FINITE ELEMENTS

AN INTRODUCTION

VOLUME I

Eric B. Becker
Graham F. Carey
J. Tinsley Oden

FINITE ELEMENTS
An Introduction

VOLUME I

ERIC B. BECKER, GRAHAM F. CAREY, and *J. TINSLEY ODEN*

Texas Institute for Computational Mechanics
The University of Texas at Austin

PRENTICE-HALL, INC., *Englewood Cliffs, New Jersey 07632*

Library of Congress Cataloging in Publication Data

Oden, John Tinsley
Finite elements.

Bibliography: p.
Includes index.
CONTENTS: v. 1. An introduction.
1. Finite element method. I. Carey, Graham F., joint author. II. Becker, Eric B., joint author. III. Title.
TA347.F5033 515.3'53 80-25153
ISBN 0-13-317057-8 (v. 1)

Editorial production supervision
and interior design by: James M. Chege

Cover design by: Edsal Enterprises

Manufacturing buyer: Joyce Levatino

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

PRENTICE-HALL INTERNATIONAL, INC., *London*
PRENTICE-HALL OF AUSTRALIA PTY. LIMITED, *Sydney*
PRENTICE-HALL OF CANADA, LTD., *Toronto*
PRENTICE-HALL OF INDIA PRIVATE LIMITED, *New Delhi*
PRENTICE-HALL OF JAPAN, INC., *Tokyo*
PRENTICE-HALL OF SOUTHEAST ASIA PTE. LTD., *Singapore*
WHITEHALL BOOKS LIMITED, *Wellington, New Zealand*

To our children:

Allison, David, Marjorie, *and* ***Elizabeth***

Varis *and* ***Tija***

Walker *and* ***Lee***

PREFACE

Our purpose in writing this book is to provide the undergraduate student of engineering and science with a concise introduction to finite element methods —one that will give a reader, equipped with little more than calculus, some matrix algebra, and ordinary differential equations, a clear idea of what the finite element method is, how it works, why it makes sense, and how to use it to solve problems of interest to him. We imposed on ourselves three constraints that we felt were of fundamental importance in designing a text of this type.

First, the treatment should not be burdened with technical details that are best appreciated by a more experienced reader. For instance, we feel that discussions of the many variants of finite element methods, detailed aspects of computational schemes for implementing these methods, and numerous applications to problem areas in which the student may have little or no interest are not appropriate in a first course on the subject at this level. Here, we present the method in a form in which the truly salient features can be exposed and appreciated. We choose to relegate those other special topics to later, more advanced volumes.

Second, we did not want to produce either a cookbook or a handbook on finite elements. Although we do give ample coverage of the operational side of finite elements, we also seek to clarify and explain the basic ideas on which these methods are founded. Without these, the student has little foundation on which to build a deeper understanding of either these concepts or their generalizations and, equally important, cannot apply the methods intelligently to difficult problems.

Finally, the book is not aimed at a specific and narrow area of application. The finite element methods are, after all, methods for solving boundary-value problems. Why should a student of, say, heat transfer or fluid mechanics be forced to master structural mechanics in order to learn something about finite elements? We are particularly sensitive to this point because the first course on this subject that we teach is populated by students with such diverse backgrounds and interests as geology, chemical engineering, mathematics, physics, civil engineering, nuclear sciences, aerospace engineering, petroleum engineering, and computer science.

We have each been working on finite element methods for nearly two decades, and this book has evolved as a result of our collective experience in teaching and studying finite elements during this period. Our experience of several years in teaching finite element methods has shown that problem solving and writing and use of simple finite element computer programs is the surest path toward understanding the method. The many exercises and programming assignments should occupy a significant part of the students' time during a semester's study from this book. We strongly believe that this time will be spent to good advantage.

The exercises vary considerably in effort required and in significance. Some of the exercises reinforce, through specific examples, ideas set forth in the text. Others extend the textual material and, in some cases, introduce concepts that, although fundamental in nature, are not necessary to an introductory treatment.

We thank our colleagues and students who have contributed to our understanding and presentation of this subject. We are particularly grateful to David Hibbitt, Linda Hayes, and Gilbert Strang who read the entire manuscript and made many helpful suggestions. We also express our appreciation to B. Palmer for typing the first draft of the manuscript and to N. Webster for assisting with revisions.

E. B. Becker
G. F. Carey
Austin, Texas J. T. Oden

THE TEXAS FINITE ELEMENT SERIES

VOLUME I

FINITE ELEMENTS: An Introduction

VOLUME II

FINITE ELEMENTS: A Second Course

VOLUME III

FINITE ELEMENTS: Computational Aspects

VOLUME IV

FINITE ELEMENTS: Mathematical Aspects

VOLUME V

FINITE ELEMENTS: Special Problems in Solid Mechanics

VOLUME VI

FINITE ELEMENTS: Fluid Mechanics

CONTENTS

1

A MODEL PROBLEM

1.1 ORIENTATION

The finite element method* is a general technique for constructing approximate solutions to boundary-value problems. The method involves dividing the domain of the solution into a finite number of simple subdomains, the finite elements, and using variational concepts to construct an approximation of the solution over the collection of finite elements. Because of the generality and richness of the ideas underlying the method, it has been used with remarkable success in solving a wide range of problems in virtually all areas of engineering and mathematical physics.

Our aim in this chapter is to give a brief introduction to several fundamental ideas which form the basis of the method. For this purpose, we confine our attention to the simplest, most transparent example: a one-dimensional, "two-point" boundary-value problem characterized by a simple linear ordinary differential equation of second order, together with a pair of boundary conditions. We shall refer to this example as our "model problem." Although the model problem is neither difficult nor of much practical interest, both its mathematical structure and our approach in formulating its finite element approximation are essentially the same as in more complex problems of

* Throughout this volume, we refer to "the finite element method," as if there were only one. There are, in fact, a variety of methods that employ an element-by-element representation of the approximate solution. Several of these are discussed in Volume II of this series.

greater significance. At many places in this chapter, we pass lightly over points of some practical and theoretical complexity, postponing until later a more thorough treatment.

1.2 THE STATEMENT OF THE MODEL PROBLEM

We begin by considering the problem of finding a function $u = u(x)$, $0 \leq x \leq 1$, which satisfies the following differential equation and boundary conditions:

$$\left.\begin{aligned} -u'' + u = x, \qquad & 0 < x < 1 \\ u(0) = 0, \qquad & u(1) = 0 \end{aligned}\right\} \tag{1.2.1}$$

Here the primes denote differentiation with respect to x ($u'' = d^2u/dx^2$). A problem such as this might arise in the study of the deflection of a string on an elastic foundation or of the temperature distribution in a rod.

The *data* of the problem consist of all the information given in advance: the domain of the solution (in this case, the domain is simply the unit interval $0 \leq x \leq 1$), the "nonhomogeneous part" of the differential equation (represented by the given function $f(x) = x$ on the right-hand side), the coefficients of various derivatives of u (in this case these are the constants -1 and $+1$), and the boundary values we demand the solution attain (in this case, zero at $x = 0$ and at $x = 1$).

The data in our model problem are "smooth"; for example, the right-hand side $f(x) = x$ and the coefficients are differentiable infinitely many times. As a consequence of this smoothness, there exists a unique function u which satisfies the differential equation at every point in the domain as well as the boundary conditions. In this particular example, it is a rather simple task to determine the exact solution to (1.2.1), $u(x) = x - (\sinh x/\sinh 1)$. However, in most technical applications, one or both of these happy features of the problem are missing—either there is *no solution* to the classical statement of the problem because some of the data are not smooth, or if a smooth solution exists, it cannot be found in closed form due to the complexity of the domain, coefficients, and boundary conditions.

As an example of the first kind of difficulty, suppose that instead of $f(x) = x$ being given as part of the data (the right-hand side of (1.2.1)), we have the problem

$$-u'' + u = \delta(x - \tfrac{1}{2}), \quad 0 < x < 1; \qquad u(0) = 0 = u(1) \tag{1.2.2}$$

where $\delta(x - \frac{1}{2})$ is the *Dirac delta*: the unit "impulse" or "point source" concentrated at $x = \frac{1}{2}$. The fact is that $\delta(x - \frac{1}{2})$ is not even a function but is rather a symbolic way of describing operations on smooth functions defined by*

$$\delta(x - \tfrac{1}{2})\phi(x) = \phi(\tfrac{1}{2})$$

for any smooth function ϕ satisfying the boundary conditions. We can convince ourselves that if any function u is to satisfy (1.2.2), then it must have a discontinuity in its first derivative u' at $x = \frac{1}{2}$; its second derivative u'' does not exist (in the traditional sense) at $x = \frac{1}{2}$ (see Exercises 1.2.3 and 1.2.4).

Something appears to be amiss! How can a function u satisfy (1.2.2) everywhere in the interval $0 < x < 1$ when its second derivative cannot exist at $x = \frac{1}{2}$ because of the very irregular data given in the problem?

The difficulty is that our requirement that a solution u to (1.2.2) satisfy the differential equation *at every point* x, $0 < x < 1$, is too strong. To overcome this difficulty, we shall reformulate the boundary-value problem in a way that will admit *weaker* conditions on the solution and its derivatives. Such reformulations are called *weak* or *variational* formulations of the problem and are designed to accommodate irregular data and irregular solutions, such as those in problem (1.2.2), as well as very smooth solutions, such as that of our model problem (1.2.1).

Whenever a smooth "classical" solution to a problem exists, it is also the solution of the weak problem. Thus, we lose nothing by reformulating a problem in a weaker way and we gain the significant advantage of being able to consider problems with quite irregular solutions. More important, weak or variational boundary-value problems are precisely the formulations we use to construct finite element approximations of the solutions. We describe such formulations of our model problem in the next section.

Examples of problems for which exact solutions cannot be found explicitly (even though they are known to exist) are found commonly in boundary-value problems in two or three dimensions. It is in the solution of such problems that the true power of the finite element method has made itself felt. The treatment of two-dimensional boundary-value problems begins in Chapter 4.

EXERCISES

1.2.1 Give an example of a physical problem for which the model problem is the mathematical statement.

* The operation $\delta(x - \frac{1}{2})\phi$ is sometimes written $\int_0^1 \delta(x - \frac{1}{2})\phi(x)\,dx = \phi(\frac{1}{2})$ for all infinitely differentiable functions satisfying the boundary conditions $\phi(0) = 0 = \phi(1)$. But even this is incorrect or, at best, only symbolic, because there exists no integrable function that can produce this action on a given smooth function ϕ!

1.2.2 Show that $u(x) = x - \sinh x/\sinh 1$ is the solution of the model problem.

1.2.3 Consider the boundary-value problem

$$-u''(x) = \delta(x - \tfrac{1}{2}), \qquad 0 < x < 1$$
$$u(0) = 0, \qquad u(1) = 0$$

where $\delta(x - \frac{1}{2})$ is the Dirac delta corresponding to a point source at $x = \frac{1}{2}$. Construct the exact solution u of this problem and sketch u and u' as functions of x. What does the graph of u'' look like? Does the classical statement of this problem given above make sense at $x = \frac{1}{2}$? Why?

1.2.4 Construct the solution u of the boundary-value problem (1.2.2) and sketch u and u' as functions of x. Comment on u'' and the meaning of the classical statement of this boundary-value problem.

1.3 VARIATIONAL STATEMENT OF THE PROBLEM

One weak statement of the model problem (1.2.1) is given as follows: find the function u such that the differential equation, together with the boundary conditions, are satisfied in the sense of weighted averages. By the satisfaction of all "weighted averages" of the differential equation, we mean that we require that

$$\int_0^1 (-u'' + u)v\,dx = \int_0^1 xv\,dx \tag{1.3.1}$$

for all members v of a suitable class of functions. In (1.3.1) the *weight function*, or *test function*, v, is any function of x that is sufficiently well behaved that the integrals make sense.*

In order to describe this weak statement of the problem more concisely, we introduce the idea of the set of all functions that are smooth enough to be considered as test functions. We will denote the set of such functions, which have zero values at $x = 0$ and $x = 1$, by the symbol H. To indicate that a function v is a member of the set H, we use the notation "$v \in H$," which is read "v belongs to H." The variational statement (1.3.1) of our prob-

* It is easy to find functions that are not smooth enough to serve as test functions. For example, if $u(x) = x - \sinh x$ and $v(x) = x^{-3}$, then neither $\int_0^1 (-u'' + u)v\,dx$ nor $\int_0^1 xv\,dx$ have finite values and (1.3.1) does not make sense. There is, however, a multitude of functions which are perfectly acceptable as test functions. The exact specification of such functions is central to the theory of the finite element method and will be discussed in detail later.

lem now assumes the more compact form: find u such that

$$\left.\begin{aligned} \int_0^1 (-u'' + u - x)v\,dx = 0 \qquad \text{for all } v \in H \\ u(0) = 0 \\ u(1) = 0 \end{aligned}\right\} \tag{1.3.2}$$

Upon reflection, it is clear that, if (1.3.2) is true, there can be no portion of finite length, however small, of the interval $0 < x < 1$ within which the differential equation (1.2.1) fails to be satisfied in an average sense. To see this, we need only hypothesize the existence of such a region and show that, as a consequence, (1.3.2) would not be satisfied. Consider the *residual*, or error, in the differential equation, defined by the function $r(x) = -u'' + u - x$. Suppose that $r(x)$ is different from zero in some small region, such as that shown in Fig. 1.1a. Corresponding to this particular $r(x)$, we can choose $v(x)$ as shown in Fig. 1.1b.* Noting that the integrand in (1.3.2) is positive in the interval $a < x < b$ and zero elsewhere, we see that the integral in (1.3.2) cannot vanish (i.e., (1.3.2) is not satisfied), so that u cannot be a solution of problem (1.2.1). Through various choices of v we can "test" the differential equation in every portion of the region of interest, so (1.3.2) does indeed require that (1.2.1) be true, on the average, over every subregion.

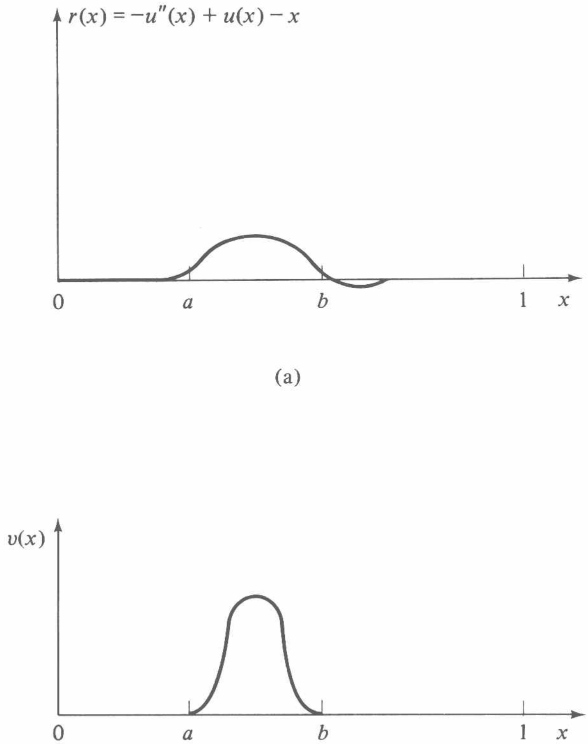
This weak statement of our problem, although seemingly less direct than the classical statement (1.2.1), has a certain appeal for those motivated by physical arguments. In modeling physical phenomena, it is often desirable to measure (or at least to consider the measurement of) the data and/or the solution of a boundary-value problem. Since any real measurement device (strain gauge, thermocouple, etc.) will have finite size, these quantities can, at best, be determined only in some average sense over small regions and not at any particular single point. The weak statement of the problem can be interpreted as assuring us that the solution will appear to be correct when tested at any location in the region with an arbitrarily small transducer.

1.3.1 A Symmetric Variational Formulation

At this stage, there are two points that should be thoroughly appreciated:

1. The weaker formulation (1.3.2) is as valid and meaningful as the original statement (1.2.1); indeed, the solution of (1.2.1) also satisfies (1.3.2) and, in fact, is *the* (only) solution of (1.3.2).

* Although we do not give the equation of $v(x)$, it is clear from the sketch that v is smooth enough to serve as a test function.



(b)

FIGURE 1.1 *Example of a residual error function* $r(x) = -u''(x) + u(x) - x$ *and a smooth test function* $v(x)$. *If* u *is the solution to (1.2.1),* r *cannot, on the average, be other than zero on any subinterval,* $a < x < b$.

2. The specification of the set H of test functions is an essential ingredient of an acceptable weak formulation.

Let us elaborate on point 2. Although it may not be immediately obvious, the test functions in variational problems such as (1.3.2) may not belong to the same class H of functions as the class $\tilde{H}$ to which the solution belongs (see Exercise 1.3.1). The set $\tilde{H}$ to which the solution u belongs is called the *class of trial functions* for such problems. Our smoothness requirements demand that we consider the pair of sets of functions, $\tilde{H}$ and H. For instance, u may be chosen from a class of functions $\tilde{H}$ which have the property that their *second* derivatives, when multiplied by a test function v, produce a function $u''v$ which is integrable over the interval $0 < x < 1$. On the other hand, no derivatives of test functions appear in (1.3.2). Thus, even though (1.3.2) is a perfectly