ASTROPHYSICS I

STARS

Richard L. Bowers

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LOS ALAMOS NATIONAL LABORATORY

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Preface to Volume I

Student interest in astronomy and astrophysics has grown dramatically during the past decade, and from it has sprung a need for modern texts reflecting the advances in theoretical and observational astronomy of the sixties and seventies. This need has previously been met largely at the introductory (descriptive) level, and at the advanced level. *Astrophysics* is intended to fill the intermediate range. Much of the material that follows developed from course material and lectures presented over a five-year period to upper-level undergraduate and graduate students in the Department of Astronomy and the Department of Physics at the University of Texas at Austin and from a one-semester course presented in the Department of Physics and Astronomy at Texas A & M University.

The text is intended for a senior-level or first-year graduate-level course in astrophysics. Volume I covers a wide range of subjects in stellar astrophysics, and Volume II treats nonstellar astrophysics. We attempt to present a relatively self-contained discussion of core topics that are established and basic to advanced work in the field. More speculative topics that have not been fully explored theoretically, or that do not rest on unambiguous observational data, have been excluded. We did so not because these topics are uninteresting but simply because of length limitations.

We have chosen to emphasize the theoretical approach to astrophysics without attempting to cover the extensive field of observational methods. Nevertheless, ample reference is made to the results of observations and to specific astrophysical systems that confirm or restrict the theory. Detailed models of most astrophysical systems involve a variety of underlying physical processes and principles, and their analysis must often be carried out on high-speed computers. Because an extensive literature on numerical methods already exists, and because their use often involves problems specific to the individual model, we do not discuss computational methods. Instead, we emphasize simple analytic models wherever possible. The results of numerical analysis are, however, incorporated as illustrations.

The subject matter is presented at a level for senior students in physics, astronomy, or physical science who have completed undergraduate course work in mechanics, modern physics, electromagnetic theory, and calculus through differential equations, or for first-year graduate students. A knowledge of descriptive astronomy as developed in any standard introductory astronomy text is also assumed.

The subject matter is organized along more or less conventional lines. The first volume, which deals with stellar astrophysics, develops many topics that are used in the parts on nonstellar astrophysics in the second volume. Part 1, particularly Chapter 1, presents a relatively nonmathematical overview of topics to be covered in more detail in the remaining parts. It serves two primary purposes. The first is to establish a framework within which the reader can structure his knowledge of astronomy and astrophysics. The second is to establish an appreciation at the order-of-magnitude level of basic astrophysical parameters, which serve as benchmarks for more detailed discussions that follow. The first chapter emphasizes order-of-magnitude arguments that, though they may often be mathematically naive, illustrate how basic physical concepts can be used to extract qualitative estimates of astrophysical parameters without recourse to elaborate numerical or analytical methods.

The remaining parts of Volume I develop in a more mathematical way the astrophysics of stellar structure and stellar evolution. Volume II introduces the astrophysics of interstellar matter, the structure and evolution of galaxies and stellar systems, and cosmology.

Problems have been included in the text to serve three purposes: (1) to supply order-of-magnitude values obtained from models developed in the text; (2) to extend the discussion once basic concepts have been presented; and (3) to supply details of derivations whose results are used in subsequent discussions. In the last case, sufficient guidance is given so that these derivations are straightforward.

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Part I

INTRODUCTION

Chapter 1

AN OVERVIEW OF STELLAR STRUCTURE AND EVOLUTION

1.1. STARS

A star is a self-gravitating ball of gas, radiating energy into space. This energy is produced mainly by thermonuclear reactions taking place in the deep interior, but may also be released during contraction or collapse of the stellar core. The star must produce energy in order to maintain enough internal pressure to support itself against its own gravitational field. Stellar structure and evolution are controlled by these two opposing effects: gravity, which tends to collapse the star, and pressure, which tends to expand it. Sometimes one or the other wins slightly, and the star may expand or contract, but there is no doubt which will win in the end. Gravitational collapse must ultimately turn the star into a cold, dead object, like the white-dwarf companion to Sirius, or perhaps into a neutron star, like the pulsar in the Crab nebula, or possibly into a black hole. It may also set off the explosive event we call a supernova. Gravitation is also responsible for the initial formation of stars out of protostellar material. As soon as a large-enough mass of this material becomes detached from its surroundings, it begins to contract, and one or more stars may form from it. Gravitation is the dominant creative force in the universe, and, because of gravitational collapse, it is the dominant destructive force as well.

Stellar Atmospheres

The Sun is a typical star, although we see far more detail on it than on any other star. When we look at a typical star, we see only the outermost layers, because stellar surface material is relatively opaque. These outer regions (the atmosphere) are the only parts generally accessible to direct observation.

Our primary source of information about a star is the light emitted from its surface. For example, from a star's visible light we can measure its total luminosity, and therefore its total energy output. We can examine the star's spectrum, which is its energy output as a function of wavelength or frequency. The analysis of stellar spectra, which makes possible the study of stellar atmospheres, is fundamental to stellar astrophysics. Such analysis gives us information on the physical structure of the atmosphere, for example, the temperatures, pressures, and densities encountered, and may indicate the presence of turbulence, convection, or magnetic fields. It enables us to deduce the chemical composition of the atmosphere. Finally, some large-scale properties of a star, such as its surface gravity and rotational velocity, can often be calculated. Needless to say, it is for the Sun that these quantities are known most accurately. However, the Sun is not an especially convenient object on which to test our understanding of stellar atmospheres, because its outer layers are turbulent, and current theories and mathematical techniques are not quite capable of handling turbulence in atmospheres.

Stellar interiors are in some ways simpler than stellar atmospheres, in other ways more complicated. The most important complication is that they are generally unobservable, although detection of neutrinos escaping directly from the core of the Sun and other stars could offer indirect observational information on their interior structure. In general, therefore, we rely heavily on theory for conclusions about stellar interiors. The theory of stellar interiors attempts to relate the intrinsic properties of the star (total mass, chemical composition, and possibly rotation) to its gross observable properties (luminosity, surface gravity, surface temperature).

We would also like to know how these quantities are likely to change with time; for example, is the Sun likely to explode as a supernova tomorrow, in a few billion years, or at all? Astronomers refer to the way that the properties of a single star change with time as *stellar evolution*. The word evolution is used differently here than it is in biology, where it refers to changes in the general characteristics of a species after several generations of individual births and deaths. This kind of change from generation to generation is also encountered in astrophysics. For example, most stars probably lose mass during some stage of their evolution. The process may be gradual (stellar winds, planetary nebula formation), more dramatic (novae, mass exchange in binary systems), or cataclysmic (supernovae). In any case, some of the ejected material may have undergone significant changes in composition through nucleosynthesis. In this way the interstellar gas is enriched, and the evolution of subsequent generations of stars will be modified. The two types of evolution are therefore related.

The time-scale on which stellar evolution takes place is obviously very long. How can we hope to verify any theories about stellar evolution when most changes require many lifetimes? The answer lies in the multitude of stars in the sky.

Although we can never hope to observe significant changes in more than a very few stars (such as variable stars, novae, or supernovae), we can say that star A is probably what star B looked like a few billion years ago, or will look like a few billion years hence. That is, we can try to relate our predictions about stellar evolution to the various types of stars we see around us, and thus be able to say that such and such a star is an object of mass M, and some (initial) chemical composition, in some stage of evolution. This line of attack is aided by the fortunate existence of star clusters whose members are so obviously physically associated that we can be confident that they all began with nearly the same initial composition at essentially the same time, and are therefore all of the same age. The major differences between them should result from the differences in their masses (and possibly from their rotational velocities and magnetic fields).

1.2. Energy Transport and Generation in Stars

One way in which stellar interiors are simpler than stellar atmospheres is in the way energy is transported from point to point. This is called *energy transfer*. Most of the time energy is transported by electromagnetic radiation. In the deep interior, the photon's mean free path is extremely small compared with the dimensions over which other stellar variables change; and this radiation has an almost isotropic distribution. Consequently, energy transport can be adequately described by the diffusion approximation. This fact greatly simplifies the analysis of radiative energy transport.

In the atmosphere, on the other hand, the radiation field is strongly anisotropic (or else no radiation would escape the star), the opacity is strongly wavelength-dependent (otherwise we would see no characteristic line spectra), and the photon mean free path is long relative to the scale height. The problem of radiative transfer in a stellar atmosphere is therefore complicated and difficult. At some stages of evolution, if the opacity of the stellar material becomes too high, the temperature gradient too large, or both, the star becomes unstable, and convection sets in. This represents an additional means of transferring energy from point to point and may cause chemical mixing. In this way, some of the chemical inhomogeneities that develop in a star may be ironed out. If convection occurs near the surface of a star (as it does in the Sun), it may extend into the visible part of the stellar atmosphere and produce additional complications, such as solar granulation, sunspots, or flares.

Stellar Energy Generation

Long before nuclear and thermonuclear energy became of interest on the Earth, astronomers realized that the source of stellar energy must be subatomic. The principal subatomic source of stellar energy is the thermonuclear conversion of hydrogen to helium in the deep interior; this is true for most stars for most of their lifetimes. Sometimes other sources of thermonuclear energy become important; sometimes gravitational energy released during collapse, or even neutrino energy release, is important. Nevertheless, it is the conversion of hydrogen to helium that establishes the time-scale of a star's life. The nuclear parameters for this reaction determine the time-scale on which the stars evolve, and thereby help to determine the time-scale of the universe.

Energy Loss (Luminosity)

The rate of energy loss from a star is important for its structure. It can be calculated for the Sun by measuring the amount of solar energy falling on a unit area above the Earth's absorbing atmosphere. From this quantity, known as the solar constant (1.374×10^6 erg $cm^{-2}sec^{-1}$), and the mean distance of the Sun from the Earth, which is defined as one astronomical unit or A.U. (one A.U. $= 1.496 \times 10^{13}$ cm), it is easily shown that the energy output for the sun is $L_\odot = 3.86 \times 10^{33}$ erg sec^{-1}. The solar luminosity $L_\odot$ represents a standard unit of energy-loss rates in all branches of astrophysics.

Problem 1.1. What collecting area for solar radiation is required to light a 100-watt light bulb, if solar energy can be converted to electrical energy with a 100 percent efficiency?

Problem 1.2. By what percentage does the solar "constant" vary during the year because of the eccentricity ($e = 0.017$) of the Earth's orbit around the Sun?

The *absolute luminosity* of a star refers to the total electromagnetic energy output at all wavelengths, ignoring the possibility that energy is being output in other forms (e.g., particles, neutrinos, gravitons). This luminosity is called the *bolometric luminosity*. In practice, we can not observe all wavelengths of the spectrum. Instead the absolute luminosity is obtained by observing the object in a restricted part of the electromagnetic spectrum, and then applying a "bolometric correction" to allow for the unobserved wavelengths.

Related to the absolute luminosity of a star is the star's effective temperature, which is defined in the following way. A unit area of a perfect radiator or *black body* at an absolute temperature T radiates energy at a rate

$$\sigma T^4 \text{ erg cm}^{-2} \text{ sec}^{-1}, \tag{1.1}$$

where σ is the Stefan-Boltzmann constant. A star does not have a well-defined surface or a well-defined surface temperature, but if it did, and if the stellar surface were a black body, the total luminosity would be

$$L = 4\pi\sigma R^2 T^4. \tag{1.2}$$

We can use this relation to define an effective temperature, T_{eff} such that, for a real star with absolute luminosity L and radius R,

$$L = 4\pi\sigma R^2 T_{eff}^4. \tag{1.3}$$

Real stars, not being black bodies, will have effective temperatures that are not the same as the atmospheric temperatures deduced by spectral analysis. Nevertheless, the differences are generally not great—perhaps a few hundred degrees—so effective temperatures can be taken as close to atmospheric temperatures for many purposes.

Problem 1.3. What is the effective temperature of the Sun? (Answer: about 6,000 K)

1.3. Stellar Time-scales

We said above that self-gravitation in effect determines the state and evolution of a star. How do we know this? To put the question more specifically, suppose there were no internal support for the Sun. How long would it take for the Sun's self-gravity to cause a significant change in its radius? We can arrive at an order-of-magnitude estimate of the answer in two ways; both are simple, instructive, and typical of many arguments we will make in later sections.

Free-fall Time-scale

First, we ask what physical quantities the answer will depend on. Since we are talking about gravitational free fall of the star, with no restraining forces, the answer depends only on the strength of the gravitational field and on the physical dimensions involved. The time-scale must therefore be a function only of the mass M and radius R of the star, and of the Newtonian gravitational constant G, which, given the mass and the radius, determines the strength of the field. Now, there is only one expression with the physical dimensions of time that can be constructed from G, M, and R. It is

$$t \sim (R^3/GM)^{1/2}. \tag{1.4}$$

Problem 1.4. Prove this by showing that the dimensions of $(R^3/MG)^{1/2}$ are time. The force law $F = -M^2G/R^2$ gives the units of G.

The answer must consist of some dimensionless constant multiplied by expression (1.4). It is a result of experience (and to some extent a tenet of faith) that dimensionless multiplying factors are usually of the order of magnitude of unity (i.e., about one); so if we were to take expression (1.4) by itself as the answer to the problem, we would usually not be wrong by more than an order of magnitude.

The other approach is possible when we can write down a mathematical model of what is happening physically. Since we assume that there is no internal pressure in the star, a small mass m at the surface will fall unhindered along a path $r(t)$ that satisfies the differential equation

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}, \tag{1.5}$$

where $r = R$ at $t = 0$. We can solve this differential equation directly, or with more insight say that during the fall of the particle over a distance R in a time t, the average value of d^2r/dt^2 must be $\approx -R/t^2$. Therefore we expect

$$-\frac{R}{t^2} \approx -\frac{MG}{R^2} \tag{1.6}$$

and hence

$$t \sim (R^3/GM)^{1/2}, \tag{1.7}$$

which is the same result as (1.4).

Problem 1.5. Imagine that equation (1.5) is the correct equation of motion for this problem, and that the surface begins with $r = R$ at $t = 0$. How long does it take for the star to collapse to half its initial size? How does this compare to the simpler estimate?

With mass and radius measured in solar units, $M_\odot = 1.99 \times 10^{33}$ g and $R_\odot = 6.96 \times 10^{10}$ cm, this time is easily shown to be

$$t_{ff} = 1.59 \times 10^3\,(M/M_\odot)^{-1/2}\,(R/R_\odot)^{3/2}\ \text{sec}. \tag{1.8}$$

For the Sun this is about 0.44 hrs, or 26.6 min. We have written t_{ff} to identify this as the free-fall time-scale.

Notice that the same result applies roughly to any situation in which gravitational fields arise from a mass M moving a distance scale R in a time t. For instance, the orbital period of the Earth about the Sun is given roughly by the same expression where the distance of the Earth from the Sun is used for R (Kepler's third law).

Problem 1.6. Think of some other astronomical situations in which this analysis can be applied, and verify that it gives the correct result.

Notice also that, since M/R^3 is proportional to the mean density, result (1.7) can be written

$$t\,\sqrt{\bar\rho} \approx 1/\sqrt{G}. \tag{1.9}$$

(For the record, the mean density of the Sun is $\bar\rho = 3M/4\pi R^3 = 1.41$ gm cm^{-3}.) This result is important for the theory of pulsating variable stars, where the balance between pressure and gravity is only maintained on average. The characteristic period of oscillation P is related to the mean density of the star by the simple relation $P\bar{\rho}^{1/2} \simeq G^{-1/2}$.

Problem 1.7. Show from the analysis above that $P\bar{\rho}^{1/2} \simeq G^{-1/2}$.

Generally, numerical estimates obtained in this way are reasonable. In any event, the result reveals the way observable properties (in this case, the collapse time) depend on characteristics of the system. For example, result (1.7) indicates that collapse time decreases slowly with increasing mass, but increases more rapidly with increasing initial radius.

Problem 1.8. Suppose that two pulsating stars have equal masses but different radii. How would you expect their periods to compare?

Clearly the free-fall time for the Sun is very fast by astronomical standards. It is much less than the evolutionary time-scale t_{ev}, which we know (from the existence of objects that old on the Earth) is at least many billions of years:

$$t_{ff} \ll t_{ev}. \tag{1.10}$$

Apparently the Sun can adjust very quickly to an imbalance in the pressure-gravity equilibrium. A star that can do so is, to a high degree of accuracy, in a state of hydrostatic equilibrium. During stages of pulsational instability, equilibrium is maintained in terms of the average over several pulsation periods. When $t_{ff} \sim t_{ev}$, the evolution is rapid, and the concept of hydrostatic equilibrium becomes inapplicable. The hydrostatic-equilibrium condition is one of the basic physical principles we will use to construct a conceptual model of a stellar interior and a stellar atmosphere.

Kelvin-Helmholtz Time-scale

The rate of stellar evolution is determined primarily by the available energy. A major energy reserve is the star's gravitational potential energy. Two masses m and m' a distance r apart have a gravitational potential energy given by the classical expression

$$\Omega = -\frac{Gmm'}{r}. \tag{1.11}$$

The total gravitational potential energy is the integral of this expression over the star. Dimensionally, it must have the form $\alpha M^2G/R$, where M is the total mass, R the radius, and α a factor usually of order unity. If this estimate is applied (with $\alpha = 1$) to the Sun, one finds

$$\begin{aligned}\Omega_{\odot} &= -M^2G/R \\ &\simeq 4 \times 10^{48} \text{ ergs.}\end{aligned} \tag{1.12}$$

Another important time-scale may now be constructed. Suppose the past energy output of the Sun could all have come from the release of gravitational potential energy during contraction from an initially very large (quasi-infinite) radius to its present size. This would release about 4×10^{48} ergs. The present luminosity of the Sun is 4×10^{33} erg sec^{-1}, and it does not seem to be changing much. At this constant rate, assuming 100 percent efficiency for the conversion of gravitational to radiative energy (which is theoretically not possible, but sets an upper limit), the Sun could have been shining at more or less its present brightness for about

$$\begin{aligned}t_K &= 4 \times 10^{48}/4 \times 10^{33} \\ &= 10^{15} \text{ sec} = 30 \text{ million yrs.}\end{aligned} \tag{1.13}$$

The subscript K identifies this as the Kelvin (or Kelvin-Helmholtz) time-scale. Although it is obvious that $t_{ev} \gg t_{ff}$ for the Sun, it is perhaps not so obvious that $t_{ev} > t_K$. Indeed, until early in this century, it was believed that the energy of the Sun did come from its gravitational contraction, and that the Sun, and presumably the Earth, was a few million years old. Because of this rather entrenched theory, Charles Darwin and Kelvin had several public disagreements when the former's *Origin of the Species* was published, because the time-scale required for biological evolution was considerably greater than t_K. At about the same time, the geologist Sir Charles Lyell maintained that the ocean floor had to be at least a billion years old, if the behavior of sediments were to be understood. With the application of radioactive dating methods to rocks by (among others) Rutherford, it became clear that the Earth itself is several billion years old, and the Sun must be at least as old as the Earth.

Problem 1.9. Review the evidence that the Earth is a few billion years old, rather than a few million.

Einstein Time-scale

In 1929, Eddington took up the crucial question of how to deal with this discrepancy, and gave the historically interesting arguments leading to the conclusion that the Kelvin-Helmholtz theory is wrong and that there must be a subatomic source of energy. Eddington also gave another interesting quantity, the total energy equivalent of the mass of the Sun (recall that Einstein published his work on special relativity in 1905–1906.) This energy equivalent is, for the Sun,

$$E = M_\odot c^2 = 1.79 \times 10^{54} \text{ ergs}.$$

If all this energy could be converted to radiation, the Sun could continue shining at its present rate for as long as

$$t_E = 4.6 \times 10^{20} \text{ sec} = 1.4 \times 10^{13} \text{ yrs}$$

(E stands for Einstein). Thermonuclear fusion, which is responsible for most of the energy release, occurs only within the inner 10 percent of the Sun's mass. If all of this matter were converted into energy, the Sun could radiate at its current rate for approximately 1.4×10^{12} years. Anticipating the result that, for the Sun, t_{ev} is about nine billion years, we can say that the efficiency of conversion of mass to energy in the Sun is about 0.007.

The evolutionary time for stars like the Sun on the main sequence is determined by a series of nuclear processes that convert hydrogen into helium. The rate for the process is set by the slowest of its steps, which is the formation of deuterium:

$$\mathrm{p} + \mathrm{p} \rightarrow \mathrm{p} + \mathrm{n} + e^+ + \nu_e \rightarrow \mathrm{d} + e^+ + \nu_e.$$

This reaction involves two processes: (1) the weak interactions, which convert a proton into a neutron; and (2) the strong interactions, which bind the neutron and proton to form deuterium. The essential characteristic of the first step is its extreme weakness. For example, the lifetime of a free neutron, which is governed by the weak interactions, is about 15.3 minutes, whereas the time-scale characteristic of the strong interactions is typically about 10^{-23} seconds.

We can estimate the main-sequence lifetime of the Sun from the assumption that deuterium formation sets the overall rate for helium formation. By the argument detailed in Chapter 7, the evolutionary time-scale for the Sun turns out to be

$$t_{ev} \approx 10^{10} \text{ yrs}. \qquad (1.14)$$

1.4. Static Configurations (Hydrostatic Equilibrium)

We have thus far emphasized the role of gravitation in the pressure-gravity equilibrium. Now consider the pressures required to support a star of mass M and radius R. A simple estimate is possible. The sphere of radius $r = 2^{-1/3}R = 0.79R$ splits a star of uniform density into two equal mass portions, each of $M/2$. The surface area of the inner sphere is $4\pi 2^{-2/3} R^2$. The total downward gravitational force at a point on the surface of this sphere is $G\,(M/2)^2/(R/2^{1/3})^2$. The pressure required for support at this point is therefore the gravitational force divided by the area, or

$$P = \frac{(M^2G/4)(2^{2/3}/R^2)}{4\pi\, 2^{-2/3} R^2} = \frac{2^{4/3}}{16\pi}\frac{M^2G}{R^4}. \qquad (1.15)$$

This result could also be obtained by dimensional arguments. We expect that stars are not uniform spheres, but we may say that pressures of the order of GM^2/R^4 will be encountered in their interiors. (Incidentally, the analysis is not restricted to stars; any self-gravitating object of mass M and radius R supported by internal pressure, such as a planet, will need pressures of this order.) For typical pressures inside the Sun,

$$P \simeq M_\odot^2 G/R_\odot^4 = 1.12 \times 10^{16} \text{ dynes cm}^{-2}. \qquad (1.16)$$

In a more exact model for the Sun, the central pressure is 1.3×10^{17} dynes cm^{-2}, and the pressure at $r = R/2$ is 5.9×10^{14} dynes cm^{-2}. At $r = 0.79R$, $P = 5.6 \times 10^{12}$ dynes cm^{-2}. A word or two of warning can be drawn from these comparisons. First, our rough estimates yield the right general order of magnitude. Second, numerical coefficients are not always of the order of magnitude of unity. For example, the factor $2^{4/3}/16\pi$

is about 0.05; so even if the uniform-density model were correct, we have "lost" almost two orders of magnitude by neglecting that factor. This is one reason why our value of 10^{16} dynes cm^{-2} is close to the central pressure of the Sun but is rather high for an "average" internal pressure. Finally, the uniform-density model is clearly too naive for the Sun, and this "geometrical" effect can contribute to the error. Nevertheless, rough estimates of physical quantities are extremely useful, and often enable us to decide what is important and what is not. Frequently they can be very good if obtained with some care.

Equations of State

Suppose we now want to know the temperature in the interior of the Sun. This involves something new, because only pressure is determined by the gravitational field. Temperature must be related to pressure by an equation of state, which can be written $P = P(\rho,T,C)$, where ρ is the density and C the chemical composition. A typical equation of state is the perfect gas law

$$P = nkT = \frac{\rho kT}{m} = \frac{\rho kT}{\mu m_H}, \qquad (1.17)$$

where n is the particle density, m is the mass per particle, $m_H = 1.67 \times 10^{-23}$ g, and μ is the mean molecular weight. The perfect gas law is a good approximation to the state of matter in most regions of most stars, but it is not always applicable. In particular, highly collapsed stars (white dwarfs or neutron stars) may have degenerate equations of state in which the pressure is much less sensitive to the temperature than in the perfect gas law. Pressure may even be independent of temperature. For example, in a highly condensed white dwarf, the equation of state is approximately

$$P = K_1\rho^{5/3}; \qquad (1.18)$$

in even more compact objects it can be approximated by

$$P = K_2\rho^{4/3}. \qquad (1.19)$$

Furthermore, in hot stars, although the matter may behave like a perfect gas, the radiation field is so strong that a significant part of the support for the star comes from radiation pressure. The radiation pressure in a thermodynamic enclosure at temperature T is given by

$$P = \frac{1}{3}aT^4. \qquad (1.20)$$

Even in nondegenerate stellar material (i.e., in ordinary stars), the total equation of state contains both gas pressure and radiation pressure, and has the form

$$P = \frac{\rho kT}{\mu m_H} + \frac{1}{3}aT^4. \qquad (1.21)$$

Stellar Temperatures

Suppose, then, that we want to estimate the interior temperature of the Sun. We will try first pure gas pressure, then pure radiation pressure, and then compare the two. For gas-pressure support, the temperature is given by (1.17):

$$T = \mu m_H P/\rho k. \qquad (1.22)$$

If we take $\rho = 3M_\odot/4\pi R_\odot^3$, and use (1.19) for the pressure $P = \alpha GM^2/R^4$, where α is a numerical factor, we find, after eliminating ρ and P, the gas temperature

$$T_g = \frac{4\pi}{3}\mu\frac{m_H G}{k}\alpha\frac{M}{R}. \qquad (1.23)$$

We must next decide what to take for the molecular weight. This is an annoying detail that is easy to guess at, but hard to work out exactly. Suppose we already know that the stellar material is mostly hydrogen and that it is almost completely ionized. There are thus two particles for every m_H, and the mean molecular weight is ½. Taking $\mu = ½$, and inserting the physical constants ($m_H = 1.67 \times 10^{-24}$ g, $k = 1.38 \times 10^{-16}$ erg deg^{-1}), we find

$$T = 4.81\,\alpha \times 10^7\text{K}$$
$$= 2.4 \times 10^7\text{K}$$

if $\alpha \simeq 0.5$. To support the Sun on gas pressure alone would require internal temperatures of the order of 20 million degrees.

Now consider the temperature required if radiation pressure were the sole support. Here (1.20) and (1.15)