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JACK K. HALE

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by
JACK K. HALE

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To Lamberto Cesari on his 70th Birthday

CONTENTS

1. Introduction

My objective in this paper is to give some of the basic results in the theory of bifurcation in differential equations. It is difficult to trace the historical development of any important concept and bifurcation theory is no exception. However, a careful study of literature shows that Poincaré [1], [2] and Liapunov [1], [2] are responsible for our present basic philosophy as well as several of the fundamental ideas of the methods that we presently employ. These two persons can be directly linked with the importance of exchanges of stability, the occurrence of complicated motions in dynamical systems, the principle of reduction to lower dimensional problems, the philosophy of genericity and the transformation theory of differential equations that is so important in obtaining approximations of the center manifold and the flow on the center manifold. In many respects, we are still exploiting the ideas of these two giants.

After the initial impetus of Poincaré and Liapunov, it is somewhat surprising that modern bifurcation theory did not appear at an earlie. date. It is perhaps true that the ideas of Liapunov connected with bifurcation theory were being developed more extensively than the corresponding ones of Poincaré. There was a very active group in the U.S.S.R. (consisting of Andronov, Vitt, Khaikin, Bogoliubov, Krylov, Leontovich, Malkin and others) working on critical cases in stability theory, nonlinear oscillations and the general theory of integral manifolds. The techniques developed from the study of these areas are fundamental ingredients in dynamic bifurcation theory (see Hale [1] for some references).

A fundamental step towards modern bifurcation theory in differential equations occurred with the definition of structural stability of Andronov and Pontrjagin [1] in 1937 and the classification of structurally stable systems in the plane. With these concepts, Andronov and Leontovich [1] were able to make precise definitions of types of bifurcation points which had the possibility of being classified. These results were applied extensively to the theory of nonlinear oscillations by Andronov, Vitt and Khaikin in 1937 (the second edition of this book is Andronov, Vitt and Khaikin [1]). As Minorsky [1] said in 1962: "Having established the initial advance in the field of nonlinear oscillations (up to 1940), the Russian scientists maintain their leadership and initiative characterized by a remarkable coordination of efforts between the mathematical and experimental parts of these fundamental researches." There were several important developments in this intervening period by Levinson [1], [2], Cartwright and Littlewood (see Cartwright [1]) on the forced van der Pol and

Lienard equations. However, in western Europe and the United States, the interest in this aspect of differential equations had never been very extensive. In addition, there was little awareness of the developments that had been made in the U.S.S.R., and, as a consequence, some duplication of effort occurred.

Since 1960, there have been extensive developments in the abstract theory of dynamical systems. At the same time, some of this theory has been applied to very interesting problems in the biological and physical sciences. In an attempt to explain phenomena that occur in nature, it has been necessary for researchers to discuss the dynamic bifurcation of specific types of equations in great detail. This has led to an exciting interaction between analytical and theoretical methods.

In this paper, we present some of the concepts and results that play an important role in these areas. When the dimension of the system is one or two, one can obtain a rather complete theory at least for general one parameter families of vector fields. For either several parameter problems or for the dimension of the system greater than two, only partial results are known. On the other hand, the results in low dimension are applicable to higher dimensional problems (even infinite dimensional ones) when the discussion is restricted to a neighborhood of an equilibrium point for which the theory of center manifolds can be employed.

The table of contents expresses in general terms the substance of this paper. The first eight sections deal with structural stability and bifurcation in the low dimensional problems mentioned above. §9 is devoted to the formulation of some of the basic problems in the qualitative theory for a special class of dynamical systems in infinite dimensions. This class is general enough to include many functional differential equations and partial differential equations. §10 is concerned primarily with some types of bifurcation that occur because the base space is infinite dimensional. Due to space, very few proofs are given. Also, there are several important omissions of topics from differential equations that are systematically used throughout, but which are not as well known as they should be. Most notable among these are the general theory of integral manifolds (for general references, see Hale [1]), the center manifold theorem (see Kelley [1]), the theory of transformation to normal forms (for references, see Bibikov [1], Br′juno [1], Henrard [1]) and the general method of averaging (for references, see Hale [1]).

The author is endebted to many colleagues and students whose ideas have been incorporated into these notes—too many to mention by name. He also acknowledges the initiative of Professor Laksmikantham in proposing the CBMS Regional Conference. Finally, Sandra Spinacci has exhibited her usual patience and understanding in the preparation of the final manuscript.

2. On the definition of bifurcation

Suppose X, Z are topological spaces, $U \subset X$ is open, Λ is an open set in a topological space and $f: U \times \Lambda \longrightarrow Z$ is a given continuous function. Let

$$S = \{(x, \lambda) \in U \times \Lambda: f(x, \lambda) = 0\}$$

be the set of solutions of the equation $f(x, \lambda) = 0$. For a fixed λ, let

$$S_\lambda = \{x: (x, \lambda) \in S\}$$

be the "cross-section" of the solution set at λ.

The basic problem is to discuss the dependence of the set S_λ on λ. In a specific problem, one has a prescription which compares S_λ with S_μ for different λ and μ. This comparison is usually made by means of an equivalence relation which divides the sets $\{S_\lambda, \lambda \in \Lambda\}$ into equivalence classes. Given the function f and an equivalence relation $\sim$, we say λ_0 is a *bifurcation point for* $(f, \sim)$ if, for any neighborhood V of λ_0, there are $\lambda_1, \lambda_2 \in V$ such that $S_{\lambda_1} \nsim S_{\lambda_2}$. This definition is less general than the one in Marsden [1].

A special case is when the equivalence relation specifies that $S_\lambda \sim S_\mu$ if the sets S_λ and S_μ are homeomorphic. This is a very convenient choice when studying the change in the structure of the set of equilibrium points in a differential equation as parameters are varied. In this case, the function f represents the vector field in a differential equation $dx/dt = f(x, \lambda)$. It is also appropriate in differential equations for the study of the set of solutions of some prescribed type; for example, periodic solutions, invariant tori, etc. In this latter case, the topological spaces are defined so that they include only those functions which exhibit this prescribed behavior and the function f could be the differential operator, $f(x, \lambda) = dx/dt - g(x, \lambda)$.

To study more general bifurcations in differential equations, the equivalence defined by homeomorphism is not sufficient. Consider a differential equation $du/dt - g(u, \lambda) = 0$ where $(u, \lambda) \in \Omega \times \Lambda$ and Ω is an open set in some Banach space E. For X, Z Banach spaces of functions from $[0, \infty)$ to E, let $U \subset X$ be defined by $U = \{u \in X: u(t) \in \Omega, t \in [0, \infty)\}$. The above equation can be written formally as $f(u, \lambda) = 0$ where $f: U \times \Lambda \longrightarrow Z$, $f(u, \lambda)(t) = du(t)/dt - g(u(t), \lambda)$. Assuming everything can be made rigorous and that all solutions are obtained in this way, a comparison of the corresponding sets S_λ and S_μ by homeomorphism will not be very interesting. Thus an alternative approach must be taken.

Suppose the differential equation generates a strongly continuous semigroup $T_\lambda(t)$, $t \geqslant 0$, on Ω. A frequently used concept of equivalence in differential equations is to say that $g(\cdot, \lambda) \sim g(\cdot, \mu)$ if there is a homeomorphism $h: \Omega \longrightarrow \Omega$ such that h maps orbits of $T_\lambda(t)$ onto orbits of $T_\mu(t)$ preserving the sense of direction in time. A vector field $g(\cdot, \lambda_0)$ is structurally stable if there is a neighborhood V of λ_0 such that $g(\cdot, \lambda) \sim g(\cdot, \lambda_0)$ for all $\lambda \in V$. Thus, λ_0 is a *bifurcation point* if λ_0 is not structurally stable.

A different but equivalent formulation of the above concept of equivalence in differential equations was introduced by Andronov and Pontrjagin [1] in 1937 for differential equations in the plane. They gave a characterization of the structurally stable vector fields which will be discussed later. Peixoto [1] generalized these results to arbitrary compact two dimensional manifolds and proved the set of structurally stable vector fields is open and dense. For some time, it was the feeling that this same property should hold true for arbitrary systems. Unfortunately, it was shown by Smale [1] that structurally stable vector fields are not dense in dimension $\geqslant 4$. Williams [1] proved the same result for $n \geqslant 3$. Since many vector fields cannot be compared by this equivalence relation, it becomes necessary to weaken the concept of equivalence. Each new definition of equivalence leads to a new type of stable vector field (ones which are equivalent to everyone in a neighborhood of it) and thus a new type of bifurcation. The ultimate goal is to have the definition restrictive enough to permit classification of the stable ones, but, at the same time, to have the stable vector fields generic; that is, the intersection of a countable sequence of open dense sets. Much of the research in finite dimensional abstract dynamical systems in the last twenty years has been devoted to this general problem. Relevant references are Smale [2], Peixoto [2], [3], Palis and Melo [1], Newhouse [1], Nitecki [1], Shub [1], Guckenheimer [1], Arnol'd [1]. In the next section, we give more specific details.

When the evolutionary equation is infinite dimensional, several new problems arise. This case will be discussed in a later section.

3. Structural stability and generic properties in $\mathbf{R}^n$

Suppose Ω is an open set in $\mathbf{R}^n$ with $\partial\Omega = \Gamma$, $\bar{\Omega} = \Omega \cup \Gamma$.

The space $C^r(\bar{\Omega}, \mathbf{R}^n)$ is the Banach space of functions bounded and continuous together with all derivatives up through order $r \geq 0$ with the norm of f in $C^r(\bar{\Omega}, \mathbf{R}^n)$ being given by the maximum of the supremum over $\bar{\Omega}$ of the norm of f and its derivatives up through order r. Let $X_n^r = X_n^r(\bar{\Omega})$ be the set of elements of $C^r(\bar{\Omega}, \mathbf{R}^n)$ which are transversal to Γ. For any $f \in X_n^r$, $r \geq 1$, the differential equation

$$\dot{x} = f(x) \tag{3.1}$$

defines a family of transformations $T_f(t)$ on $\bar{\Omega}$ satisfying the semigroup property with $T_f(t)x_0 = x(t, x_0)$, where $x(t, x_0)$ is the solution of (3.1) with $x(0, x_0) = x_0$. Furthermore, for each $x_0 \in \bar{\Omega}$, there are an $\alpha_{x_0} \leq 0$, $\beta_{x_0} \geq 0$, such that the maximal interval of definition of $T_f(t)x_0$ is $[\alpha_{x_0}, \beta_{x_0})$. The number α_{x_0} is either that value α_{x_0} where $T_f(\alpha_{x_0})x_0 \in \partial\Omega = \Gamma$ or $-\infty$ and in this case the interval $[\alpha_{x_0}, \beta_{x_0})$ is $(-\infty, \beta_{x_0})$. The number β_{x_0} is defined in a similar way in the positive direction. The operator $T_f(t)$ on $\bar{\Omega}$ satisfies $T_f(0) = I$, the identity, $T_f(t + s)x = T_f(t)T_f(s)x$ for those t, s for which it is meaningful and $T_f(t)x$ has continuous derivatives up through order r in t, x.

The *orbit* $\gamma_f(x)$ *of f through x* is

$$\gamma_f(x) = \bigcup \{T_f(t)x,\ t \in [\alpha_x, \beta_x]\}.$$

The *ω-limit set* $\omega_f(x)$ and *α-limit set* $\alpha_f(x)$ *of the orbit* $\gamma_f(x)$ are defined by

$$\omega_f(x) = \bigcap_{\tau \geq 0} \text{cl} \bigcup_{t \geq \tau} T_f(t)x, \qquad \alpha_f(x) = \bigcap_{\tau \leq 0} \text{cl} \bigcup_{t \leq \tau} T_f(t)x.$$

An *equilibrium point* or *critical point* of f is a zero of f. A *periodic orbit* of f is an orbit which is a closed curve. A set $M \subset \bar{\Omega}$ is *invariant* if, for each $x \in M$, $T_f(t)x$ is defined for $t \in (-\infty, \infty)$ and belongs to M for $t \in (-\infty, \infty)$. This implies $T_f(t)M = M$ for $t \in (-\infty, \infty)$.

The vector fields in (3.1) are chosen from X_n^r; that is, are transversal to Γ, in order to eliminate technical difficulties with points of contact on Γ. We are discussing the vector fields in $\mathbf{R}^n$, but many of the remarks hold for vector fields on compact manifolds M.

DEFINITION 3.1. Two vector fields f, g in X_n^r, $r \geq 1$, are *equivalent*, $f \sim g$, if there is a homeomorphism $h: \bar{\Omega} \longrightarrow \bar{\Omega}$ such that h maps the orbits defined by f homeomorphically onto the orbits defined by g with the sense of direction in time preserved. An $f \in X_n^r$

is said to be ***structurally stable*** if there is a neighborhood U of f such that $f \sim g$ for every $g \in U$. An $f \in X_n^r$ is a *bifurcation point* if f is not structurally stable.

Two important remarks need to be made about this definition. Definition 3.1 would not be meaningful without the condition $r \geqslant 1$. In fact, for $r = 0$, given any vector field f that has an isolated zero at x_0 and any $\epsilon > 0$, there are a $\delta > 0$ and a function g such that $|f - g| < \epsilon$ and $g(x) = 0$ for $|x - x_0| < \delta$. Therefore, no f with an isolated zero could be structurally stable.

In Definition 3.1, it is tempting to require that the mapping h be a diffeomorphism. However, if $f(0) = 0$, $g(0) = 0$, $\partial f(0)/\partial x = A$, $\partial g(0)/\partial x = B$, and $f \sim g$ in a neighborhood of zero, then one can show (see Peixoto **[2]**, **[4]**) that the eigenvalues of A and B must be proportional. Since one can always make a small perturbation that will change one eigenvalue of A and not the other, it follows that no vector field with a zero could be structurally stable. Thus, the Definition 3.1 would have little meaning. If x_0 is a critical point of f and $A = \partial f(x_0)/\partial x$, then x_0 is said to be *hyperbolic* if the real parts of the eigenvalues of A have nonzero real parts. The point x_0 is a *saddle point of order k*, if it is hyperbolic and there are k eigenvalues of A with positive real parts. The term *saddle point* without the designation of the order will refer to any saddle point of order k with $k \neq 0$ or n.

If $n = 2$, a saddle point of order 1 corresponds to the usual definition of saddle point. For $n = 2$, a saddle point of order 0 or 2 corresponds to a node or focus depending upon whether the eigenvalues of A are real or complex.

If γ is a *periodic orbit* of f, then one can define a Poincaré map near γ in the following way. For any arc C transversal to γ at p_0 and any $p \in C$ sufficiently near p_0, there is a unique $\tau(p) > 0$ such that $T_f(\tau(p))p \in C$, $T_f(t)p \notin C$ for $0 < t < \tau(p)$. The map $p \mapsto T_f(\tau(p))p$ is called the Poincaré map $\pi(p)$. This map in C^r and $\pi(p_0) = p_0$. The periodic orbit γ is *hyperbolic* if no eigenvalue of $\partial\pi(p_0)/\partial p$ has modulus one.

If $n = 2$, the periodic orbit γ is hyperbolic if $d\pi(p_0)/dp \neq 1$. It is instructive to give an equivalent definition in terms of the vector field itself. If $\gamma = \{\phi(t), t \in \mathbf{R}\}$ where $\phi(t)$ is periodic of least period ω and $\dot{\phi}(t) = f(\phi(t))$ then the *linear variational equation* for ϕ is

$$\dot{y} = A(t)y, \qquad A(t) = \partial f(\phi(t))/\partial x. \tag{3.2}$$

One characteristic multiplier of this ω-periodic system is 1 since $\dot{\phi}$ satisfies (3.2). If $X(t)$ is a principal matrix solution of (3.2), then the product of the multipliers is equal to det $X(\omega)$. Thus, if $\rho_\gamma = \exp \omega\sigma_\gamma$, σ_γ real, is the other multiplier, then

$$\sigma_\gamma = \frac{1}{\omega}\int_0^\omega \operatorname{tr} A(s)\,ds. \tag{3.3}$$

One can then easily show that γ is hyperbolic if and only if $\sigma_\gamma \neq 0$, unstable if $\sigma_\gamma > 0$ and asymptotically orbitally stable if $\sigma_\gamma < 0$.

For two dimensional systems, the following result of Andronov and Pontrjagin **[1]**, Peixoto **[1]**, completely solves the problem of structural stability in X_2^r.

THEOREM 3.2. *If* $\Sigma_2^r \subset X_2^r$, $r \geq 1$, *is the set of structurally stable vector fields in* X_2^r, *then* $f \in \Sigma_2^r$ *if and only if the following conditions are satisfied*:

(i) *The critical points of f are hyperbolic.*

(ii) *The periodic orbits of f are hyperbolic.*

(iii) *There is no orbit of f with both the* α- *and* ω-*limit sets being saddle points.*

Furthermore, Σ_2^r *is open and dense in* X_2^r.

The fact that a structurally stable vector field must satisfy (i)–(iii) is very easy to prove. However, the converse is more difficult and relies heavily upon the following result of Hartman [1], [3] Grobman [1], and its extension to diffeomorphisms which is valid in the space of n-dimensional vector fields X_n^r.

THEOREM 3.3 (HARTMAN-GROBMAN). *If* $f \in X_n^r$, $r \geq 1$, $f(x_0) = 0$, *and the eigenvalues of* $A = \partial f(x_0)/\partial x$ *have nonzero real parts then, in a neighborhood of* x_0, f *is equivalent to the linear equation* $\dot{x} = Ax$.

In Theorem 3.2, the fact that X_2^r is open follows from the definition and the fact that it is dense follows from an argument in transversality theory. See Peixoto [1] for a complete proof.

Condition (i), the Implicit Function Theorem and the compactness of $\bar{\Omega}$ imply that $f \in \Sigma_2^r$ has only a finite number of critical points. Using (ii), (iii) and similar arguments, one shows there is only a finite number of periodic orbits.

The simplicity of the description of the structurally stable systems in two dimensions given by Theorem 3.2 permits a complete classification in terms of certain distinguished graphs (see Peixoto [4]).

To what extent does Theorem 3.2 hold in dimension $n \geq 3$? As remarked earlier, the *structurally stable systems are not dense in* X_n^r *for* $n \geq 3$. This was proved by Smale [1] for $n \geq 4$ and by Williams [1] for $n \geq 3$. However, there are structurally stable systems in every dimension and on every type of n-dimensional manifold.

Even though Σ_n^r is not dense, it is very important to classify structurally stable vector fields and to find "simple" classes of vector fields which are generic. Let us turn first to the problem of genericity.

The concepts (i), (ii) in Theorem 2 have meaning in $\mathbf{R}^n$. Also, (iii) can be extended in the following way. For any hyperbolic critical point or periodic orbit of a vector field $f \in X_n^r$, one can define the global stable and unstable manifolds in the following way. The stable (unstable) manifold of a hyperbolic critical point x_0 is the set of $x \in \bar{\Omega}$ such that $T_f(t)x \longrightarrow x_0$ as $t \longrightarrow +\infty$ $(-\infty)$. Similar definitions are given for a periodic orbit.

In $\mathbf{R}^2$, condition (iii) is then equivalent to the statement that the stable and unstable manifolds of all critical points and periodic orbits intersect transversally. One can then ask if the vector fields in X_n^r which satisfy these properties are generic in X_n^r. The answer is yes and is the famous theorem of Kupka [1] and Smale [3].

THEOREM 3.4 (KUPKA-SMALE). *The set of vector fields in X_n^r for which the critical points and periodic orbits are hyperbolic with stable and unstable manifolds intersecting transversally is generic.*

Any vector field satisfying the conditions of Theorem 3.4 will be called a Kupka-Smale (KS) vector field. They can have only a finite number of critical points with the proof being the same as in two dimensions. However, in contrast to two dimensions, there can be an infinite number of periodic orbits if the dimension is $\geqslant 3$ (for an example, see Nitecki [1], Palis and deMelo [1]).

The KS vector fields are dense, but all KS vector fields cannot be structurally stable since the structurally stable systems are not dense in dimension $\geqslant 3$. To find a subset of the KS vector fields which are structurally stable, one must put some further restrictions on the behavior of the α- and ω-limit sets of orbits.

For $f \in X_n^r$, let

$$L_\alpha(f) = \{p\colon p \in \alpha(q) \text{ for some } q\}, \qquad L_\omega(f) = \{p\colon p \in \omega(q) \text{ for some } q\}.$$

DEFINITION 3.5. Suppose $f \in X_n^r$. A point $p \in \Omega$ is a *wandering point* of f if there are a neighborhood V of p and $t_0 > 0$ such that if $|t| > t_0$, then $T_f(t)V \cap V = \emptyset$. In the contrary case, p is a *nonwandering point of f.* The *set of nonwandering points of f* is denoted by $\Omega(f)$.

In Definition 3.5, the notation $|t| > t_0$ means for all $t \geqslant t_0$ and all $t < -t_0$ as long as the orbit is defined.

We remark that $\Omega(f) \supset L_\alpha(f) \cup L_\omega(f)$, but it is easy to construct examples for which the inclusion is proper (see, for example, Palis and deMelo [1]).

DEFINITION 3.6. A vector $f \in X_n^r$ is *Morse-Smale (MS)* if it is KS with a finite number of critical points and periodic orbits with $\Omega(f)$ equal to the set of critical points and periodic orbits.

Some of the basic results on Morse-Smale systems are due to Smale [4], Palis [1] and Palis and Smale [1]. They are summarized in the following theorem which is also valid for vector fields on any compact manifold.

THEOREM 3.7. (1) *The set of MS systems is open and nonempty in X_n^r for any n.*

(2) *Any $f \in MS$ is structurally stable.*

(3) *The set of gradient vector fields which are MS is open and dense in the set of all gradient vector fields.*

Since the MS systems are structurally stable, they cannot be dense in dimension $n \geqslant 3$. On the other hand, one can ask if there are any other structurally stable systems which are not MS. One way to answer this question is to construct a structurally stable system with infinitely many periodic orbits.

To see how such a situation might arise, suppose $\Omega \subset \mathbf{R}^3$ and $f \in X_3^r(\bar{\Omega})$ has a hyperbolic periodic orbit γ. Let $W^s(\gamma)$, $W^u(\gamma)$ be the stable and unstable manifolds for γ and let π be a Poincaré map of some transversal r of γ at p and $W_r^s(\gamma) = W^s(\gamma) \cap r$, $W_r^u(\gamma) = W^u(\gamma) \cap r$;

that is, that part of the stable and unstable manifolds in the transversal r. Then $W_r^s(\gamma)$, $W_r^u(\gamma)$ are the local stable and unstable manifolds of the point p as a fixed point of the diffeomorphism π. There is the possibility that $W_r^s(\gamma) \cap W_r^u(\gamma)$ contains points other than the fixed point p of π. Any such point q is called *homoclinic* to p. A point q is called *transverse homoclinic* to p if $W_r^s(\gamma)$ is transversal to $W_r^u(\gamma)$ at q. If q is *transverse homoclinic* to p, then the behavior of the stable and unstable manifold is very bad. In fact, since $\pi W_r^s(\gamma) \subset W_r^s(\gamma)$, $\pi W_r^u(\gamma) \subset W_r^u(\gamma)$ and $q \in W_r^s(\gamma) \cap W_r^u(\gamma)$, $q \neq p$, we must have $\pi^n q \in W_r^s(\gamma) \cap W_r^u(\gamma)$ for all $n = 0, \pm 1, \pm 2, \ldots$ and $\pi^n q \longrightarrow p$ as $n \longrightarrow \infty$. If, in addition, q is transverse homoclinic to p, continuity of the map π implies that the picture near p must be something like the one in Figure 1. The arrows do not represent the direction of a flow as for vector fields, but only that points move in the direction indicated under iterates of π. In Figure 1, we have only indicated some of the complications that are arising from looking at the forward evolution of the unstable manifold. The same type of thing must occur with the stable manifold. Note that there will be infinitely many transverse intersections in any neighborhood of the homoclinic point q.

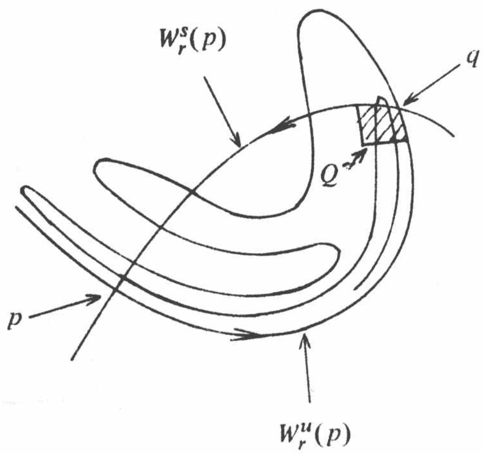

FIGURE 1.

This phenomenon was observed by Poincaré [1]. Birkhoff [1] proved that every transverse homoclinic point is the limit of periodic points (that is, points x such that $\pi^n x = x$ for some integer n) and indicated some of the random behavior that occurs near these points. Smale [1] carried the analysis even further. We briefly describe the results following Moser [2]. If A is a finite or countably infinite sequence of symbols, let S be the collection of doubly infinite sequences $s = \{s_k, k = 0, \pm 1, \ldots\}$ with each $s_k \in A$. The shift automorphism σ on S is defined by $\sigma s = \{\tilde{s}_k, k = 0, \pm 1, \ldots\}$, $\tilde{s}_k = s_{k-1}$ for all k.

Near a homoclinic point q, we can construct a small quadrilateral Q, two of its sides consisting of parts of $W_r^u(p)$, $W_r^s(p)$ and the others parallel to the tangents of these sets at q (see Figure 1). For any point $\alpha \in Q$, let $k = k(\alpha)$ be the smallest positive integer such that

$\pi^k(\alpha) \in Q$, if it exists. Let $D(\tilde{\pi})$ be the set of $\alpha \in Q$ for which such a k exists and define $\tilde{\pi}\alpha = \pi^k(\alpha)$ for $\alpha \in D(\tilde{\pi})$. The map $\tilde{\pi}$ is called the *transversal map* of π for the quadrilateral Q.

THEOREM 3.8. *If π is a C^∞-diffeomorphism of the plane with a point q transverse homoclinic to a hyperbolic fixed point p, then in a neighborhood of q, the transversal map $\tilde{\pi}$ of a quadrilateral possesses an invariant set I homeomorphic to the sequence space S with an infinite number of symbols by a map $\tau: S \longrightarrow I$ such that $\tilde{\pi}\tau = \tau\sigma$. Also, there is an integer k, an invariant set $\tilde{I}$ of π^k and a homeomorphism $\tilde{\tau}: S \longrightarrow \tilde{I}$, where S is the sequence space of a finite number of symbols, such that $\pi^k\tilde{\tau} = \tilde{\tau}\sigma$.*

Note the difference in the two conclusions in the theorem. In the first part, the set I is invariant for $\tilde{\pi}$ and $\tilde{\pi}$ is equivalent on I to the shift automorphism on an infinite number of symbols. In the second part, the set $\tilde{I}$ is invariant under a fixed power k of π itself and π^k is equivalent on $\tilde{I}$ to the shift automorphism on a finite number of symbols.

It follows immediately from Theorem 3.8 that there are infinitely many periodic points in a neighborhood of the transverse homoclinic point and they are dense. Also, there is a random behavior to the orbits on the invariant set I (or $\tilde{I}$) since knowing the early terms of a sequence tells nothing about the later terms of a sequence.

For examples of transverse homoclinic points in celestial mechanics, see Moser [2]. Transverse homoclinic points also occur in structurally stable systems as we shall see below. More examples in second order nonautonomous differential equations will be given later when we are studying analytical methods in bifurcation theory. Now, we prefer to continue the general survey.

To describe further aspects of the theory, it is convenient to work with $\mathrm{Diff}^r(M)$, $r \geqslant 1$, the space of diffeomorphisms with derivatives up through order r on a smooth compact manifold M. This can be related to differential equations in several ways. One of the most important is through a Poincaré map for periodic orbits as described above. More generally, if $f \in \mathcal{X}_n^r(\bar{\Omega})$, $M \subset \bar{\Omega}$ is compact and, for each $x \in M$, there is a $\tau(x) > 0$ such that $T_f(\tau(x))x \in M$, $T_f(t)x \notin M$, $0 < t < \tau(x)$, then the map $x \mapsto T_f(\tau(x))x$ is in $\mathrm{Diff}^r(M)$ if it has the required number of derivatives.

If $g \in \mathrm{Diff}^r(M)$, a point $p \in M$ is a periodic point of g if there is a positive integer $n = n(p)$ such that $T^n p = p$. The periodic orbit is *hyperbolic* if no eigenvalue of $\partial g(p)/\partial x$ has modulus one. For each hyperbolic periodic point p of g, one can define the global stable manifold $W^s(p)$ and unstable manifold $W^u(p)$ in a manner similar to the definitions for vector fields.

We now give an example due to Thom which was an inspiration for many further developments in dynamical systems. In $\mathbf{R}^2$ identify the points (x, y), $(x + m, y + n)$ for all integers m, n. Any unit square with integer vertices may be identified with the torus T^2 and any mapping of the plane into itself yields a mapping of T^2 into T^2 in the obvious way. Let L be a 2×2 matrix with integer coefficients, determinant 1 and real eigenvalues. The eigenvalues are then λ, λ^{-1} with $\lambda < 1$ irrational. This implies that the linear subspaces E^s, E^u generated respectively by the eigenvectors for λ, λ^{-1} have irrational slope. For any

$x \in \mathbf{R}^2$, each of the lines $x + E^s$, $x + E^u$ is invariant under L. The map L on $\mathbf{R}^2$ generates a natural map π on T^2 obtained from the above identification of T^2 with unit squares in $\mathbf{R}^2$ with integer coefficients. If $p = \pi x \in T^2$, $x \in \mathbf{R}^2$, let $W^s(p) = \pi(x + E^s)$, $W^u(p) = \pi(x + E^u)$. If p is a periodic point of π, then $W^s(p)$, $W^u(p)$ are respectively the stable and unstable manifolds of p. Since the slopes of the linear subspaces E^s, E^u are irrational, the sets $W^s(p)$ and $W^u(p)$ are dense in T^2 for every $p \in T^2$. Furthermore, it is not difficult to show that every point of intersection of these sets is a point of transversal intersection. Also, the points of intersection are dense in T^2. In particular, there is a dense set of points transverse homoclinic to the critical point $p = \pi(0)$. Since each transverse homoclinic point is the limit of periodic points, it follows that the periodic points of π are dense in T^2. A simple direct proof of this last result is contained in Palis and deMelo [1, p. 171].

With $\pi: T^2 \longrightarrow T^2$ defined as above, Anosov [1], [2] showed that π is structurally stable. A more elementary proof was given by Moser [1], [2]. Since π is structurally stable and contains infinitely many periodic orbits, this necessarily implies that the Morse-Smale systems are not dense in the set of structurally stable systems.

The above example was generalized by Anosov [2] in the following way.

DEFINITION 3.9. Let M be a compact manifold. An $f \in \mathrm{Diff}^r(M)$, $r \geqslant 1$, is an *Anosov diffeomorphism* if the tangent space at each point x of M is a direct sum $E_x^s \oplus E_x^u$ invariant under the derivative Df; that is, $Df_x E_x^s = E_{f(x)}^s$, $Df_x E_x^u = E_{f(x)}^u$ and there is a Riemannian metric on M and a constant $\lambda \in (0, 1)$ such that $|Df_x v| \leqslant \lambda |v|$, $|Df_x^{-1} u| \leqslant \lambda |u|$ for all $x \in M$, $v \in E_x^s$, $u \in E_x^u$.

Anosov [2] has shown that these diffeomorphisms are structurally stable. A simpler proof was given by Moser [1], [2]. For a discussion of the restrictions that are imposed on the manifold M in order for it to admit an Anosov diffeomorphism, see Palis and deMelo [1].

The next important step in the abstract theory of dynamical systems was taken by Smale [2] by defining systems which satisfy Axiom A. Suppose $f \in \mathrm{Diff}^r(M)$ and $\Lambda \subset M$ is a closed invariant set. The set Λ is said to have a hyperbolic structure if the tangent space at each point $x \in \Lambda$ is the direct sum $E_x^s \oplus E_x^u$ invariant under Df and there are a Riemannian metric and $\lambda \in (0, 1)$ such that $|Df_x v| \leqslant \lambda |v|$, $|Df_x^{-1} u| \leqslant \lambda |u|$ for $x \in \Lambda$, $v \in E_x^s$, $u \in E_x^u$. If Λ is hyperbolic, it is possible to define stable and unstable manifolds for the set Λ by looking at asymptotic orbits.

DEFINITION 3.10. $f \in \mathrm{Diff}^r(M)$ satisfies *Axiom A* if the set of nonwandering points $\Omega(f)$ is hyperbolic and the periodic points of f are dense in $\Omega(f)$.

If f satisfies Axiom A, Smale [2] has shown that $\Omega(f) = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$ where each Ω_j is closed invariant and transitive; that is, has a dense orbit. Robinson [1] has shown that any $f \in \mathrm{Diff}^r(M)$, $r \geqslant 1$, is structurally stable if it satisfies Axiom A and all stable and unstable manifolds intersect transversally. An $f \in \mathrm{Diff}^r(M)$ is said to be *absolutely stable* if there are a neighborhood $V(f) \subseteq \mathrm{Diff}^r(M)$ of f and a constant $K > 0$ such that, for every $g \in V(f)$, there is a homeomorphism h of M such that $hf = gh$ and $|h - I|_0 < K|f - g|_0$ where $|\cdot|_0$ designates the norm in C^0. Results of Franks [1], Guckenheimer [1] and Mañé [1] show that f is absolutely stable if and only if it satisfies Axiom A and all stable and unstable manifolds intersect transversally. Mañé (unpublished) has also recently shown that Axiom A is implied by structural stability and a technical condition on the characteristic exponents of Liapunov on $\Omega(f)$.

It is also possible to study structural stability restricted to the set of nonwandering points. More specifically, $f \in \mathrm{Diff}^r(M)$ is said to be *Ω-stable* if there is a neighborhood $V(f)$ of f such that, for every $g \in V(f)$, there is a homeomorphism $h: \Omega(f) \longrightarrow \Omega(g)$ such that $hf = gh$ on $\Omega(f)$. If f satisfies Axiom A, $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_k$, then a *cycle* of Ω is a sequence $p_1 \in \Omega_{k_1}, \ldots, p_s \in \Omega_{k_s} = \Omega_{k_1}$ such that $W^s(p_i) \cap W^u(p_{i+1}) \neq \varnothing$, $1 \leqslant i \leqslant s-1$. Smale [2] showed that Axiom A and no cycles imply Ω-stable. Palis [2] has shown that any f satisfying Axiom A is not Ω-stable if it has a cycle. It is not known if f Ω-stable implies that it must satisfy Axiom A.