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La Comisión Organizadora.

CONTENIDO

ON THE STRUCTURE OF HIGHER TERMS OF THE SPECTRAL SEQUENCE OF A FIBRE SPACE

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1. Introduction

Let (E, B, p) denote a fibre space[1] with B arcwise connected and (E_r, d_r), $r = 1, 2, \ldots$, the associated spectral sequence. Furthermore, let $F = p^{-1}(b)$, $b \in B$ denote a fixed fibre. Then a well-known result of Leray-Serre states that $E_1 = C(B, H(F))$, the singular chains of B with $H(F)$ as coefficients, and $d_1 : E_1 \to E_1$ is the boundary operator $\partial : C(B, H(F)) \to C(B, H(F))$ in the sense of local coefficients, where $\pi_1(B, b)$ operates on $H(F)$ in the usual manner. Hence, $E_2 = H(B, H(F))$ where the homology is in the sense of local coefficients. In case, $\pi_1(B, b) = 0$, therefore, $E_2 = H(B, H(F))$ where the coefficient group $H(F)$ is taken in the ordinary sense. In [2], the authors extended this latter result and showed that in case B was r-connected that $E_i = H(B, H(F))$ for $i = 2, \cdots, r+1$ and $d_i = 0$ for $i = 2, \cdots r$. The purpose of this paper is to extend this result still further. An alternate way of stating the above Leray-Serre result is that if $\pi_0(B) = 0$, then E_2 depends only upon B and the action of $\pi_1(B, b)$ on $H(F)$. Here we will show that if B is $r-1$ connected then $E_i = H(B, H(F))$ for $2 \leqq i \leqq r$ and E_{r+1} depends only upon B and the action of $\pi_r(B, b)$ on $H(F)$. More precisely, we first show that $\pi_r(B, b)$ and $H(F)$ are *paired* to $H(F)$, $r > 1$. Then, in case B is $r-1$ connected, $E_i = H(B, H(F))$ for $i = 2, \cdots, r$ $d_i = 0$ for $i = 2, \cdots, r-1$ and $d_r : E_r = H(B, H(F)) \to E_r$ is given by the cap product

$$d_r(h) = \gamma \cap h, \qquad h \in H(B, H(F))$$

where γ is the characteristic cohomology class of B and the cap product is defined in terms of the pairing of $\pi_r(B, b)$ and $H(F)$ to $H(F)$[2].

REMARK. The corresponding result for singular cohomology is also valid, where cup product replaces cap product and $\pi_r(B)$ and $H^*(F, G)$ (the cohomology group of F with coefficients in G) are suitably paired.

2. Preliminaries

2.1. FIBRE SPACES. In this paper we employ the concept of fibre space as given in [1] and for the reader's convenience we recall the basic definitions. Given a triple (E, B, p) where $p : E \to B$ is a map, let Ω_p denote the subset of $E \times B^I$ given by

$$\Omega_p = \{(e, \omega) \in E \times B^I : p(e) = \omega(0)\}.$$

† Due to the untimely death of Professor Hurewicz, the second-named author has prepared this joint account of their research and accepts full responsibility for its accuracy.

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[1] In this paper we used the term fibre space in the sense of [1], see §2.

[2] This simple description of d_r was suggested by Norman Steenrod.

Then, we have a natural map $\tilde{p} : E^I \to \Omega_p$ given by

$$\tilde{p}(\alpha) = (\alpha(0),\ p(\alpha))$$

where $p(\alpha)(t) = p(\alpha(t))$, $0 \leqq t \leqq 1$. Finally, we say that (E, B, p) is a *fibre space* provided $\tilde{p} : E^I \to \Omega_p$ admits a cross section, i.e. a map $\Lambda : \Omega_p \to E^I$ such that $\tilde{p} \circ \Lambda = 1$.[3] The map Λ is referred to as a lifting function. It is easy to show that any two lifting functions are homotopic in the class of lifting functions, i.e. given any two lifting functions Λ_0, Λ_1, there exists a homotopy $H : \Omega_p \times I \to E^I$ such that

(i) $H_0 = \Lambda_0, H_1 = \Lambda_1$

(ii) $\tilde{p}H[(e, \omega), t] = (e, \omega), t \in I, (e, \omega) \in \Omega_p$.

The following fact will also be used in the sequel. Let $\tilde{E}$ denote the space of paths in E emanating from a fibre F, i.e.

$$\tilde{E} = \{\alpha \in E^I : \alpha(0) \in F\}$$

where $F = p^{-1}(b)$, $b \in B$. Then, if Λ is a lifting function for the fibre space (E, B, p), let $\tilde{\Lambda} : \tilde{E} \to \tilde{E}$ be given by

$$\tilde{\Lambda}(\alpha) = \Lambda(\alpha(0), p\alpha),\ \alpha \in \tilde{E}.$$

Then one shows easily that $\tilde{\Lambda}$ is homotopic to the identity map $1 : \tilde{E} \to \tilde{E}$.

2.2. Singular theory based on cubes. Let X denote a topological space and, employing the notation in Serre [3], $Q_n(X)$ the free abelian group generated by singular n-cubes in X. Letting $D_n(X)$ denote the subgroup generated by degenerate n-cubes, we set $C_n(X) = Q_n(X)/D_n(X)$. Then $C(X) = \sum_n C_n(X)$ is called the group of singular chains in X (based on cubes) with integral coefficients. For an arbitrary coefficient group G we set $C_n(X, G) = C_n(X) \otimes G$ and $C(X, G) = \sum_n C_n(X, G)$. Also, we set $C^n(X, G) = \mathrm{Hom}\ (C_n(X), G)$. $C^*(X, G) = \sum_n C^n(X, G)$ is then the group of singular cochains with coefficients in G.

The boundary operator ∂ in $C(X)$ is given by

$$\partial u = \textstyle\sum_{i=1}^n (-i)^i [\lambda_i^1 u - \lambda_i^0 u]^4$$

where u is a singular n-cube and

$$(\lambda_i^\varepsilon u)(x_1, \cdots, x_{n-1}) = u(x_1, \cdots, x_{i-1}, \varepsilon, x_i, \cdots, x_{n-1})$$

for $\varepsilon = 0, 1$; $1 \leqq i \leqq n$. Employing ∂, $C(X, G)$ and $C^*(X, G)$ become chain and cochain complexes, respectively, and we have therefore the singular homology and cohomology groups of X, namely

$$H(X, G) = \textstyle\sum_n H_n(X, G),\ H^*(X, G) = \sum_n H^n(X, G).$$

[3] This definition is easily seen to be equivalent to assuming the validity of the Covering Homotopy Theorem for all spaces as applied to (E, B, p) and hence is stronger than the definition of fibre space in the sense of Serre.

[4] This ∂ differs in sign from that employed in [3].

2.3. CAP PRODUCTS. Let u denote a singular n-cube in X. Following Serre [3], we define certain faces of u as follows: Let H denote a subset of p elements from the set of indices $\{1, \cdots, n\}$ and K the complement of H, containing, therefore, q elements where $p + q = n$. Let $\varphi_K : K \to \{1, \cdots, q\}$ denote a strictly monótone function. For $\varepsilon = 0$ or 1 we let $\lambda_H^\varepsilon u$ denote the following q-face of u

$$(\lambda_H^\varepsilon u)(x_1, \cdots, x_q) = u(y_1, \cdots, y_n)$$

where

$$y_i = \varepsilon \qquad \text{for} \quad i \in H$$

$$y_i = x_{\varphi_K(i)} \qquad \text{for} \quad i \in K.$$

Also, set sgn $H = (-1)^\nu$ where ν is the number of pairs (i, j), $i \in H, j \in K$ such that $i > j$.

Suppose now that the groups G_1 and G_2 are paired to G. For $g_1 \in G_1, g_2 \in G_2$, let (g_1, g_2) denote the element of G obtained from pairing g_1 and g_2. For $f^q \in C^q(X, G_1)$, u_{p+q} a singular cube in $C_{p+q}(X)$, set

$$f^q \cap u_{p+q} \otimes g_2 = \textstyle\sum_H \operatorname{sgn} H \, \lambda_K^1 u \otimes (f^q(\lambda_H^0 u), g_2), \; g_2 \in G_2.$$

It is not difficult to show the usual cap product identity

$$\partial(f^q \cap u_{p+q} \otimes g_2) = (-1)^p \, \delta f^q \cap u_{p+q} \otimes g_2 + f^q \cap \partial u_{p+q} \otimes g_2$$

where δ is the differential operator in $C^*(X, G_1)$. Therefore, the pairing of $C^q(X, G_1)$ and $C_{p+q}(X, G_2)$ to $C_p(X, G)$ leads to a pairing of $H^q(X, G_1)$ and $H_{p+q}(X, G_2)$ to $H_p(X\ G)$.

REMARK. The above cap product differs, at the homology level, from the definition (adapted to cubical theory) given in Eilenberg [4] by a factor of $(-1)^{pq}$ where n is the dimension of the second factor. In comparison with the above definition of $f^q \cap u_{p+q} \otimes g_2$, the Eilenberg definition (adapted to cubical theory) would read

$$f^q \cap u_{p+q} \otimes g_2 = \textstyle\sum_H \operatorname{sgn} H \, \lambda_K^0 u \otimes (f^q(\lambda_H^1 u), g_2).$$

2.4. THE SPECTRAL SEQUENCE. Let (E, B, p) denote a fibre space. We filter $A = C(E)$ singular chains of E (integral coefficients) just as in Serre [3]. If u is a singular n-cube in X, a coordinate index i, $1 \leq i \leq n$ is called db (degenerate base) for u if

$$pu(x_1, \cdots, x_i, \cdots, x_n) = pu(y_1, \cdots, y_i, \cdots, y_n)$$

for arbitrary x_i, y_i. Otherwise i is called a pb (proper base) coordinate for u. Then, we set

$$\dim_r u = \max \text{ pb coordinate index for } u.$$

The filtration

$$0 = A^{-1} \subseteqq A^0 \subseteqq \cdots \subseteqq A^p \subseteqq A^{p+1} \subseteqq \cdots$$

is obtained by letting A^p denote the subgroup of A generated by singular cubes u such that $\dim_r u \leq p$. The spectral sequence associated with this filtration can be obtained as follows: $E_r^{p,q}$, $1 \leq r < \infty$, is the image of i_*, where

$$i_* : H_n(A^p, A^{p-r}) \to H_n(A^{p+r-1}, A^{p-1}), \qquad n = p + q$$

is the map induced by the natural injection i. Furthermore, the differential operator

$$d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$$

is just the boundary operator

$$\partial_* : H_n(A^{p+r-1}, A^{p-1}) \to H_{n-1}(A^{p-1}, A^{p-r-1})$$

for the triple $(A^{p+r-1}, A^{p-1}, A^{p-r-1})$ restricted to $E_r^{p,q}$, $\quad n = p + q$.

2.5. Extended r-skeleton of an n-cell. Let I^n denote an n-cell $n \geqq 1$ and r an integer $1 \leqq r \leqq n$. As in §2.3, let H denote a subset of r elements from $\{1, \cdots, n\}$ and K the complement of H. Let $\alpha : K \to \{0, 1\}$ denote a function on K taking the values 0 or 1. Then set

$$F_K^\alpha = \{x \in I^n : x_k = \alpha(k) \quad \text{for} \quad k \in K\}.$$

Then, F_K^α is an ordinary r-face of I^n. Now, let $\bar{H}$ denote a set of $r + 1$ elements from $\{1, \cdots, n\}$ and $\bar{K}$ its complement. Let (i, j) denote a fixed pair of indices in $\bar{H}$ and $\bar{\alpha} : \bar{K} \to \{0, 1\}$ a function on K with 0 or 1 as values. Set

$$F_{\bar{K}}^{\bar{\alpha}}(i, j) = \{x \in I^n : x_k = \bar{\alpha}(k) \quad \text{for} \quad k \in \bar{K} \text{ and } x_i = x_j\}.$$

We call $F_{\bar{K}}^{\bar{\alpha}}(i, j)$ a *diagonal* r-face of I^n. Let F_r denote the set union of all the ordinary r-faces and $\bar{F}_r$ the set union of all the diagonal r-faces. Then $F_r^* = F_r \cup \bar{F}_r$ will be called the *extended* r-skeleton of I^n.

Now, let X denote a topological space and $Q(X) = \sum_n Q_n(X)$ as in §2.2. Let $Q^r(X)$ denote the subgroup of $Q(X)$ generated by singular cubes

$$u : (I^n, F_r^*) \to (X, x_0)$$

where x_0 is a fixed element of X. Then, set

$$C^r(X) = (Q^r(X))/(Q^r(X) \cap D(X)).$$

The following lemmas will be used in the sequel and their proofs, follow standard lines.

Lemma 1. *If X is r-connected,* i.e. $\pi_i(X, x_0) = 0$, $i \leqq r$, *then the chain complexes $C^r(X)$ and $C(X)$ are chain equivalent and hence the singular homology groups of X may be based on singular cubes whose extended r-skeleton lies at the fixed point x_0.*

Lemma 2. *If (E, B, p) is a fibre space with B r-connected, then the singular homology groups of E may be based on singular cubes whose extended r-skeleton lies in the fibre $F = p^{-1}(b)$, $b \in B$ a fixed base point.*

2.6. A homotopy addition lemma. Let F denote the extended $n - 1$ skeleton of I^n, $n > 1$, and let

$$u : (I^n, F) \to (X, x_0)$$

X a space, $x_0 \in X$, denote a map. Then u represents an element α in $\pi_n(X, x_0)$. Furthermore, let

$$u_k : (I^n, \dot{I}^n) \to (X, x_0)$$

denote the map given by

$$u_k(x_1, \cdots, x_n) = u(y_1, \cdots, y_n)$$

where

$$y_i = x_n x_i \qquad i < k$$

$$y_k = x_n$$

$$y_i = x_n x_{i-1} \quad i > k.$$

Since u maps the extended $n - 1$ skeleton of I^n into x_0, u_k maps I^n into x_0 and hence u_k represents an element α_k in $\pi_n(X, x_0)$. The following lemma is then valid. Its proof follows standard lines and is omitted.

LEMMA. $\sum_{k=1} (-1)^{n-k} \alpha_k = \alpha$.

3. Pairing $\pi_n(B)$ and $H(F, G)$ to $H(F, G)$, $n > 1$

Let (E, B, p) denote a fibre space, $F = p^{-1}(b)$, $b \in B$, a fibre and $\pi_n(B, b) = \pi_n(B), n > 1$, a homotopy group of B. Let $\alpha \in \pi_n(B)$ with representative $f : (I^n, \dot{I}^n) \to (B, b)$. Also, let $v : I^q \to F$, $q \geq 0$, denote a singular q-cube in F. We define a singular $q + n - 1$ cube (f, v) in F as follows. For $x \in I^{q+n-1}$, set ${}^{n-1}x = (x_1, \cdots, x_{n-1})$, $x^q = (x_n, \cdots, x_{q+n-1})$. Then, if Λ is any lifting function for (E, B, p) set

$$(f, v)(x) = \Lambda[v(x^q), \omega({}^{n-1}x)](1)$$

where $\omega({}^{n-1}x)$ is the loop in B given by

$$\omega({}^{n-1}x)(t) = f({}^{n-1}x, t).$$

If now, $h \in H_q(F, G)$ is a homology class with representative cycle $z = \sum_j v_j \otimes g_j$, we denote by $(\alpha, h) \in H_{q+n-1}(F, G)$ the homology class containing the cycle $(f, z) = \sum_j (f, v_j) \otimes g_j$. It is easy to show that (f, z) is indeed a cycle and (α, h) is independent of the representatives f and z chosen, as well as the lifting function Λ employed. Furthermore, bilinearity, namely

$$(\alpha + \alpha', h) = (\alpha, h) + (\alpha', h)$$

$$(\alpha, h + h') = (\alpha, h) + (\alpha, h')$$

follows easily and hence $\pi_n(B)$ and $H_q(F, G)$ are paired to $H_{n+q-1}(F, G)$.

An alternate description of this pairing may be given as follows. Let Ω denote the loop space of B based at $b \in B$. Then, any lifting function Λ gives rise to a

$$\bar{\Lambda} : \Omega \times F \to F$$

map as follows:

$$\bar{\Lambda}(\omega, x) = \Lambda(x, \omega)(1), \qquad \omega \in \Omega, \qquad x \in F.$$

Applying the Künneth Theorem, we obtain induced homomorphisms

$$\bar{\Lambda}_* : H_p(\Omega) \otimes H_q(F, G) \to H_{p+q}(F, G).$$

Then, the pairing of $\pi_n(B)$ and $H(F, G)$ to $H(F, G)$ is given by the composition homomorphism

$$\pi_n(B) \otimes H(F, G) \xrightarrow{i \otimes 1} \pi_{n-1}(\Omega) \otimes H(F, G) \xrightarrow{j \otimes 1} H_{n-1}(\Omega) \otimes H(F, G) \xrightarrow{\overline{\Lambda}_*} H(F, G),$$

where $i : \pi_n(B) \to \pi_{n-1}(\Omega)$ is the standard natural isomorphism and $j : \pi_{n-1}(\Omega) \to H_{n-1}(\Omega)$ is the Hurewicz homomorphism. The two descriptions are easily seen to yield identical pairings. We shall, however, have need for the explicit form of the pairing given initially in terms of representatives of homotopy and homology classes.

4. The basic map and identity

4.1. The basic map. We assume in this section that B is a fixed arcwise connected topological space and $b \in B$ a fixed base point. Let $\tilde{B}$ be the space of paths in B starting at b, i.e. $\tilde{B} = B^I(0, b)$. $\tilde{B}$ is a fibre space over B with map $\xi : \tilde{B} \to B$ given by

$$\xi(\omega) = \omega(1), \qquad \omega \in \tilde{B}.$$

Let $C(B)$ and $C(\tilde{B})$ denote the singular chains of B and $\tilde{B}$, respectively. We define a dimension preserving homomorphism (not a chain map)

$$\varphi : C_n(B) \to C_n(\tilde{B})$$

as follows. Let u denote a singular n-cube in B, and let p denote an index between 1 and n, i.e. $1 \leqq p \leqq n$ and $q = n - p$. As in §2.3 let H denote a subset of p elements from $\{1, \cdots, n\}$ and K its complement. For such an H we first define a homomorphism

$$\varphi_H^p : C_n(B) \to C_n(\tilde{B})$$

as follows. Let $\alpha : H \to \{1, \cdots, p\}$, $\beta : K \to \{p + 1, \cdots, n\}$ denote increasing functions. For $x \in I^n$, set

$$y(x) = (y_1, \cdots, y_n), \qquad z(x) = (z_1, \cdots, z_n)$$

where

$$\begin{cases} y_i = x_{\beta(i)}, & z_i = 1 \text{ for } i \in K \\ y_i = 0, & z_i = x_{\alpha(i)} \text{ for } i \in H. \end{cases}$$

Then, let α_x denote the arc

$$\alpha_x(t) = \begin{cases} u[2ty(x)], \ t \leqq 1/2 \\ u[(2 - 2t)y(x) + (2t - 1)z(x)], \ t \geqq 1/2 \end{cases}$$

where u is a given n-cube. Then, set

$$\varphi_H^p u(x) = \alpha_x$$

and the homomorphism

$$\varphi_H^p : C(B) \to C(\tilde{B})$$

is defined. To obtain φ set

$$\varphi^p = \sum_H \operatorname{sgn} H\, \varphi_H^p \quad \text{and} \quad \varphi = \sum_p \varphi^p.$$

Note that the definition of φ depends on n but is not displayed in the notation. Let

$$0 = A^{-1} \subseteq A^0 \subseteq \cdots \subseteq A^p \cdots$$

denote the filtration of $G(\tilde{B})$ relative to the fibering $(\tilde{B}, B, \xi)$. Then if u is an n-cube in B, we make the following important observation

$$\varphi^p u \in A^p = A^{n-q} \qquad 1 \leqq p \leqq n.$$

4.2. THE BASIC IDENTITY. Let u denote a fixed n-cube and p, q, H and K as above. Suppose also that the indices in H and K are denoted by

$$H : i_1 < \cdots < i_p, \qquad K : j_1 < \cdots < j_q.$$

Then the following lemmas can be verified easily from the definitions.

LEMMA 1. *For* $k \leqq p$,

$$\lambda_k^0 \varphi_H^p u = \varphi_{H^*}^{p-1} \lambda_{i_k}^0 u$$

and $(-1)^k \operatorname{sgn} H = (-1)^{i_k} \operatorname{sgn} H^*$ *where*

$$H^* = \{i_1, \cdots, i_{k-1}, i_{k+1} - 1, \cdots, i_p - 1\}.$$

LEMMA 2. *For* $1 \leqq k \leqq q$,

$$\lambda_{p+k}^0\, \varphi_H^p u = \lambda_{j_k - k+1}^1\, \varphi_{H^*}^{p+1} u$$

and $(-1)^{p+k} \operatorname{sgn} H = (-1)^{j_k - k+1} \operatorname{sgn} H^*$ *where* $H^* = H \cup j_k$.

LEMMA 3. *If the q-skeleton of u is at a fixed point $b \in B$, then for* $1 \leqq k \leqq p$

$$\lambda_{p+k}^1\, \varphi_H^p u = \varphi_{H^*}^p\, \lambda_{j_k}^1 u$$

and $(-1)^{p+k} \operatorname{sgn} H = (-1)^{j_k} \operatorname{sgn} H^*$ *where* $H^* = \{i_1^*, \cdots, i_p^*\}$ *and* $i_e^* = i_e$ *or* $i_e - 1$ *according as* $i_e < j_k$ *or* $i_e > j_k$.

Now, suppose that u is an n-cube whose $r - 1$ skeleton lies at $b \in B$. Then for any $q \leqq r$, the following basic identity I_φ is valid,

$$(I_\varphi) \quad \partial\varphi u - \varphi\partial u = S_q(u) + R_q(u) + \partial \sum_{j \leqq n-q-1} \varphi^j u - \sum_{j \leqq n-q-1} \varphi^j \partial u$$

where

$$S_q(u) = \sum_{k=1}^{n-q} \sum_H (-1)^k \operatorname{sgn} H\, [\lambda_k^1\, \varphi_H^{n-q}\, u - \lambda_k^0\, \varphi_H^{n-q} u]$$

$$R_q(u) = \sum_{k=1}^{q} \sum_H (-1)^{n-q+k} \operatorname{sgn} H\, [\lambda_{n-q+k}^1 \varphi_H^{n-q}\, u - \varphi_{H^*}^{n-q}\, \lambda_{j_k}^1 u]$$

where in the expression $R_q(u)$, H^* and j_k have the following meaning: If $H = \{i_1 < \cdots < i_{n-q}\}$, and $K = \{j_i < \cdots < j_q\}$ is its complement, $H^* = \{i_1^*, \cdots, i_{n-q}^*\}$ where $i_m^* = i_m$ or $i_m^* = i_m - 1$ according as $i_m < j_k$ or $i_m > j_k$. j_k is, of course, already indicated as an element of K.

The proof of this identity is immediate by induction on q, making use of the previous lemmas. I_φ immediately implies

LEMMA. *If the $r - 1$ skeleton of a singular n-cube u in B lies at $b \in B$ then*

$$\partial \varphi u - \varphi \partial u \in A^{n-r}$$

and

$$\partial \varphi u - \varphi \partial u = R_r(u) \text{ modulo } A^{n-r-1}.$$

5. Application of the basic map to fibre spaces

5.1. THE INDUCED MAP ψ. Let (E, B, p) denote a fibre space, B arcwise connected, $b \in B$ a fixed base point and $p^{-1}(b) = F$ the fibre over b. The homomorphism

$$\varphi : C(B) \to C(\tilde{B})$$

of the preceding section induces a homomorphism

$$\psi : C(B) \otimes C(F) \to C(E)$$

as follows. Let u denote a p-cube in B, v a q-cube in F. For $(x, y) \in I^{p+q}$, $x \in I^p$, $y \in I^q$ set

$$\psi^i_H(u \otimes v)(x, y) = \Lambda\,[v(y), \varphi^i_H u(x)](1)$$

where $1 \leq i \leq p$ and H is a subset of i indices from $\{1, \cdots, p\}$ as in the previous section and Λ is any lifting function for (E, B, p). Then set

$$\psi^i = \textstyle\sum_H \psi^i_H, \qquad \psi = \sum_{i=1}^p \psi^i.$$

We note that

$$\psi : C_p(B) \otimes C_q(F) \to C_{p+q}(E)$$

depends on p and q but they will not be displayed in the notation.

Now, let

$$0 = A^{-1} \subseteq A^0 \subseteq \cdots \subseteq A^p \subseteq \cdots$$

denote the filtration of $C(E)$ as in §2.4. Then, we note that

$$\psi^i : C_p(B) \otimes C(F) \to A^i \quad 1 \leq i \leq p.$$

5.2. THE IDENTITY I_ψ. The basic identity I_ϕ implies easily a corresponding identity for ψ which we state as the

FUNDAMENTAL LEMMA. *If u is a p-cube in B whose $r - 1$ skeleton lies at a fixed point $b \in B$, then*

$$(I_\psi) \quad \partial\psi(u \otimes v) - \psi\partial(u \otimes v) = R_r(u \otimes v) \text{ modulo } A^{p-r-1}$$

where v is a singular cube in F, $R_r(u \otimes v) \in A^{p-r}$ and

$$R_r(u \otimes v) = \textstyle\sum_{k=1}^r \sum_H (-1)^{p-r+k} \operatorname{sgn} H[\lambda^1_{p-r+k}\, \psi^{p-r}_H(u \otimes v) - \psi^{p-r}_{H^*}(\lambda^1_{j_k} u \otimes v)]$$

where H ranges over subsets of $\{1, \cdots, p)$ which contain $n - r$ elements.

As in §4.2, H^* and j_k have the following meaning. Let K denote the complement of a given H. We write the elements of K in increasing order $j_1 < \cdots < j_r$, thus determining j_k. If $H = \{i_1, \cdots, i_{p-r}\}$, $H^* = \{i_1^*, \cdots, i_{p-r}^*\}$ where $i_m^* = i_m$ or $i_m^* = i_m - 1$ according as $j_k > i_m$ or $j_k < i_m$.

5.3. The main result. Now, in the chain complex $C_p(B) \otimes C(F)$ introduce the boundary operator

$$\partial_F(b \otimes f) = (-1)^p b \otimes \partial f, \qquad b \in C_p(B), \qquad f \in C(F).$$

Furthermore, ψ induces homomorphisms

$$\psi_0 : C_p(B) \otimes C(F) \rightarrow A^p/A^{p-1}.$$

Since B is arcwise connected, we apply the Fundamental Lemma for $r = 1$ and see that

$$\psi_0 \, \partial_F = \partial \psi_0$$

and hence ψ_0 induces homomorphisms

$$\psi_1 : C_p(B) \otimes H_q(F) \rightarrow H_{p+q}(A^p, A^{p-1}) = E_1^{p,q}.$$

Now, let u denote a $p + q$ cube in E such that $\dim_r u \leqq p$. As in Serre [3], Bu will denote the p-cube in B, Fu the q-cube in F given by

$Bu(x_1, \cdots, x_p) = p \cdot u(x_1, \cdots, x_p, y_1, \cdots, y_q)$ for any choice of $(y_1, \cdots, y_q)$.

$Fu(x_1, \cdots, x_q) = u(0, \cdots, 0, x_1, \cdots, x_q)$.

Next, define

$$\theta : A^p \rightarrow C_p(B) \otimes C(F)$$

by setting

$$\theta(u) = Bu \otimes Fu$$

for u a generator of A^p. Then θ induces

$$\theta_0 : A^p/A^{p-1} \rightarrow C_p(B) \otimes C(F).$$

It is easy to see that $\partial_F \theta_0 = \theta_0 \partial$ and hence θ_0 induces

$$\theta_1 : H_{p+q}(A^p, A^{p-1}) = E_1^{p,q} \rightarrow C_p(B) \otimes H_q(F).$$

Theorem *ψ_0 and θ_0 form a chain equivalence and hence ψ_1 and θ_1 are isomorphisms onto.*

Proof. The proof is given in the Appendix, §1.

Now, let us assume that for the fibre space (E, B, p), B is $r - 1$ connected with $r > 1$. We may then assume that the singular chains of E are generated by singular cubes whose extended $r - 1$ skeletons lie in F, and that the singular chains of B are generated by singular cubes whose extended $r - 1$ skeletons lie at the fixed base point b. As a matter of interest, the case $r = 1$ which gives the Leray-Serre result is given in the Appendix, §2. Define

$$\partial_B : C_p(B) \otimes H(F) \rightarrow C_{p-1}(B) \otimes H(F)$$

by

$$\partial_B u \otimes h = (\partial u) \otimes h.$$

Then, applying the Fundamental Lemma, it is easy to see that the following diagram commutes

$$\begin{array}{ccc} C(B) \otimes H(F) & \xrightarrow{\psi_1} & E_1 \\ \partial_B \downarrow & & \downarrow d_1 \\ C(B) \otimes H(F) & \xrightarrow[\psi_1]{} & E_1 \end{array}$$

and, recalling that $H(E_j) = E_{j+1}$, ψ_1 induces an isomorphism

$$\psi_2 : H(B, H(F)) \to E_2$$

which is a special case of the Leray-Serre result. Now, again applying the Fundamental Lemma, we see, step by step, that ψ_2 induces isomorphisms

$$\psi_i : H(B, H(F)) \to E_i$$

for $2 \leqq i \leqq r$ and the composition maps

$$d_i \cdot \psi_i : H(B, H(F)) \to E_i \to E_i$$

are 0 for $2 \leqq i \leqq r - 1$, and hence $d_i = 0$ for $2 \leqq i \leqq r - 1$ (in case $r > 2$) Next, we investigate the structure of the differential operator d_r. It should be remarked, that $\theta_1 : E_1 \to C(B) \otimes H(F)$ also induces, step by step, isomorphisms

$$\theta_i : E_i \to H(B, H(F)) \qquad 2 \leqq i \leqq r.$$

Consider, the composition

$$H(B, H(F)) \xrightarrow{\psi_r} E_r \xrightarrow{d_r} E_r \xrightarrow{\theta_r} H(B, H(F)).$$

Take a homology class $h \in H_p(B, H_q(F))$ and let z denote a representative cycle of h such that $z = \sum_{i,j} \mu_{i,j} u_i \otimes v_{i,j}$, with μ_{ij} integers, u_i p-cells in B, $v_{i,j}$ q-cells in F and $\sum_j \mu_{i,j} v_{i,j}$ is a cycle in F for each i, representing a homology class in F which we denote by h_i. Then, employing the Fundamental Lemma, $d_r \psi_r(h) \in E_r^{p-r,q+r-1}$ is determined entirely by

$$\textstyle\sum_{i,j} \mu_{ij} R_r(u_i - v_{ij}) = R_r(z).$$

In order to determine the structure of $\theta_r d_r \psi_r(h)$ we look at the image of $R_r(u \otimes v)$ under $\theta : A^{p-r} \to C_{p-r}(B) \otimes C(F)$ where u and v are as in the Fundamental Lemma §5.2. Let H denote a subset of $p - r$ indices from $\{1, \cdots, p\}$ and K be its complement. Then $f_H = \lambda_H^0 u$ represents an element of $\pi_r(B, b)$, and f_H maps the extended $r - 1$ skeleton of I^r into b. Following §2.6 let

$$f_{H,k}(x_1, \cdots, x_r) = f_H(x_r x_1, \cdots, x_r x_{k-1}, x_r, x_r x_k, \cdots, x_r x_{r-1}).$$

Letting $e : I^r \to b$ denote the natural representative of $0 \in \pi_r(B, b)$, set

$$\bar{f}_{H,k} = f_{H,k} + e$$

where the addition occurs in the r^{th} coordinate. Then, the following lemma is easy to verify directly from definitions.