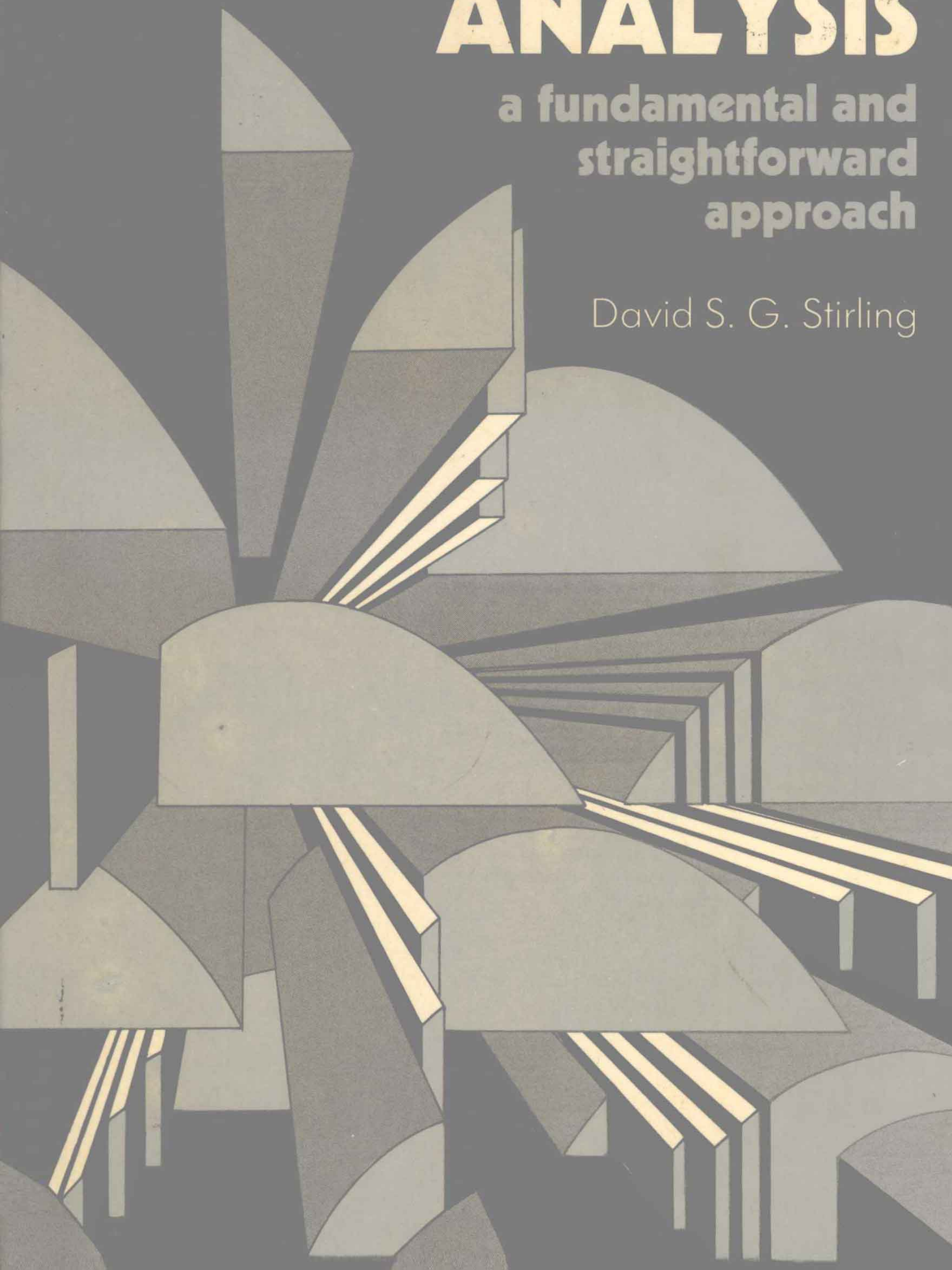
Ellis Horwood Series in **MATHEMATICS AND ITS APPLICATIONS**

MATHEMATICAL ANALYSIS

a fundamental and straightforward approach

David S. G. Stirling



MATHEMATICAL ANALYSIS:
A Fundamental and Straightforward Approach

DAVID S. G. STIRLING, B.Sc., Ph.D.
Lecturer in Mathematics, University of Reading

ELLIS HORWOOD LIMITED
Publishers · Chichester

Halsted Press: a division of
JOHN WILEY & SONS
New York · Chichester · Brisbane · Toronto

First published in 1987 by
ELLIS HORWOOD LIMITED
Market Cross House, Cooper Street,
Chichester, West Sussex, PO19 1EB, England
The publisher's colophon is reproduced from James Gillison's drawing of the ancient Market Cross, Chichester.

Distributors:
Australia and New Zealand:
JACARANDA WILEY LIMITED
GPO Box 859, Brisbane, Queensland 4001, Australia
Canada:
JOHN WILEY & SONS CANADA LIMITED
22 Worcester Road, Rexdale, Ontario, Canada
Europe and Africa:
JOHN WILEY & SONS LIMITED
Baffins Lane, Chichester, West Sussex, England
North and South America and the rest of the world:
Halsted Press: a division of
JOHN WILEY & SONS
605 Third Avenue, New York, NY 10158, USA

British Library Cataloguing in Publication Data
Stirling, David S. G.
Mathematical analysis. —
(Ellis Horwood series in mathematics and its applications)
1. Calculus
I. Title
515 QA303

Library of Congress Card No. 87–13500

ISBN 0–85312–996–7 (Ellis Horwood Limited — Library Edn.)
ISBN 0–7458–0262–1 (Ellis Horwood Limited — Student Edn.)
ISBN 0–470–20903–8 (Halsted Press)

Printed in Great Britain by Unwin Bros. of Woking

MATHEMATICAL ANALYSIS:
A Fundamental and Straightforward Approach

Mathematics and its Applications

Series Editor: G. M. BELL, Professor of Mathematics,
King's College London (KQC), University of London

Statistics and Operational Research

Editor: B. W. CONOLLY, Professor of Operational Research,
Queen Mary College, University of London

Mathematics and its applications are now awe-inspiring in their scope, variety and depth. Not only is there rapid growth in pure mathematics and its applications to the traditional fields of the physical sciences, engineering and statistics, but new fields of application are emerging in biology, ecology and social organisation. The user of mathematics must assimilate subtle new techniques and also learn to handle the great power of the computer efficiently and economically.

The need for clear, concise and authoritative texts is thus greater than ever and our series will endeavour to supply this need. It aims to be comprehensive and yet flexible. Works surveying recent research will introduce new areas and up-to-date mathematical methods. Undergraduate texts on established topics will stimulate student interest by including applications relevant at the present day. The series will also include selected volumes of lecture notes which will enable certain important topics to be presented earlier than would otherwise be possible.

In all these ways it is hoped to render a valuable service to those who learn, teach, develop and use mathematics.

Mathematics and its Applications

Series Editor: G. M. BELL, Professor of Mathematics, King's College London (KQC), University of London

Artmann, B.	**The Concept of Number**
Arczewski, K. & Pietrucna, J.	**Mathematical Modelling in Discrete Mechanical System**
Arczewski, K. and Pietrucha, J.	**Mathematical Modelling in Continuous Mechanical Systems**
Bainor, D.D. & Konstantinov, M.	**The Averaging Method and its Applications**
Balcerzyk, S. & Joszefiak, T.	**Commutative Rings**
Balcerzyk, S. & Joszefiak, T.	**Noetherian and Krull Rings**
Baldock, G.R. & Bridgeman, T.	**Mathematical Theory of Wave Motion**
Ball, M.A.	**Mathematics in the Social and Life Sciences: Theories, Models and Methods**
de Barra, G.	**Measure Theory and Integration**
Bell, G.M. and Lavis, D.A.	**Co-operative Phenomena in Lattice Models Vols. I & II**
Berkshire, F.H.	**Mountain and Lee Waves**
Berry, J.S., Burghes, D.N., Huntley, I.D., James, D.J.G. & Moscardini, A.O.	**Mathematical Modelling Courses**
Berry, J.S., Burghes, D.N., Huntley, I.D., James, D.J.G. & Moscardini, A.O.	**Mathematical Methodology, Models and Micros**
Berry, J.S., Burghes, D.N., Huntley, I.D., James, D.J.G. & Moscardini, A.O.	**Teaching and Applying Mathematical Modelling**
Broun, R.	**Modern Topology**
Burghes, D.N. & Borrie, M.	**Modelling with Differential Equations**
Burghes, D.N. & Downs, A.M.	**Modern Introduction to Classical Mechanics and Control**
Burghes, D.N. & Graham, A.	**Introduction to Control Theory, including Optimal Control**
Burghes, D.N., Huntley, I. & McDonald, J.	**Applying Mathematics**
Burghes, D.N. & Wood, A.D.	**Mathematical Models in the Social, Management and Life Sciences**
Butkovskiy, A.G.	**Green's Functions and Transfer Functions Handbook**
Butkovskiy, A.G.	**Structural Theory of Distributed Systems**
Cao, Z-Q., Kim, K.H. & Roush, F.W.	**Incline Algebra and Applications**
Cerny, I.	**Complex Domain Analysis**
Chorlton, F.	**Textbook of Dynamics, 2nd Edition**
Chorlton, F.	**Vector and Tensor Methods**
Cohen, D.E.	**Computability and Logic**
Crapper, G.D.	**Introduction to Water Waves**
Cross, M. & Moscardini, A.O.	**Learning the Art of Mathematical Modelling**
Cullen, M.R.	**Linear Models in Biology**

Series continued at back of book

Table of Contents

MATHEMATICAL ANALYSIS:
A Fundamental and Straightforward Approach

Mathematics and its Applications

Series Editor: G. M. BELL, Professor of Mathematics,
King's College London (KQC), University of London

Statistics and Operational Research

Editor: B. W. CONOLLY, Professor of Operational Research,
Queen Mary College, University of London

Mathematics and its applications are now awe-inspiring in their scope, variety and depth. Not only is there rapid growth in pure mathematics and its applications to the traditional fields of the physical sciences, engineering and statistics, but new fields of application are emerging in biology, ecology and social organisation. The user of mathematics must assimilate subtle new techniques and also learn to handle the great power of the computer efficiently and economically.

The need for clear, concise and authoritative texts is thus greater than ever and our series will endeavour to supply this need. It aims to be comprehensive and yet flexible. Works surveying recent research will introduce new areas and up-to-date mathematical methods. Undergraduate texts on established topics will stimulate student interest by including applications relevant at the present day. The series will also include selected volumes of lecture notes which will enable certain important topics to be presented earlier than would otherwise be possible.

In all these ways it is hoped to render a valuable service to those who learn, teach, develop and use mathematics.

Mathematics and its Applications

Series Editor: G. M. BELL, Professor of Mathematics, King's College London (KQC), University of London

Artmann, B.	**The Concept of Number**
Arczewski, K. & Pietrucna, J.	**Mathematical Modelling in Discrete Mechanical System**
Arczewski, K. and Pietrucha, J.	**Mathematical Modelling in Continuous Mechanical Systems**
Bainor, D.D. & Konstantinov, M.	**The Averaging Method and its Applications**
Balcerzyk, S. & Joszefiak, T.	**Commutative Rings**
Balcerzyk, S. & Joszefiak, T.	**Noetherian and Krull Rings**
Baldock, G.R. & Bridgeman, T.	**Mathematical Theory of Wave Motion**
Ball, M.A.	**Mathematics in the Social and Life Sciences: Theories, Models and Methods**
de Barra, G.	**Measure Theory and Integration**
Bell, G.M. and Lavis, D.A.	**Co-operative Phenomena in Lattice Models Vols. I & II**
Berkshire, F.H.	**Mountain and Lee Waves**
Berry, J.S., Burghes, D.N., Huntley, I.D., James, D.J.G. & Moscardini, A.O.	**Mathematical Modelling Courses**
Berry, J.S., Burghes, D.N., Huntley, I.D., James, D.J.G. & Moscardini, A.O.	**Mathematical Methodology, Models and Micros**
Berry, J.S., Burghes, D.N., Huntley, I.D., James, D.J.G. & Moscardini, A.O.	**Teaching and Applying Mathematical Modelling**
Broun, R.	**Modern Topology**
Burghes, D.N. & Borrie, M.	**Modelling with Differential Equations**
Burghes, D.N. & Downs, A.M.	**Modern Introduction to Classical Mechanics and Control**
Burghes, D.N. & Graham, A.	**Introduction to Control Theory, including Optimal Control**
Burghes, D.N., Huntley, I. & McDonald, J.	**Applying Mathematics**
Burghes, D.N. & Wood, A.D.	**Mathematical Models in the Social, Management and Life Sciences**
Butkovskiy, A.G.	**Green's Functions and Transfer Functions Handbook**
Butkovskiy, A.G.	**Structural Theory of Distributed Systems**
Cao, Z-Q., Kim, K.H. & Roush, F.W.	**Incline Algebra and Applications**
Cerny, I.	**Complex Domain Analysis**
Chorlton, F.	**Textbook of Dynamics, 2nd Edition**
Chorlton, F.	**Vector and Tensor Methods**
Cohen, D.E.	**Computability and Logic**
Crapper, G.D.	**Introduction to Water Waves**
Cross, M. & Moscardini, A.O.	**Learning the Art of Mathematical Modelling**
Cullen, M.R.	**Linear Models in Biology**

Series continued at back of book

MATHEMATICAL ANALYSIS: A Fundamental and Straightforward Approach

DAVID S. G. STIRLING, B.Sc., Ph.D.
Lecturer in Mathematics, University of Reading

ELLIS HORWOOD LIMITED
Publishers · Chichester

Halsted Press: a division of
JOHN WILEY & SONS
New York · Chichester · Brisbane · Toronto

First published in 1987 by
ELLIS HORWOOD LIMITED
Market Cross House, Cooper Street,
Chichester, West Sussex, PO19 1EB, England
The publisher's colophon is reproduced from James Gillison's drawing of the ancient Market Cross, Chichester.

Distributors:
Australia and New Zealand:
JACARANDA WILEY LIMITED
GPO Box 859, Brisbane, Queensland 4001, Australia
Canada:
JOHN WILEY & SONS CANADA LIMITED
22 Worcester Road, Rexdale, Ontario, Canada
Europe and Africa:
JOHN WILEY & SONS LIMITED
Baffins Lane, Chichester, West Sussex, England
North and South America and the rest of the world:
Halsted Press: a division of
JOHN WILEY & SONS
605 Third Avenue, New York, NY 10158, USA

British Library Cataloguing in Publication Data
Stirling, David S. G.
Mathematical analysis. —
(Ellis Horwood series in mathematics and its applications)
1. Calculus
I. Title
515 QA303

Library of Congress Card No. 87-13500

ISBN 0-85312-996-7 (Ellis Horwood Limited — Library Edn.)
ISBN 0-7458-0262-1 (Ellis Horwood Limited — Student Edn.)
ISBN 0-470-20903-8 (Halsted Press)

Printed in Great Britain by Unwin Bros. of Woking

Table of Contents

Author's Preface

Analysis tackles the issues which were fudged in the development of the calculus. With the recent trend away from formal proof in school, it may not be evident to students beginning higher education that there is a problem to be attended to here. Indeed, some school leavers have seen virtually none of the ideas of proof and do not necessarily accept that it is a vital part of mathematics. This book was written in acknowledgement that most present-day students have this background.

The main aim of the book is to present the accepted core material of analysis in such a way that the development appears fairly natural to the reader. A detailed discussion of the real number system, which is necessarily technical, is postponed until other matters have highlighted the need for it, while I have tried to maximise the number of results whose value can be appreciated from a standpoint other than that of the analyst, so that the subject is not seen as merely self-serving. This approach, while it could not be sustained throughout a degree course, seems to be correct for the start of a subject. The technical jargon of analysis cannot sensibly be avoided but it can be minimised and I have taken the view that a definition is not worth the sacrifice of memory unless it is used often.

The principal difference between this book and many others is that attention is devoted not only to giving proofs but to indicating how one might construct these proofs, a rather different process from appreciating the final product. The completeness of the real number system is assumed in the form of Dedekind's axiom of continuity, because this is more plausible than some of its immediate consequences.

Logically, this book presumes no knowledge of calculus, but it would be rather pointless to start analysis without that, and I have tacitly relied on calculus for some of the motivation. This is particularly true of Chapter 11, on functions of several variables, where the experience of grappling with the problems which arise in practice is a necessary supplement to the theory.

The book contains many problems for the reader to solve, designed to illustrate the main points or to force attention onto the subtler ones. Tackling these problems is an essential part of reading the book although the starred problems may be regarded as optional, being more difficult or more peripheral than the others.

I should like to thank my colleagues at Reading, especially David White, for comments and useful conversations over the years and the students who

have been subjected to this course for their comments. In particular, I am grateful to Michael Sewell for the final impetus which made me write it, to Robin Dixon for help with the diagrams, to Joyce Bird and Rosemary Pellew for deciphering my handwriting and typing it so expertly and to Ellis Horwood and his staff for their editorial and production cooperation.

Reading, February 1987 David Stirling

CHAPTER 1

The Need for Proof

> "If a man will begin with certainties, he shall end in doubts; but if he will be content to begin with doubts, he shall end in certainties."
>
> Francis Bacon.

Although mathematics is usually thought of as a science, it differs from most of science in one important respect—it is not based on empirical results which may later be altered by improved evidence. Thus physics, for example, is based on the prediction of various consequences of the basic 'laws' of the subject, but since these laws are derived from experiment and observation, they may be revised from time to time if their consequences turn out to conflict with what happens in the real world. Mathematics, on the other hand, is not based on experiment but is a more abstract creation whose results are true in a way that is not subject to later revision. A mathematical result, once established, is known to be true without reservation.

At its heart, mathematics is about numbers, which are already an abstraction: the thing that two sheep and two apples have in common (the 'two-ness') is abstract. Once we have accepted the idea of number, we have to find out the properties of the system we have created, that is, deduce them from our basic ideas. After we have discovered these, they remain true for all time and are not subject to the periodic revision that occurs with scientific laws. (Nevertheless, although the mathematics remains constant, it is at least conceivable that revisions in scientific laws could dramatically alter its usefulness.)

Having said that mathematics is a man-made structure in which results are deduced from some basic properties, let us consider the sort of processes involved. To fix our ideas here, let us look at a particular set of problems:

(i) $\sqrt{(x+3)} = \sqrt{(1-x)} + \sqrt{(1+x)}$,
(ii) $\sqrt{(x+3)} = \sqrt{(1-x)} - \sqrt{(1+x)}$,
(iii) $\sqrt{(x+3)} = \sqrt{(1+x)} - \sqrt{(1-x)}$.

Before starting, recall that the $\sqrt{y}$ sign denotes the non-negative square root of y.

Let us start with the first equation, $\sqrt{(x+3)} = \sqrt{(1-x)} + \sqrt{(1+x)}$. Squaring both sides gives

$$x + 3 = 1 - x + 2\sqrt{(1-x)}\cdot\sqrt{(1+x)} + (1+x),$$

and, on rearranging, we obtain

$$x + 1 = 2\sqrt{(1 - x)}\cdot\sqrt{(1 + x)}.$$

Squaring now yields

$$x^2 + 2x + 1 = 4 - 4x^2$$

whence, in turn,

$$5x^2 + 2x - 3 = 0,$$
$$(5x - 3)(x + 1) = 0,$$
$$x = 3/5 \quad \text{or} \quad x = -1.$$

We conclude that the solutions should be 3/5 and -1. If we are suspicious of this we can always test these values in equation (i) to check that the equation is satisfied. For example, letting $x = 3/5$ gives $\sqrt{(x + 3)} = \sqrt{(18/5)} = 3\sqrt{(2/5)}$ while

$$\sqrt{(1 - x)} + \sqrt{(1 + x)} = \sqrt{(2/5)} + \sqrt{(8/5)} = 3\sqrt{(2/5)}$$

so that $x = 3/5$ is indeed a solution. A simpler calculation shows that $x = -1$ is also a solution, and we have completed the problem.

Now let us try equation (ii): $\sqrt{(x + 3)} = \sqrt{(1 - x)} - \sqrt{(1 + x)}$. Squaring gives $x + 3 = 1 - x - 2\sqrt{(1 - x)}\cdot\sqrt{(1 + x)} + 1 + x$ which we rearrange to give $x + 1 = -2\sqrt{(1 - x)}\cdot\sqrt{(1 + x)}$. Squaring again yields $x^2 + 2x + 1 = 4 - 4x^2$ which, as before, has solutions $x = 3/5$ and $x = -1$. If we now test $x = -1$ we obtain $\sqrt{(x + 3)} = \sqrt{2}$ while $\sqrt{(1 - x)} - \sqrt{(1 + x)} = \sqrt{2} - \sqrt{0} = \sqrt{2}$ so that $x = -1$ satisfies equation (ii). However, putting $x = 3/5$ gives $\sqrt{(x + 3)} = \sqrt{(18/5)} = 3\sqrt{(2/5)}$ while $\sqrt{(1 - x)} - \sqrt{(1 + x)} = \sqrt{(2/5)} - \sqrt{(8/5)} = \sqrt{(2/5)} - 2\sqrt{(2/5)} = -\sqrt{(2/5)}$ so in this case only $x = -1$ is a solution of (ii). Our method has produced one true solution and a spurious one.

If we consider equation (iii), $\sqrt{(x + 3)} = \sqrt{(1 + x)} - \sqrt{(1 - x)}$, and apply the same method we obtain, after squaring, the equation $x + 3 = 1 + x - 2\sqrt{(1 + x)}\cdot\sqrt{(1 - x)} + 1 - x$ which simplifies as in equation (ii) to give $x = 3/5$ or $x = -1$. In this case, testing $x = -1$ yields $\sqrt{(x + 3)} = \sqrt{2}$ and $\sqrt{(1 + x)} - \sqrt{(1 - x)} = -\sqrt{2}$ while putting $x = 3/5$ gives $\sqrt{(x + 3)} = 3\sqrt{(2/5)}$ and $\sqrt{(1 + x)} - \sqrt{(1 - x)} = \sqrt{(2/5)}$; both 'solutions' are spurious.

What is happening here? The method we have used to solve these equations is capable of introducing completely spurious numbers, so that we seem to need to check the answers. Since we do not normally need to check answers (except to correct the very human failing of making mistakes—and there are none above), why should we need to here? The resolution of this apparent difficulty is tackled in Chapter 2.