Ergebnisse der Mathematik und ihrer Grenzgebiete 95

A Series of Modern Surveys in Mathematics

S. I. Adian

The Burnside Problem and Identities in Groups

Springer-Verlag Berlin Heidelberg New York

S. I. Adian

The Burnside Problem and Identities in Groups

Translated from the Russian
by John Lennox and James Wiegold

Springer-Verlag
Berlin Heidelberg New York 1979

Sergei I. Adian
Steklov Mathematical Institute, Moscow, USSR

John Lennox, James Wiegold
University College, Cardiff, Wales, U.K.

Title of the Russian Original Edition:
Problema Bernsaida i tozhdestva v gruppakh
Publisher: Nauka, Moscow, 1975

AMS Subject Classification (1970): 20 E 10, 20 F 50

ISBN 3-540-08728-1 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-08728-1 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data. Adian, S. I. The Burnside problem and identities in groups (Ergebnisse der Mathematik und ihrer Grenzgebiete; 95). Translation of Problema Bernsaida i tozhdestva v gruppakh. Bibliography: p. Includes indexes. 1. Burnside problem. 2. Groups, Theory of – 3. Groups, Theory of – Generators. I. Title. II. Title: Identities in groups. III. Series. QA171.A313. 512'.22. 78-16896

Printed in Germany
Printing and binding: Konrad Triltsch, Würzburg
2141/3140-543210

Ergebnisse der Mathematik und ihrer Grenzgebiete 95

A Series of Modern Surveys in Mathematics

Preface to the English Edition

Three years have passed since the publication of the Russian edition of this book, during which time the method described has found new applications.

In [26], the author has introduced the concept of the periodic product of two groups. For any two groups G_1 and G_2 without elements of order 2 and for any odd $n \geqslant 665$, a group $G_1 \textcircled{n} G_2$ may be constructed which possesses several interesting properties. In $G_1 \textcircled{n} G_2$ there are subgroups $\bar{G}_1$ and $\bar{G}_2$ isomorphic to G_1 and G_2 respectively, such that $\bar{G}_1$ and $\bar{G}_2$ generate $G_1 \textcircled{n} G_2$ and intersect in the identity. This operation "$\textcircled{n}$" is commutative, associative and satisfies Mal'cev's postulate (see [27], p. 474), i.e., it has a certain hereditary property for subgroups. For any element x which is not conjugate to an element of either $\bar{G}_1$ or $\bar{G}_2$, the relation $x^n = 1$ holds in $G_1 \textcircled{n} G_2$. From this it follows that when G_1 and G_2 are periodic groups of exponent n, so is $G_1 \textcircled{n} G_2$. In addition, if G_1 and G_2 are free periodic groups of exponent n the group $G_1 \textcircled{n} G_2$ is also free periodic with rank equal to the sum of the ranks of G_1 and G_2. I believe that groups having many interesting properties can be constructed using this notion of periodic product. For example, it has been proved recently that a periodic product $G_1 \textcircled{n} G_2$ is a simple group if and only if each of the groups G_1 and G_2 coincides with the subgroup generated by its n-th powers.

Using a modification of our methods, one can prove that the word problem and the conjugacy problem are solvable for any finitely presented group which has only defining relations of the form $A^n = 1$ with elementary periods A and given odd $n \geqslant 665$.

A contradiction within the system of parameter conditions, on which Britton based his argument in [25], was demonstrated in the introduction to the Russian edition. It was clear to me at that time that this mistake was connected with the principal difficulties inherent in the Burnside Problem, and would prevent Britton from completing his proof with any ease. In fact, Britton has not yet published a correction of the mistake in his proof.

In conclusion I would like to express my sincere thanks to Professors James Wiegold and John Lennox who took on themselves the onerous task of translating this book.

Moscow, May 15, 1978 S. Adian

Translators' Preface

We would like to thank Professor Adian for willingly giving us so much of his time while the translation was being prepared. His help and patience over difficult points in the Russian were invaluable, and we are very grateful.

Thanks, too, to the University of Bielefeld, in particular to Professor Jens Mennicke, for inviting one of us (J.W.) to the Burnside meeting of July 1977. Not only was this an enjoyable visit, but it also made it possible to have many long discussions with Professor Adian concerning the translation.

Finally, our thanks to Springer-Verlag for their unfailing courtesy at all times.

Preface

To the memory of
P. S. Novikov

This book is based on a special course that the author delivered to the Faculty of Mechanics and Mathematics at Moscow University in the academic years 1971/72 and 1972/73. It presents a new and improved version of the method of investigating groups with an identical relation of the form $x^n = 1$ evolved by P.S. Novikov and the author for solving Burnside's problem on periodic groups, first published in the joint paper [5]. The distinguishing feature of that method is the proof of a large number of assertions (more than a hundred) by simultaneous induction over a natural parameter. Comparing now with [5], certain new concepts are introduced here, the definitions of a number of the old concepts are altered somewhat, and a large number of new lemmas are added. These changes have made it possible to simplify the proof significantly and to reduce the bound for n from $n \geqslant 4381$ to $n \geqslant 665$. I have succeeded in giving the definitions of all the main concepts, which also go by induction on a natural parameter, in the first chapter. Undoubtedly, this facilitates the reading of the book, since the reader has the opportunity of grasping the definitions of the concepts before beginning to take the proof apart. No special knowledge is required of the reader. If any difficulties arise in a first reading, it is recommended that cumbersome proofs of individual lemmas be omitted, and appeal made to the subject index.

After giving the solution of the Burnside problem, we shall prove the results contained in the author's lecture to the International Congress of Mathematicians in Nice [9]. Moreover, a construction is given of finitely generated torsion-free groups such that every pair of cyclic subgroups have non-trivial intersection. This is a non-commutative analogue of the additive group of rational numbers. There is reason to suppose that the methods of the book will find application to the solution of other problems in the theory of infinite groups.

A recent unsuccessful attempt was made in [25] to give a simpler solution of the Burnside problem. In that paper the author follows, in the main, the original scheme proposed by P. S. Novikov in 1959 (see [4]), which was based on a use of transformations of cyclic words and the method of V. A. Tartakovskiĭ (see [24]). As in [5], the proof of a large number of the assertions proceeds by simultaneous induction over a natural parameter. However, several of the concepts that were essential in [5] are omitted from [25]. For example, it does not contain the concept of mutual normalisability in given rank, which was central in [5], nor that of cascade of rank α. In [25] the proof rests on so-called parametric conditions, which involve a system of some hundreds of equalities and inequalities in 302 parameters. The consistency of this system is not proved in [25]. More than that, an analysis carried out jointly by Ju. I. Hmelevskiĭ, the editor of this book, and myself, shows that the system of parametric conditions used in [25] is contradictory*.

[25] is therefore erroneous.

I am heartily grateful to Ju. I. Hmelevskiĭ, who read the manuscript with great care, checked all the proofs, made a number of useful remarks, and was of essential help to me in the analysis of [25].

*For example, the conditions

$u_4 = u_1 + r_{25}$ (p. 145, line 10 from below),

$r_{25} \geqslant u_{37} + 54/e$ (p.283, line 4 from below),

$u_{37} > 14\alpha + 214/e$, where $\alpha = \varepsilon_{30} + u_{13} + 6u_4$ (p.221, lines 11 and 12 from below)

give an obvious contradiction: $u_4 > r_{25} > u_{37} > u_4$.

Introduction

In 1902, Burnside [1] formulated the following problem:

"Is every group with a finite number of generators and satisfying an identical relation $x^n = 1$ finite?"

This problem is known as the *Burnside problem for groups of finite exponent*, and it remained open for a considerable time. The negative solution was obtained in a joint paper of P. S. Novikov and the author [5], where it was shown that, for every odd $n \geqslant 4381$ and every $m > 1$, there exists an infinite group $\Gamma(m, n)$ on m generators and satisfying the identical relation $x^n = 1$.

Up to then, a positive answer to the Burnside problem had been obtained for $n \leqslant 3$ (see [1]), $n = 4$ (see [2]) and $n = 6$ (see [3]). The existence of a periodic group on two generators having no bound on the orders of its elements was established in [12].

In order to describe the group $\Gamma(m, n)$, a classification of periodic words in a group alphabet was introduced in [5], and a theory constructed of transformation of words corresponding to an identical relation $x^n = 1$ for fixed odd $n \geqslant 4381$.

In the first five chapters of this book we present an improved version of this theory for odd exponent $n \geqslant 665$. On the basis of this theory we prove in Chapter VI the existence of infinite groups of odd exponent $n \geqslant 665$. In that Chapter we also give the proofs of all the results about the properties of free groups of odd exponent $n \geqslant 665$ that were published in [6, 7, 8, 10]. Chapter VII contains an account of the applications of our method to questions not connected with periodic groups. These applications were published in [10, 11].

The Chapters of the book are divided into sections, and the sections into subsections. Reference to the assertion or definition contained in subsection 16 of §5 of Chapter II, for example, will be made in the form II.5.16. Reference to the Chapter will not be made within the Chapter itself.

In order to avoid numerous repetitions in the execution of the complicated simultaneous induction to which Chapters II-V are devoted, we shall not cite the separate formulations of the inductive assumptions (and we shall not even number them separately, as was done in [5]). In referring to one or other inductive assumption, we shall directly indicate the corresponding assertion that is formulated and proved for larger values of the inductive parameter in the succeeding chapters and sections. For example, if assertion *IV.1.7* is encountered in the text at a stage before it is proved in rank α, this means that the assertion obtained from IV.1.7 on replacing the inductive parameter α by a suitable $\beta \leqslant \alpha - 1$ is assumed to be proven at that stage, in accordance with the inductive assumption. We shall distinguish such references from all others by writing them in *italics*.

All assertions considered are trivial when $\alpha = 0$. This saves us from special considerations in verifying the first steps of the induction.

Contents

Chapter I. Basic Concepts and Notation

We shall use the following notation for logical connectives:

$\exists, \forall$ the existence and universal quantifiers,
& conjunction ("and"),
$\vee$ disjunction ("or"),
$\neg$ negation ("not"),
$\Rightarrow$ implication ("if ..., then, ..."),
$\Longleftrightarrow$ logical equivalence ("if and only if").

The symbol $\rightleftharpoons$ denotes equality by definition, and it is used to introduce notation for certain expressions.

If X and Y are elements of a set $\mathscr{M}$, then for brevity we write $(X \in \mathscr{M})$ & $(Y \in \mathscr{M})$ as $X, Y \in \mathscr{M}$.

The symbol $\subset$ is used to denote inclusion of one set in another, $\cup$ for union of sets and $\cap$ for intersection of sets. The empty set is denoted by $\varnothing$.

We fix an integer $m > 1$ and an odd number $n \geqslant 665$, which will be unchanged for the duration of Chapters I-VI. In addition, we shall use the fixed values of the following numerical parameters:

$$p = 9, \quad p_1 = 17,$$
$$q_1 = 2p_1 + 3 = 37, \quad q_2 = q_1 + p_1 = 54, \quad q = q_2 + 2p_1 + 2 = 90\,.$$

§ 1. Words and Occurrences

We shall consider words in a group alphabet

$$a_1, a_2, \ldots, a_m, a_1^{-1}, a_2^{-1}, \ldots, a_m^{-1}\,. \tag{1}$$

The empty word is denoted by Λ. Letter-for-letter equality of two words X and Y is denoted by $X \mathrel{\overset{\circ}{=}} Y$.

1.1. Two letters a_i and a_i^{-1} for given i are said to be *mutually inverse*. If

$$A \mathrel{\overset{\circ}{=}} b_1 b_2 b_3 \ldots b_j \ldots b_{r-1} b_r\,,$$

where each b_j is one of the letters in (1), then the word

$$b_r^{-1} b_{r-1}^{-1} \ldots b_j^{-1} \ldots b_3^{-1} b_2^{-1} b_1^{-1},$$

with $(a_i^{-1})^{-1} \rightleftharpoons a_i$, is called the *inverse* of A and is denoted by A^{-1}. Clearly, $A^{-1} \mathrel{\overline{\underline{\circ}}} E^{-1}\, D^{-1}$ whenever $A \mathrel{\overline{\underline{\circ}}} DE$.

We denote the length of the word A by $\partial(A)$, that is, $\partial(A)$ is the number of letters comprising A. In particular, $\partial(\Lambda) = 0$.

1.2. A word E is said to *occur* in a word X if there exist words R and Q such that $X \mathrel{\overline{\underline{\circ}}} REQ$. If the word R (the word Q) is empty, then E is said to be a *start* (an *end*) of X.

Clearly, one of any two starts of a given word X is a start of the other, and similarly one of two ends of A is an end of the other.

One and the same word E may occur in a given word X in different places. In order to distinguish between two different occurrences of E in X we use an extra symbol $*$. If X is a word in an alphabet not containing the letter $*$, and $X \mathrel{\overline{\underline{\circ}}} REQ$, then we shall call the word $R * E * Q$ an *occurrence of E in X*.

If $V \rightleftharpoons R * E * Q$, we denote $Q^{-1} * E^{-1} * R^{-1}$ by V^{-1}.

The word E is said to be the *base* of the occurrence $R * E * Q$. If V denotes the occurrence $R * E * Q$, we shall write $E = \mathrm{Bas}\,(V)$.

We shall consider only occurrences with non-empty bases. An occurrence of the form $* E *$ will be identified with the word E.

Upper-case Latin letters U, V, W, with or without suffices, will be used to denote occurrences in words in the alphabet (1).

1.3. Let $P * E * Q$ and $R * D * S$ denote two occurrences in one and the same word, *i.e.*, $PEQ \mathrel{\overline{\underline{\circ}}} RDS$. All the relations defined in this subsection are meaningful only for occurrences in one and the same word.

We shall say that the occurrence $P * E * Q$ is *contained in* $R * D * S$ if $\partial(R) \leqslant \partial(P)$ and $\partial(S) \leqslant \partial(Q)$, that is, if R is a start of P and S is an end of Q. If in addition $P \mathrel{\overline{\underline{\circ}}} R$ (or $Q \mathrel{\overline{\underline{\circ}}} S$), we say that $P * E * Q$ is a *start* (or an *end*) of $R * D * S$, or else that $R * D * S$ *starts* (or *ends*) with the occurrence $P * E * Q$.

We say that the occurrences $P * E * Q$ and $R * D * S$ *intersect* if there is an occurrence V with nonempty base that is contained in $P * E * Q$ and in $R * D * S$. We shall call the maximal such occurrence V (with respect to length of base) the *common part* of $P * E * Q$ and $R * D * S$, or their *intersection*. If $P * E * Q$ is contained in $R * D * S$, their common part is $P * E * Q$.

If $\partial(P) < \partial(R)$ and $\partial(S) < \partial(Q)$, we say that the occurrence $P * E * Q$ *lies to the left* of $R * D * S$, and write $P * E * Q < R * D * S$. If PE is a start of R, we shall say that $P * E * Q$ *lies strictly to the left* of $R * D * S$, and write $P * E * Q \ll R * D * S$.

If $P * E * Q < R * D * S$, then neither of the occurrences $P * E * Q$ and $R * D * S$ is contained in the other; conversely, if neither is contained in the other, then $P * E * Q < R * D * S$ or $R * D * S < P * E * Q$.

If $P * E * Q \ll R * D * S$, then $P * E * Q$ and $R * D * S$ do not intersect; conversely, if they do not intersect, then either $P * E * Q \ll R * D * S$ or $R * D * S \ll P * E * Q$.

The *union* of occurrences $P * E * Q$ and $R * D * S$ is the occurrence containing both of them that has base of shortest length. Suppose that U is the union of $P * E * Q$ and $R * D * S$. If $P * E * Q$ is contained in $R * D * S$, then $U = R * D * S$. If $P * E * Q < R * D * S$, then $P * E * Q$ is a start of U, and $R * D * S$ is an end of it.

1.4. Let $P * E * Q$ and $P_1 * E * Q_1$ be occurrences in words X and Y and $V \rightleftharpoons R * C * S$ an occurrence in X contained in $P * E * Q$. Then there exist words A and B such that $R \overline{\underline{\circ}} PA$, $S \overline{\underline{\circ}} BQ$ and $E \overline{\underline{\circ}} ACB$. In such a case the occurrence $P_1A_1 * C * BQ_1$ in Y is denoted by

$$\phi(V; P * E * Q, P_1 * E * Q_1) .$$

For any two occurrences W and W_1 with the same base, the function $V_1 \rightleftharpoons \phi(V; W, W_1)$ sets up a one-to-one mapping of the set of occurrences in W onto the set of all occurrences in W_1.

If $V_1 = \phi(V; W, W_1)$, then $V = \phi(V_1; W_1, W)$.

Let W_1, W_2, W_3 be occurrences with the same base and suppose that V_1 is contained in W_1. If $V_2 = \phi(V_1; W_1, W_2)$ and $V_3 = \phi(V_2; W_2, W_3)$, then $V_3 = \phi(V_1; W_1, W_3)$.

Clearly, the function $V_1 = \phi(V; W, W_1)$ preserves the relations $<$, $\ll$ and carries the common part (union) of two occurrences contained in W to the common part (union) of their images in W_1.

§ 2. Periodic Words

For any integer $t > 0$, let A^t stand for the word $AA \ldots A$, with A repeated t times. For $t < 0$ we set $A^t \rightleftharpoons (A^{-1})^{-t}$. Finally, for any word A we set by definition

$$A^0 = \Lambda .$$

We shall call a word of the form $A_1A^tA_2$, where A_2 is a start of A, A_1 is an end of A and $\partial(A_1A^tA_2) > 2\partial(A)$, a *periodic word with period A*. The set of all periodic words with period A is denoted by Per(A). For empty A_1 (or A_2), we call A the *left* (or *right*) *period* of $A_1A^tA_2$.

Clearly, if $A \overline{\underline{\circ}} B^r$, then Per($A$) $\subset$ Per(B) .

2.1. We say that a word B is a *cyclic shift* of a word A if $A \overline{\underline{\circ}} PQ$ and $B \overline{\underline{\circ}} QP$ for some P and Q.

Clearly, if B is a cyclic shift of A, then Per(A) = Per(B). If $X \in$ Per(A), B occurs in X and $\partial(B) = \partial(A)$, then B is a cyclic shift of A.

2.2. *If $AB \overline{\underline{\circ}} BA$, then there is a word D such that $A \overline{\underline{\circ}} D^t$ and $B \overline{\underline{\circ}} D^r$ for some $t, r \geqslant 0$.*

We may assume that $\partial(A) \geqslant \partial(B)$. If B is empty, then $B \overline{\underline{\circ}} A^0$. So we assume that B is nonempty and that the assertion is true whenever $\partial(AB) < j$, and prove it for $\partial(AB) = j$. Suppose that $AB \mathrel{\overline{\underline{\circ}}} BA$. Then for some C we have $A \mathrel{\overline{\underline{\circ}}} BC$ and $CB \mathrel{\overline{\underline{\circ}}} BC$. Since $\partial(CB) = j - \partial(B)$, the inductive assumption gives that $C \mathrel{\overline{\underline{\circ}}} D^k$ and $B \mathrel{\overline{\underline{\circ}}} D^r$ for some D, k and r. In that case $A \mathrel{\overline{\underline{\circ}}} D^{k+r}$.

2.3. *If $A^t A' \mathrel{\overline{\underline{\circ}}} B^r B'$, where A' is a start of A, B' is a start of B and $\partial(A^t A') \geqslant \partial(AB)$, there exists a word D such that $A \mathrel{\overline{\underline{\circ}}} D^k$ and $B \mathrel{\overline{\underline{\circ}}} D^s$ for some k and s.*

Suppose that $\partial(A) \geqslant \partial(B)$. Since A is a start of $B^r B'$, we have $A \mathrel{\overline{\underline{\circ}}} B^{r_1} B_1$, where $r_1 > 0$ and $B \mathrel{\overline{\underline{\circ}}} B_1 B_2$. Cancelling A on the left of the original equality, we get $A^{t-1} A' \mathrel{\overline{\underline{\circ}}} B_2 B^{r-r_1-1} B'$, where now $B_1 B_2$ is a start of the left hand side and $B_2 B_1$ is a start of the right hand side. Thus $B_1 B_2 \mathrel{\overline{\underline{\circ}}} B_2 B_1$. By 2.2, there exists a word D such that $B_1 \mathrel{\overline{\underline{\circ}}} D^{s_1}$ and $B_2 \mathrel{\overline{\underline{\circ}}} D^{s_2}$ for some s_1, s_2. We can take $s \rightleftharpoons s_1 + s_2$ and $k \rightleftharpoons r_1 (s_1 + s_2) + s_1$.

2.4. Suppose that $X \in \mathrm{Per}(A)$. The occurrence of $P * E * Q$ in X is said to be *interior relative to the period A* if $\partial(P) \geqslant 8\,\partial(A)$ and $\partial(Q) \geqslant 8\,\partial(A)$. We denote the set of all occurrences of this sort in X by $\mathrm{Inn}(X, A)$.

2.5. Suppose that $X \in \mathrm{Per}(A)$ again. Two occurrences $V \rightleftharpoons FP * E * QG$ and $W \rightleftharpoons FR * E * SG$ in one and the same word FXG, where F and G are arbitrary words, are said to *correspond in phase relative to the period A* if there is an integer r such that

$$\partial(R) - \partial(P) = r\,\partial(A)\,.$$

The words F and G here may be empty. For $r = 0$ we have that $W \mathrel{\overline{\underline{\circ}}} V$. If $r > 0$ (or $r < 0$) we shall say that W is *the result of shifting V to the right (to the left) by r periods A.*

We denote by $\mathrm{Corr}_A(V, W)$ the predicate that is valid if and only if V and W are occurrences in some word FXG that correspond in phase relative to period A, where now $X \in \mathrm{Per}(A)$. It is clear that the relation $\mathrm{Corr}_A(V, W)$ is symmetric and transitive.

2.6. We extend the concept of correspondence in phase defined in 2.5 to occurrences in different periodic words with given period. Two occurrences $P * E * Q$ and $R * E * S$ in the words $X \in \mathrm{Per}(A)$ and $Y \in \mathrm{Per}(A)$ are said to *correspond in phase* if one of the words P and R is an end of the other and one of Q and S is a start of the other.

We remark that if U and V are two occurrences in a given word $X \in \mathrm{Per}(A)$, then $\mathrm{Corr}_A(U, V)$ implies that they correspond in phase in the sense just defined if we take $Y \rightleftharpoons X$.

2.7. A word A is said to be *simple* if it cannot be represented in the form D^r for $r > 1$.

If A is a non-empty word, then there is a simple word B such that $A \mathrel{\overline{\underline{\circ}}} B^t$ for some $t \geqslant 1$.

This is proved by induction on $\partial(A)$. If A is not simple, then $A \mathrel{\overline{\underline{\circ}}} D^r$ for some D and some $r > 1$. Since $\partial(D) < \partial(A)$, the inductive hypothesis gives the existence of a simple B such that $D \mathrel{\overline{\underline{\circ}}} B^k$, where $k \geqslant 1$. Thus $A \mathrel{\overline{\underline{\circ}}} B^{rk}$.

2.8. *If AB is a simple word, then BA is simple.*

It is enough to prove this for the case where B consists of a single letter a. Assume that $aA \mathrel{\overline{\underline{\circ}}} D^r$, where $r > 1$. Then for some E, $D \mathrel{\overline{\underline{\circ}}} aE$, so that $Aa \mathrel{\overline{\underline{\circ}}} (Ea)^r$.

2.9. *Suppose that* $A^t A_1 \mathrel{\overline{\underline{\circ}}} B^r B_1$, *where* $\partial(A^t A_1) \geqslant \partial(AB)$, A_1 *is a start of* A *and* B_1 *is a start of* B. *If* A *is simple, then* $B \mathrel{\overline{\underline{\circ}}} A^k$ *for some* k.

Indeed, by 2.3 there is a D such that $A \mathrel{\overline{\underline{\circ}}} D^s$ and $B \mathrel{\overline{\underline{\circ}}} D^k$. Clearly, we may take $s > 0$. Since A is a simple word, $s = 1$, that is, $D \mathrel{\overline{\underline{\circ}}} A$.

§ 3. Aperiodic Words

In what follows we shall need an infinite sequence

$$t_1, t_2, t_3, \ldots, t_i, t_{i+1}, \ldots \tag{2}$$

whose terms are either 1 or 2, such that for each i the word $t_1 t_2 \ldots t_i$ does not have any occurrence of a non-empty word of the form E^3. We mention now a method of constructing such a sequence, suggested by Aršon [13].

3.1. Consider all permutations of three symbols, 1,2,3:

$$\begin{array}{ccc} 1\ 2\ 3, & & 3\ 2\ 1, \\ 2\ 3\ 1, & & 1\ 3\ 2, \\ 3\ 1\ 2, & & 2\ 1\ 3. \end{array}$$

We call the permutations in the left-hand column *odd*, and those in the right-hand column *even*. The odd permutations are numbered by their first elements, and the even ones by their last elements. Every even permutation is the mirror image of the odd permutation with the same numeral.

We shall construct words A_i by induction on i, for $i \geqslant 1$. Set

$$A_1 = 1 .$$

If the word A_i has been constructed already, and $A_i = h_1 h_2 \ldots h_r$, then we let A_{i+1} stand for the result of replacing every symbol h_j in A_i by the even or odd permutation with numeral h_j, according to the parity of j. Let us write down the first few A_i:

$$\begin{aligned} A_2 &= 123\,, \\ A_3 &= 123\ 132\ 312\,, \\ A_4 &= 123\ 132\ 312\ 321\ 312\ 132\ 312\ 321\ 231\,. \end{aligned}$$

We have introduced a *triple* of symbols here so as to be able to survey the whole word more easily. It is clear that A_i is a start of A_{i+1} in all cases.

3.2. *There is no occurrence of a nonempty word of the form EE in any* A_i, $i \geqslant 1$.

Proceed by induction on i. Suppose that it is true for A_i. Decompose A_{i+1} into triples: the number of them is the length of A_i. We shall call the occurrences of these triples in A_{i+1} the *constituent triples* of A_{i+1}.

By definition of A_i, there is no sequence in it of two triples of the same parity. Moreover, it follows from the inductive assumption that there is no sequence of two triples with one and the same numeral.

Suppose that $A_{i+1} \eqcirc PEEQ$, where E is non-empty. We show by an analysis of cases that this assumption leads to a contradiction.

Let $V \rightleftarrows R * abc * S$ be the last triple intersecting the occurrence $U \rightleftarrows P * E * EQ$.

If $\partial(E) = 1$, then E is different from a and b, and since $E \eqcirc c$, we get that there is a triple to the right of V and having the same numeral. Consequently $\partial(E) > 1$.

If $\partial(E) = 2$, then $E \eqcirc bc$, and then to the right of V there stands a triple of the same parity. Thus $\partial(E) \geqslant 3$.

Assume that V is an end of the occurrence $P * E * EQ$, that is, $E \eqcirc E_1\, abc$ and $R \eqcirc PE_1$.

Suppose that $\partial(E) = 3j$. If j is odd, then the triple abc occurs in the decomposition of A_i both as an even and as an odd triple. If j is even, then A_i has a subword of the form D^2, where $\partial(D) = j$, which contradicts the inductive assumption.

Suppose that $\partial(E) = 3j + 1$. Then $\partial(E_1ab)$ is a multiple of three, that is, there is a triple $V_1 \rightleftarrows PEE_2 * cab * cQ$ in the decomposition of A_{i+1}. It is easy to convince oneself in this case that c is a start of E. In fact, if E_2 is not empty, the triple immediately to the left of V is of the form $PE_3 * bac * abcEQ$, that is, $E_2 \eqcirc E_3ba$, whence it follows by analogy with the preceding case that the triple immediately to the left of V_1 is of the form $PEE_4 * cba * cabcQ$, *etc*. But c cannot be a start of E, since then the word c^2 would occur in A_{i+1}.

Finally, suppose that $\partial(E) = 3j + 2$. Then $\partial(E_1a)$ is a multiple of 3, that is, the decomposition of A_{i+1} contains a triple of the form $PEE_2 * cba * bcQ$, where $E_2cb \eqcirc E_1$. If E_2 is not empty, then a is an end of E_2. As in the preceding case, we can convince ourselves that c must be a start of E, which is impossible.

Thus we have shown that V cannot be an end of the occurrence $P * E * EQ$. It remains to consider the following two cases:

$$E \eqcirc E_1a \eqcirc bcE_2 \eqcirc bcE_3a \tag{3}$$

and

$$E \eqcirc E_1ab \eqcirc cE_2 \eqcirc cE_3ab\,,$$

where $PE_1 \eqcirc R$. Since these cases are analogous, we may restrict attention to the first one. Suppose that (3) is satisfied.

If $\partial(E) = 3j$, then $P * bc * E_2EQ$ is an end of some constituent triple, that is, $P \eqcirc P_1a$. Then $P_1 * abcE_3 * abcE_3aQ$ ends with some constituent triple, which is impossible, as was proved above.

If $\partial(E) = 3j + 1$, then by (3), $\partial(bcE_3)$ is a multiple of three. Consequently, the occurrence $P * bcE_3 * aEQ$ begins with the triple bca, that is, $E_3 \eqcirc aE_4$. Then to the right of V there stands the triple $PEbc * acb * E_5Q$, that is, $E_3 \eqcirc abcE_5$. Thus the second triple contained in $P * bcE_3 * aEQ$ also ends with a. In exactly the same way