

differential
equations IN
ENGINEERING
PROBLEMS

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DIFFERENTIAL EQUATIONS IN ENGINEERING PROBLEMS

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PREFACE

Mathematics is an indispensable tool to engineers. As a tool, however, it is a means to an end, and for this reason the mathematical training of engineers must be significantly different from that of mathematicians. This does not mean that engineers should only be taught final results and formulas; it does mean that much of the rigor upon which mathematicians quite properly insist must be cast aside, and that the emphasis must always be on the physical aspects of the problem. Rigorous proofs must often be replaced by plausibility arguments. On the other hand, the foundation must be solid enough so that conclusions may be reached with certainty and pitfalls avoided.

Based partly on this philosophy, the authors have been giving a series of courses in engineering mathematics at Columbia University for a number of years. This book is based on the material taught and the notes used by the senior author in the second course of this sequence. The material is suitable for courses in the junior year, although in some engineering curricula lack of time forces its postponement to a later period. It is assumed that prospective students are familiar with the calculus and certain topics of algebra such as determinants and the solution of algebraic equations of arbitrary degree.

One of the difficult problems in a book of this kind is the choice of material. In deciding what should be included and what should be omitted, the guiding thought has been the *need of engineers in their work*. Certain topics and methods which are important to mathematicians but do not find frequent application in engineering have thus been left out. The authors feel that a frank and searching analysis of the real needs of engineers requires the abandonment of some traditional ideas.

The use of engineering examples to introduce new topics and the need for new techniques is another guideline of the book. This method establishes the need for new mathematical material before its study is begun and thus stimulates interest. Engineers always demand sound reasons before capital outlays (in this case, time and energy) are made; they are not satisfied to study the solution of an equation merely because the author suggests that it be considered. Not all students and readers will be equally interested in all physical problems; that is why, especially in the early chapters, a number of physical problems leading to the same differential equation are used.

The first two chapters provide background material. A brief review of some of the fundamental relations of the calculus, mechanics, electricity, and thermodynamics is given in the first chapter so that physical problems may later be introduced without undue digression. The second chapter deals with the important problem of formulating the differential equations; unless the reader can do this, his command of the solution of differential equations would clearly be wasted.

The next four chapters deal with linear equations with constant coefficients—the type of equation which arises (although frequently only as a rough approximation) in the great majority of engineering applications. Two chapters are devoted to the classical method of solution of these equations. A brief introduction to the Laplace transform approach is given in Chapter 5, not only because it provides a simple method of solving ordinary differential equations with given initial conditions, but because the same method can at a later stage be used to solve partial differential equations and difference equations. The method further provides a convenient way of characterizing linear systems by “transfer functions” and “system functions.” The sixth chapter takes up simultaneous linear equations.

The following three chapters are concerned with methods of handling certain types of linear equations with variable coefficients and some nonlinear equations. The use of series expansions is discussed in Chapter 7, Bessel equations and Bessel functions in Chapter 8, and nonlinear equations and elliptic integrals in Chapter 9.

The use of Fourier series in the solution of differential equations is covered in Chapter 10. A review of the Fourier series concepts is given in this chapter, although it is expected that the reader has previously dealt with such series. The last chapter briefly introduces the solution of boundary value problems involving more than one independent variable by separation of variables. Since this method, when applicable, reduces the problem to the solution of ordinary differential equations, it can well be included in courses which are intended to cover ordinary differential equations only.

Extensive problem sets are given at the end of each chapter. Some of these are routine practice exercises; some are applied problems requiring the analysis of a physical system; and some represent extensions of the theory discussed in the book. The authors feel that a good working knowledge of the text material can be gained only through the actual solution by the student of a large number of problems. Answers are given to over half of the problems.

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Chapter 1

INTRODUCTION

1.1 Definition

The solution of many physical problems aims at the determination of the numerical value or values of a physical quantity. Such mathematical solutions usually require the evaluation of the roots of algebraic or transcendental equations by the methods of algebra or the calculus.

In other types of problems, it is required to find, instead, the *law of variation* of a physical quantity depending on other variable physical quantities; hence the mathematical solution of the problem involves an *unknown function* of one or more variables. When the problem involves not only an unknown function, but also the rates of change of this function with respect to the variables, its mathematical formulation leads to a relationship between the unknown function, its derivatives, and the independent variables. Such an equation, involving an unknown function and one or more of its derivatives, is called a *differential equation*.*

It is the aim of this book to show how certain physical problems may be formulated and solved by means of differential equations.

1.2 Introductory example

A body B has a temperature u_0 of 200°F at a given time and a temperature u_1 of 150°F one hour later. It is required to determine the relationship between the temperature of B and the time t , and to evaluate the temperature of B one hour after the last measurement under the assumption that the surrounding air is kept at a constant temperature u_a of 74°F .

In order to set up the problem in mathematical terms, it will be necessary to remind the reader of two empirical facts about heat and temperature with which he is familiar even if not formally trained in thermodynamics:

1. Heat flows from a higher to a lower temperature; i.e., when a body is surrounded by a colder medium, the body cools; when a body is surrounded by a warmer medium, the body gets warmer.

* The differentials of the unknown function rather than its derivatives appear sometimes in the differential equation.

2. A body changes temperature more or less rapidly depending upon the difference between its temperature and the temperature of the surrounding medium. The greater this difference, the faster the change in temperature of the body.

Various hypotheses could be made concerning the relationship between difference in temperature and speed of temperature change, but the simplest is to assume that the speed of temperature change is proportional to the temperature difference. This fundamental hypothesis, called *Newton's law of cooling*, has been proved experimentally to be correct within a range of temperatures including those of the present problem. We shall therefore take it as the fundamental law behind the thermodynamic phenomenon under study.

In order to transform the statement of our problem into mathematical terms, we shall let

t = time in hours measured from the instant at which the temperature of B is 200°F ,

$u(t)$ = temperature of B at time t ,

u_a = air temperature, assumed constant and equal to 74°F ,

k = constant of proportionality (assumed positive) between the speed of temperature change and the temperature difference $u - u_a$.

It is known from the calculus that the speed or rate of change of a function $u(t)$ is measured by its *first derivative* du/dt . Newton's law is therefore equivalent to the statement that at any time t the quantity du/dt is proportional to the temperature difference $u(t) - u_a$. We now notice that, since heat flows from a higher to a lower temperature, and since initially u_0 is higher than u_a , the difference $u(t) - u_a$ will always be positive, and B will actually become cooler with time. Hence the rate of change du/dt will always be negative, and with a positive k , Newton's law becomes

$$\frac{du}{dt} = -k[u(t) - u_a]. \quad (1.2.1)$$

Equation (1.2.1), involving the unknown function $u(t)$ and its first derivative, is a differential equation.

In order to obtain $u(t)$ from Eq. (1.2.1), note that, u_a being constant,

$$\frac{d(u - u_a)}{dt} = \frac{du}{dt},$$

divide both sides of Eq. (1.2.1) by $(u - u_a)$ and multiply both sides by dt :

$$\frac{d(u - u_a)}{(u - u_a)} = -k dt. \quad (a)$$

The two variables u and t appear in the left- and right-hand member of Eq. (a), respectively, and have thus been "separated."

Integrating both sides of Eq. (a) and remembering that

$$\int \frac{dx}{x} = \ln x + c,$$

where $\ln x$ stands for the natural log of x , we obtain

$$\ln(u - u_a) + c_1 = -kt + c_2$$

or, letting $c_2 - c_1 = c$,

$$\ln(u - u_a) = -kt + c. \quad (b)$$

Raising the natural logarithmic base e to the equal powers $\ln(u - u_a)$ and $-kt + c$, this equation gives

$$u - u_a = e^{-kt+c},$$

from which, letting $e^c = C$,

$$u(t) = u_a + Ce^{-kt} \quad (1.2.2)$$

since $e^{-kt+c} = e^{-kt}e^c$.

The value of the *arbitrary constant* C is easily obtained by means of the *initial condition* stating that the value of the temperature at $t = 0$ equals u_0 . Substituting u_0 for u in the left-hand member and zero for t in the right-hand member of Eq. (1.2.2), and remembering that by definition $e^0 = 1$, we obtain

$$C = u_0 - u_a,$$

by means of which

$$u(t) = u_a + (u_0 - u_a)e^{-kt}. \quad (1.2.3)$$

Equation (1.2.3) entirely solves our problem when k is known. If k is unknown, we must make use of an additional condition such as the one stating that in the present case u equals u_1 at $t = 1$:

$$u_1 = u_a + (u_0 - u_a)e^{-k}.$$

Rewriting this equation in the form

$$e^k = \frac{u_0 - u_a}{u_1 - u_a}$$

and taking logarithms on both sides, we obtain

$$k = \ln \frac{u_0 - u_a}{u_1 - u_a}. \quad (1.2.4)$$

In the present numerical example there results

$$k = \ln \frac{200 - 74}{150 - 74} = \ln 1.66 = 0.51$$

and

$$u(t) = 74 + 126e^{-0.51t}. \quad (c)$$

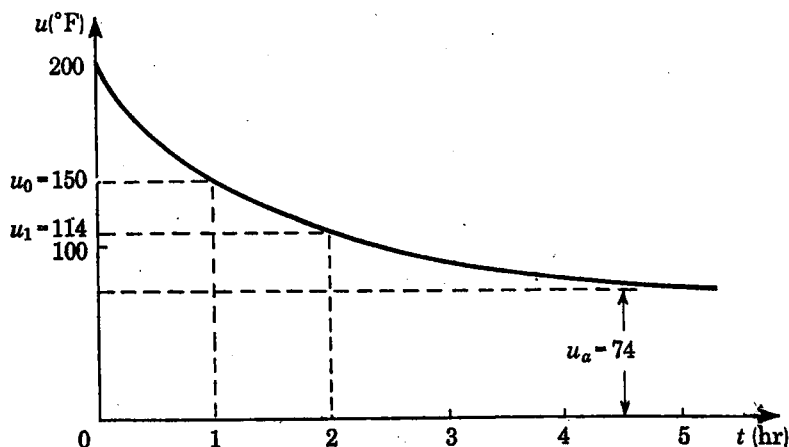


Figure 1.1

The value of the temperature at any time t may now be computed by Eq. (c); for instance, the required temperature at $t = 2$ is 114°F . Equation (c) also shows that it will take an infinite time for the body to reach the temperature of the surrounding medium (74°F) since $e^{-\infty} = 0$.

It is often convenient to present the solution of a differential equation in graphical form. The function $u(t)$ of Eq. (c) is given in Fig. 1.1.

1.3 Terminology

It is convenient to introduce at this point some of the terms more commonly used in connection with differential equations.

A differential equation involving one unknown function of a *single independent variable* and some of its derivatives is called *ordinary*. A differential equation is called *partial* when it involves a function of

more than one variable and some of its *partial* derivatives. The *order* of a differential equation is the order of the *highest* derivative appearing in the equation. The *degree* of a differential equation is the exponent to which the *highest* derivative appearing in it is raised, after clearing the equation of fractions and radicals (if possible). For example,

$$2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - 4y = 0 \quad (a)$$

is an ordinary differential equation of the second order and the first degree. It is also called a differential equation *with constant coefficients* because the coefficients of the various derivatives and of y are independent of x .

The equation

$$\left(\frac{\partial z}{\partial x}\right)^2 + 2x\frac{\partial z}{\partial y} + (x-y)z = 3e^{x+y} \quad (b)$$

is a partial differential equation of the first order and the second degree in the unknown function $z(x,y)$. It is said to have *variable coefficients*, since some of its coefficients are functions of the independent variables x and y .

A differential equation is said to be *linear* when it is of first degree in the unknown function and in each of its derivatives, i.e., when each term of the equation contains one and only one derivative, or the unknown function itself to the first power. Equations (a) and (b) are nonlinear; the equation

$$\frac{d^2y}{dx^2} + 3x^3\frac{dy}{dx} - 4y = 0 \quad (c)$$

is linear.

A differential equation is called *homogeneous* when at least one derivative or the unknown function itself appears in each term of the equation. Equation (a) is homogeneous, while Eq. (b) is nonhomogeneous because the term in its right-hand member does not contain z or its derivatives.*

* A first-order equation is also called homogeneous (in a different meaning of this word) when it can be written in the form $dy/dx = M(x,y)/N(x,y)$ and the functions M and N are homogeneous of the same order in the two variables x and y , i.e., are such that

$$M(tx, ty) = t^n M(x, y); \quad N(tx, ty) = t^n N(x, y).$$

In this case the equation may also be written

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

Any function which, when substituted in both members of a differential equation, makes them identically equal, (i.e., equal for *all* values of the independent variable or variables, at least in a certain interval) is called a *solution* or an *integral* or a *particular integral* of the equation. The function defined by Eq. (c) of Sec. 1.2 is a particular integral of the differential equation (1.2.1).

Equation (1.2.2) represents a whole family of solutions of Eq. (1.2.1), and in fact the particular integral given by Eq. (c) was obtained from this family by giving to the arbitrary constant C the value $u_0 - u_a = 126$. A careful study of the derivation of Eqs. (1.2.1) and (1.2.2) will show that this family of integrals is the solution of *any* problem involving a negative rate of change proportional to the function, with a coefficient of proportionality equal to k . For this reason, this family of solutions is called the *general solution* of the differential equation. It will also be noticed that Eq. (1.2.1) is of the first order and that its general solution contains one essential arbitrary constant. (The two constants of integration c_1 and c_2 are not essential since their difference only appears in the solution.)

It is proved in the theory of differential equations that *the general solution of an ordinary differential equation of order n contains n arbitrary constants and, under widely satisfied conditions, is unique*. In other words, there is only one general solution of each ordinary differential equation, and this solution contains as many essential constants as the order of the equation. Hence if a function is found which satisfies the differential equation and has the correct number of arbitrary constants, this function is necessarily the general solution of the equation.

The arbitrary constants may appear in complicated ways in the general solution of a nonlinear differential equation, but are always multiplying constants appearing to the first power in the general solution of linear equations.

Particular integrals are generally found by attributing specific numerical values to the arbitrary constants of the general solution. Hence the statement of a problem leading to an n th order ordinary differential equation must contain information relative to n *initial or boundary conditions* by means of which the values of the n constants may be determined. This is always feasible in the case of linear equations, in which we shall be mainly interested. In the introductory example of Sec. 1.2 a single initial condition was needed to evaluate the single arbitrary constant C . The unknown constant k was not an arbitrary constant, but a parameter characteristic of the

physical phenomenon; it appeared in the differential equation and did not arise in the process of its solution. In order to evaluate it, an additional condition was required.

Some nonlinear differential equations have solutions which cannot be derived from their general solution by attributing particular values to the arbitrary constants. Such solutions are called *singular solutions*.

Certain physical problems involving two or more unknown functions of one or more variables lead to *systems of simultaneous differential equations* which may be ordinary or partial depending upon the number of variables. The solution of such systems does not differ essentially from the solution of single equations.

1.4 Solution of physical problems by means of differential equations

We are now in a position to outline the various steps involved in the solution of physical problems by differential equations.

1. One or more independent variables and one or more unknown functions of these variables must be identified in the statement of the problem. It will be necessary to choose units, origins, and sometimes positive and negative signs for these quantities. In the introductory problem of Sec. 1.2, the independent variable was the time t , and the unknown function, the temperature, was $u(t)$. The time unit was the hour and the origin of time was chosen at the instant of the first temperature measurement. The unit of temperature was the degree Fahrenheit. No sign had to be attributed to either t or u , since both quantities were essentially positive in the ranges considered by the problem.

2. The verbal statement of the problem will always be found to express, directly or indirectly, one or more laws of nature, i.e., some basic hypothesis proved true by experiment. These may be laws of physics, chemistry, economics, biology, etc.; in fact, of any branch of science. In the introductory problem of Sec. 1.2, we made use of Newton's cooling law, a fundamental hypothesis of thermodynamics.

3. The translation into mathematical terms of the law or laws contained in the statement of the problem leads to the setting up of one or more differential equations governing the class of problems under consideration. In our introductory problem, Newton's law became Eq. (1.2.1), which represents all heat flow problems with a temperature rate proportional to temperature.

4. The differential equation (or system of simultaneous equations) is solved by purely mathematical methods, leading to its general

solution whenever classical procedures are used in connection with ordinary differential equations. In Sec. 1.2, the solution of Eq. (1.2.1) led to the general solution given by Eq. (1.2.2).

5. Making use of additional information contained in the statement of the problem, it must be possible to set up the conditions which determine the values of the arbitrary constants and hence to evaluate the particular integral satisfying both the differential equation (or equations) and these conditions. The initial temperature condition determined the value of the arbitrary constant C of Eq. (1.2.2) and led to the particular integral given in Eq. (1.2.3).

More modern procedures, like the Laplace transform, sometimes allow direct evaluation of the particular integral satisfying given conditions, without passing through the general solution. In cases where an analytical solution cannot be obtained, the particular integral may be determined graphically, numerically, or even experimentally in approximate fashion.

6. The mathematical solution thus obtained must be interpreted in physical terms by examining the analytical properties of the function and of its derivatives or by making tables or charts for them. These results may be used to predict the behavior of physical systems of the type under consideration and may save considerable amounts of time, energy, and money. For example, Eq. (1.2.3) indicated that the body B would never reach the temperature of the surrounding air because of the behavior of the function e^{-kt} .

The most difficult step in this process of solution of physical problems consists in the translation of the physical problem into mathematical terms, since it requires a clear understanding of the physical laws involved as well as a certain familiarity with mathematical manipulations. For this reason some of the fundamental relations and assumptions used to set up simple differential equations from physical problems will be reviewed in the following sections. *The reader is referred to books on the calculus, mechanics, electricity, and heat, for a thorough treatment of these topics, and may omit the reading of these sections if he is sufficiently familiar with these subjects.*

1.5 Geometrical applications of the calculus

SLOPE AND DIFFERENTIAL. A single-valued function y of the variable x is represented in the Cartesian plane x,y by a curve which is cut only once by any parallel to the y axis (Fig. 1.2). Consider two points P and P_1 on this curve with coordinates x,y and $(x + \Delta x)$,