is a further subsequence such that the $L_Y(x_j)$ converge in Y. By Bishop 1, the limit of the $L_Y(x_j)$ is a purely k-dimensional complex analytic variety which is thus a finite union of leaves in Y. By transversality $M \cap T_{\delta_i}(M_i)$ is a finite union of closed nonsingular leaves in $T_{\delta_i}(M_i)$. On the other hand, M is a connected subset which contains both $x \in M_i$ and some point of $\partial T_{\delta_i}(M_i)$ (in a compact space, the limit of closed connected subsets is connected). However the previous two sentences state incompatible facts. Thus eventually the $L_\epsilon(x_j) \subset T_{\delta_i}(M_i)$. Consider again the covering of $T_{\delta_i}(M_i)$ by a finite number of foliated charts. The intersections of these charts with S^{2n-1}_ϵ are foliated charts of $T_{\delta_i}(M_i)$. The volumes (respectively $2k$-dimensional and $(2k-1)$-dimensional) of the plaques are bounded above and below. Thus since $\mathrm{Vol}_{2k-1}(L_\epsilon(x_j)) \to \infty$, there exists some chart that the number of intersections of the $L_\epsilon(x_j)$, $j = 1, \ldots \infty$, with the chart is unbounded. Thus the $L(x_j)$ have unbounded $2k$-volumes. This contradicts the assumption (1.2). The proof is complete.

This lemma states that the phenomenon of [26] cannot occur in the present context. On the contrary, the structure of the foliation on S^{2n-1}_ϵ is regular. In particular, the results of [8, 9] are valid, and we obtain the following corollary.

Corollary. *Each M_i, $i = 1, \ldots, r$, has an arbitrarily small open tubular neighborhood $\tau_{\delta_i}(M_i)$ in S^{2n-1}_ϵ such that $\tau_{\delta_i}(M_i)$ and its closure $\overline{\tau}_{\delta_i}(M_i)$ are saturated and in $\tau_{\delta_i}(M_i)$ all holonomy groups are finite.*

(The subscript δ_i is not necessarily a distance, but is only an index for the neighborhood.) In particular the holonomy group of M_i in $\tau_{\delta_i}(M_i)$ is finite. By Cartan's theorem [6],, we may find coordinates of a transversal to M_i in $\tau_{\delta_i}(M_i)$ such that the holonomy group Γ_i is a subgroup of $U(n-k)$, and by a result of Haefliger thesis (see [22]), the foliation in $\tau_{\delta_i}(M_i)$ is obtained locally by suspending this representation. Note that those leaves corresponding to fixed points of the holonomy group have nontrivial holonomy, so there is an open dense set of leaves that have trivial holonomy. The leaf space of the foliation in $\tau_{\delta_i}(M_i)$ is the germ of the complex analytic space $(\mathbb{C}^{n-k}/\Gamma, 0) = S_i$ [6]. Fixing the model in $\tau_{\delta_i}(M_i)$ of the foliation given by the suspension of Γ_i, the fact that the foliation in $\mathcal{U}$ has leaves with finite volume implies, by an argument similar to the one of Lemma 1, that the number of leaves of $\tau_{\delta_i}(M_i)$ which belong to the same leaf in $\mathcal{U}$ is bounded by some number N.

Given any neighborhood $\mathcal{U}$ of the origin, we construct a subneighborhood. Choose ϵ so that the $D_\epsilon^{2n} \subset \mathcal{U}$ and so that all the components Σ_i of Σ are the cones of their links M_i in D_ϵ^{2n}.

Lemma 2. *Suppose $\{z_j\}$, $j = 1, 2, \ldots$, is a sequence of points in D_ϵ^{2n} converging to $z \in \Sigma$. Let $L(z_j)$ be the leaf containing z_j in D_ϵ^{2n}, and let L be the limit of any subsequence of the $L(z_j)$. Then $L \subset \Sigma$.*

Proof. If not there is $y \in L - \Sigma$. Let $y_j \in L(z_j)$ converge to y. Since each $L(z_j)$ is connected, L is connected (limit of connected closed sets is connected), and contains both y and the origin. Consider $D_{\epsilon'}^{2n}$ for ϵ' slightly larger than ϵ (close enough to ϵ that $D_{\epsilon'}^{2n} \subset \mathcal{U}$ and the cone structure for the separatrix Σ' in $D_{\epsilon'}^{2n}$ still holds). Let $L'(z_j)$ and L denote the leaves and the limit (possibly with respect to a subsequence), respectively, in the interior of $D_{\epsilon'}^{2n}$. By Bishop 1, L' is a purely k-dimensional variety and $L \subset L'$. Thus y is connected to the origin in L'. Thus there is an irreducible component $L'_0 \subset L' - \Sigma'$. Note that $0 \notin L'_0$. Thus L'_0 intersects Σ' somewhere in $D_{\epsilon'}^{2n}$. However this is impossible since the foliation is nonsingular off of the origin. The lemma is proved.

For each $i = 1, \ldots, r$, consider the saturation $\mathcal{L}_{D_\epsilon^{2n}}(\tau_{\delta_i}(M_i))$. A leaf $L \in \mathcal{L}_{D_\epsilon^{2n}}(\tau_{\delta_i}(M_i))$ can intersect S_ϵ^{2n} at points not in $\tau_{\delta_i}(M_i)$. By Lemma 2, δ_i can be made small enough that all $L \in \mathcal{L}_{D_\epsilon^{2n}}(\tau_{\delta_i}(M_i))$ intersect S_ϵ^{2n} transversally. Suppose each δ_i has been so chosen. Let $\mathcal{V}$ be the the interior of the union of the $\mathcal{L}_{D_\epsilon^{2n}}(\tau_{\delta_i}(M_i))$, $i = 1, \ldots, r$. Relabel $\Sigma \cap \mathcal{V}$ to Σ.

Lemma 3. *$\mathcal{V}$ is a connected neighborhood of Σ.*

Proof. Lemma 2 implies $\mathcal{V}$ is a neighborhood (consider a sequence $\{z_j\}$ converging to the origin). Consider the connected component of $\mathcal{V}$ containing the origin. This component contains all of Σ. By transversality, the closure of the component contains all of $\overline{\Sigma}$, hence each $\tau_{\delta_i}(M_i)$, hence each $\mathcal{L}_{D_\epsilon^{2n}}(\tau_{\delta_i}(M_i))$. By transversality again, the component contains all of $\mathcal{V}$.

Lemma 4. *Suppose $\tau_{\delta_j}(M_j) \cap \mathcal{L}_{D_\epsilon^{2n}}(\tau_{\delta_i}(M_i)) \neq \emptyset$ for all sufficiently small $\tau_{\delta_i}(M_i)$. Then $\tau_{\delta_j}(M_j) \cap \mathcal{L}_{D_\epsilon^{2n}}(\tau_{\delta_i}(M_i))$ contains a neighborhood of M_j.*

Proof. Consider the relation $R \subset \overline{\tau}_{\delta_i}(M_i) \times \overline{\tau}_{\delta_j}(M_j)$. Namely $(p, q) \in R$ if either p and q are on the same leaf in D_ϵ^{2n} or if $p \in M_i$ and $q \in M_j$. We claim this is a closed relation. For suppose $\{(p_i, q_i)\}$, $i = 1, 2, \ldots$, is a sequence of points in R converging to (p, q). We can suppose the sequence of leaves $\{L(p_i) = L(q_i)\}$ converges in D_ϵ^{2n}. If $p \in M_i$ or $q \in M_j$, then $(p, q) \in R$ by Lemma 2. Otherwise, extending to $D_{\epsilon'}^{2n}$ and using, as above, connectivity, Bishop 1 and the nonsingularity off of Σ, we see that p and q lie on the same leaf and hence $(p, q) \in R$. We have shown that R is closed and hence compact in $\overline{\tau}_{\delta_i}(M_i) \times \overline{\tau}_{\delta_j}(M_j)$. By projection to $\overline{\tau}_{\delta_i}(M_i)$, we see the sets

$$R_i = \{p \in \overline{\tau}_{\delta_i}(M_i) : (p, q) \in R \text{ for some } q\} \subset \overline{\tau}_{\delta_i}(M_i),$$
$$R_j = \{q \in \overline{\tau}_{\delta_i}(M_i) : (p, q) \in R \text{ for some } p\} \subset \overline{\tau}_{\delta_j}(M_j),$$

are closed. We study the points $(p, q) \in R$ with $q \in \partial R_j$ (where boundaries are with respect to S_ϵ^{2n}). One possibility for such a (p, q) is that $q \in \partial \overline{\tau}_{\delta_j}(M_j)$. A second possibility is that $p \in \partial \overline{\tau}_{\delta_i}(M_i)$. A third possibility that is $p \in \tau_{\delta_i}(M_i) - M_i$ *and* $q \in \tau_{\delta_j}(M_j) - M_j$. We claim the third is in fact not possible. For suppose there is such a (p, q) on a common leaf L. Then L has a saturated tubular neighborhood Y with finite holonomy [8, 9]. Each leaf in Y intersects both $\tau_{\delta_i}(M_i)$ and $\tau_{\delta_j}(M_j)$. Thus R_i contains a neighborhood of p and R_j contains a neighborhood of q. That is, $p \notin \partial R_i$ and $q \notin \partial R_j$ and the claim is proved. Now suppose the lemma is false. Then there exist points $q \in \partial R_j - M_j$

arbitrarily close to M_j. However, by the claim just proved, the leaf $L(q)$ for any such q must satisfy $L(q) \cap \tau_{\delta_i}(M_i) \subset \partial\tau_{\delta_i}(M_i)$. However this contradicts Lemma 2. The result is proved.

Let $\mathcal{L}^{\circ}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i))$ denote the intersection of $\mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i))$ with the interior of D^{2n}_ϵ.

Lemma 5. *For any $i = 1, 2, \ldots, r$, $\mathcal{L}^{\circ}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i))$ is a connected neighborhood of Σ.*

Proof. Note that $\cup_{i=1}^{r}\mathcal{L}^{\circ}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i)) = \mathcal{V}$. We claim that if $\mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i)) \cap \mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_j}(M_j)) \neq \emptyset$ and $\mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i)) \cap \mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_k}(M_k)) \neq \emptyset$, then also $\mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_j}(M_j)) \cap \mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_k}(M_k)) \neq \emptyset$. For $\mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i)) \cap \mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_j}(M_j)) \neq \emptyset$ is equivalent to $\mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i)) \cap \mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_j}(M_j)) \neq \emptyset$, which is equivalent to $\mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i)) \cap \tau_{\delta_j}(M_j) \neq \emptyset$. By Lemma 4, $\mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i)) \cap \tau_{\delta_j}(M_j)$ contains a neighborhood of M_j, as does $\mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_k}(M_k)) \cap \tau_{\delta_j}(M_j)$. Hence $\mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i)) \cap \mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_k}(M_k)) \neq \emptyset$ and thus $\mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i)) \cap \mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_k}(M_k)) \neq \emptyset$, as claimed. Accordingly we may define an equivalence relation among the indices $i = 1, 2, \ldots, r$; namely, two indices i and j are equivalent if $\mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i)) \cap \mathcal{L}_{D^{2n}_\epsilon}(\tau_{\delta_j}(M_j)) \neq \emptyset$. If there is more than one equivalence class, $\mathcal{V}$ is decomposed into the disjoint union of two open subsets. Since $\mathcal{V}$ is connected, this cannot be and there is only one equivalence class. Thus any $\mathcal{L}^{\circ}_{D^{2n}_\epsilon}(\tau_{\delta_i}(M_i))$ is an open set containing Σ and the lemma is proved.

At this point we can improve Lemma 2.

Corollary. *Suppose $\{z_j\}$, $j = 1, 2, \ldots$, is a sequence of points in $\mathcal{V}$ converging to $z \in \Sigma$. Then any convergent subsequence of the leaves $L(z_j)$ converges to all of Σ.*

Proof. If not, there is more than one equivalence class in the proof of Lemma 5.

Lemma 6. *For $i = 1, 2, \ldots$, let z_i and w_i be points in $\mathcal{V}$ on the same leaf. Suppose the sequences $\{z_i\}$ and $\{w_i\}$ converge to z and w in $\mathcal{V}$ respectively, and $z \notin \Sigma$. Then w and z lie on the same leaf.*

Proof. A subsequence of the leaves $L(z_i)$ converges to an analytic space L of dimension k by Bishop 1. Since each $\overline{L}(z_i)$ intersects S^{2n}_ϵ transversally, the limit of the $\overline{L}(z_i)$ is $\overline{L}$. As a limit of closed connected subsets, $\overline{L}$ is connected, and since the foliation is nonsingular off of 0, $\overline{L}$ is a leaf in D^{2n}_ϵ. By transversality, L is a leaf in $\mathcal{V}$. However $w \in L$, and the lemma is proved.

Recall that S_i is the leaf space of $\tau_{\delta_i}(M_i)$. We introduce an equivalence relation $\sim$ in each S_i; namely, $p \sim q$ if the leaves in $\tau_{\delta_i}(M_i)$ represented by p and q are contained in the same leaf in $\mathcal{V}$. By Lemma 6, this equivalence relation is closed and Hausdorff. Since it is holomorphic, the orbit space $T_i = S_i/\sim$ has the structure of a complex analytic space [12]. Recall that the germ of the leaf space $S_i = (\mathbb{C}^{n-k}/\Gamma, 0)$ as germs of analytic varieties. Thus the germ of T_i is a quotient of $(\mathbb{C}^{n-k}/\Gamma, 0)$. Denote these germs by $g(S_i)$ and $g(T_i)$. There is a natural analytic map $\tilde{G}_{ji}: g(S_i) \to g(T_j)$; namely a class of a leaf L in S_i is mapped to the class of L in T_j. Lemma 5 implies $\tilde{G}_{ji}$ is defined on the germ. Moreover $\tilde{G}_{ji}$ factors to a natural analytic map $G_{ji}: g(T_i) \to g(T_j)$. Clearly G_{ij} is the inverse of G_{ji}. Thus all the $g(T_i)$ are naturally isomorphic. Finally let $g(\Sigma)$ be the germ of the leaf space of a neighborhood of Σ in $\mathcal{V}$ and hence in the interior of D^{2n}_ϵ, with the added equivalence that all of Σ is identified to a point. From Lemma 5, we see that $g(\Sigma)$ is naturally isomorphic to any $g(T_i)$.

Thus we have detailed the structure of $g(\Sigma)$ and also proved our theorem.

5. A final remark

Let $D_i = D_i^{2(n-k)}$ be a disk transversal to the foliation in $\tau_{\delta_i}(M_i)$, with center $0_i = D_i \cap M_i$. The projection maps $D_i \to S_i$ are the quotients of the holonomy group Γ_i of M_i (which is the same as the holonomy of Σ_i, since Σ_i is a cone over M_i). The projection maps $D_i \to S_i \to T_i$ are surjective finite holomorphic mappings. Thus in the complement of nowhere dense closed analytic subsets, they are coverings. Assume that they are Galois coverings, and let $\mathcal{G}$ be the group of deck transformations. The elements of $\mathcal{G}$ are bounded holomorphic functions on the complement of a nowhere dense analytic subset, so by Riemann's extension theorem, they extend to biholomorphisms of the D_i. These extended elements preserve 0_i, but only the elements of $\Gamma_i \to \mathcal{G}$ correspond to the holonomy of M_i. For example, in example 2.4, the separatrices Σ_1 and Σ_2 are the axes with $\Gamma_1 = Z/pZ$, $\Gamma_2 = Z/qZ$, and $\mathcal{G} = Z/pqZ$. It would be interesting to know how the foliations in $\tau_{\delta_i}(M_i)$ amalgamate to form the foliation in $\mathcal{V}$.

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