STUDIES IN MATHEMATICS AND ITS APPLICATIONS

23

SINGULAR PERTURBATIONS I

Spaces and Singular Perturbations on Manifolds without Boundary

L.S. Frank

NORTH-HOLLAND

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Introduction

Asymptotic analysis, which started as a mathematical tool for the treatment of special problems in mathematical physics affected by the presence of characteristic small or large parameters, has been rapidly developing during the last decennia, acquiring more and more global features and penetrating into different fields of mathematics and applied sciences.

Although, originally, asymptotic analysis had a rather heuristic character, it was realized that, in order to guarantee the validity of formal asymptotic expansions, rigorous mathematical theories (especially uniform error estimates) were needed to ensure further development and the applicability of existing formal techniques. The latter stimulated a vigorous growth of asymptotic analysis as an integral part of pure and applied mathematics.

Singular perturbations being one of the central topics in the asymptotic analysis, they play also a special role as an adequate mathematical tool for describing several important physical phenomena, such as propagation of waves in media in the presence of small energy dissipations or dispersions, appearance of boundary or interior layers in fluid and gas dynamics, as well as in the elasticity theory, semi-classical asymptotic approximations in quantum mechanics, phenomena in the semi-conductor devices theory and so on.

Elliptic and, more generally, coercive singular perturbations are of special interest for the asymptotic solution of problems, which are characterized by the boundary layer phenomena, as, for instance in the theory of thin buckling plates, elastic rods and beams.

A perturbation is said to be singular since its structure and the nature of the phenomena which it describes is completely different from the ones which are proper to the corresponding reduced problem. For instance, considering a gas flow around an obstacle in fluid dynamics in the situation when the dimensionless viscosity parameter (the inverse of the Reynolds number) is small, one has a mathematical model (the Navier-Stokes equations) which reflects the physical boundary layer phenomenon in a neighborhood of the obstacle, while setting the viscosity equal to zero one gets a different mathematical model (the Euler-Lagrange equations), in which this phenomenon is completely lost.

Considering a stochastic model which is a superposition of a deterministic process and of a "white noise" of a small level (described as a Wiener process with a small variance), one comes to the Kolmogorov-Chapman parabolic equation with a small diffusion term for the density of the stochastic process in question; it is a singular perturbation of the reduced hyperbolic equation, which describes the deterministic situation.

Other examples come from the theory of elastic rods or beams. If an elastic beam at rest is subjected to a strong pulling out longitudinal force described by a large dimensionless parameter, then using this parameter and setting it equal to infinity, one can simplify the mathematical model, getting a reduced differential equation, which only partially refects the physical phenomenon. Indeed, for instance, in the case when the beam at rest is simply supported by its end points, the natural boundary conditions would tell that at the end points the displacements and the momenta of the forces applied must be zero. However for the reduced equation (which is of the second order) it is possible to have only the displacements vanishing, while the momenta of the forces at the end points (not necessarily zero) are determined *a posteriori*. In fact, a boundary layer phenomenon in a neighborhood of the end points of the beam is present in this situation and should not be neglected.

The linear singular perturbation theory and its possible applications is the topic of this volume. Let $\mathcal{A}^\varepsilon$ be such a perturbation which is usually a differential (or integro-differential) operator affected by the presence of a small parameter $\varepsilon \in (0, \varepsilon_0)$. One is interested in solving the equation

$$\mathcal{A}^\varepsilon u_\varepsilon = f, \qquad \varepsilon \in (0, \varepsilon_0), \tag{1}$$

where f is a given second member.

It is (impicitly or explicitly) assumed that the reduced equation (defined in a natural way and usually much simpler than (1)):

$$\mathcal{A}^0 u_0 = f, \tag{2}$$

can be uniquely solved.

Then one is interested in getting a convergent series

$$u_\varepsilon = \sum_{k \geq 0} \varepsilon^k u_k \tag{3}$$

for the solution u_ε of (1), and that is usually not possible, since, as a consequence of a singular nature of the perturbation $\mathcal{A}^\varepsilon$, the solutions of (1) do not depend analytically on ε even in the case when $\mathcal{A}^\varepsilon$ is a real analytic function of $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ valued in some operator space.

Giving up the convergence, one asks for an asymptotic convergence of the series on the right hand side of (3), i.e. for each integer $N > 0$ one would like to have in a certain sense:

$$u_\varepsilon - \sum_{0 \leq k < N} \varepsilon^k u_k = O(\varepsilon^N), \quad \text{as } \varepsilon \to 0. \tag{4}$$

Usually, formal asymptotic expansion techniques allow to produce a relatively simple algorithm for computing recursively the coefficients u_k, $k \geq 0$, in the asymptotic expansion (3) or, even for more complicated forms of such an expansion, taking into account, for instance, the boundary layer phenomenon.

A very important question, which arises afterwards, is a proof of the asymptotic convergence like (4) (or in a different form, appropriate to the situation considered). The only reasonable way to ensure the asymptotic convergence of approximate solutions to the solution of (1) is to have uniform *a priori* estimates for u_ε, i.e. uniform upper bounds for the norm of the inverse operator $(\mathcal{A}^\varepsilon)^{-1}$ (whose existence is, in fact, a part of the problem) as an operator from an appropriate data space $\mathcal{D}_\varepsilon$ into the solution space $\mathcal{H}_\varepsilon$,

$$(\mathcal{A}^\varepsilon)^{-1} : \mathcal{D}_\varepsilon \to \mathcal{H}_\varepsilon, \quad \varepsilon \in (0, \varepsilon_0). \tag{5}$$

Such a question is not merely a matter of mathematical rigor, but it is crucial for the entire "raison d'être" of the formal techniques which may, eventually, allow to determine uniquely u_k, $k \geq 0$, even in the situation, when the solution to (1) does not exist or is not uniquely defined without additional restrictions.

For being specific, consider several examples.

Example 1. Let $q(x)$ ($x \in \mathbf{R}^3$) be a real valued infinitely differentiable function and assume that $q(x) \equiv q(\infty)$ for $|x| \geq r$, $r > 0$ being sufficiently large. Consider the following singular perturbation:

$$A^\varepsilon u := u - \varepsilon^2 \mathrm{div}(q(x) \mathrm{grad}\; u) \tag{6}$$

and the corresponding equation

$$A^\varepsilon u_\varepsilon = f, \tag{7}$$

where f is a given infinitely differentiable function with compact support, i.e. $f(x)$ vanishes outside of some ball in $\mathbf{R}^3$, and $u_\varepsilon(x)$ is supposed to vanish at infinity.

The natural reduced operator A^0 for A^ε is the identity, so that $u_0 = f$, if u_ε in (7) admits an asymptotic expansion.

Furthermore, introducing the differential operator:

$$B(x, \partial_x) := \operatorname{div}(q(x)\operatorname{grad}) = \nabla \cdot (q(x)\nabla), \tag{8}$$

one can formally write an asymptotic expansion for u_ε in the form:

$$u_\varepsilon \sim \sum_{k \geq 0} \varepsilon^{2k} u_{2k}(x), \quad u_{2k}(x) := (B(x, \partial_x))^k f(x), \tag{9}$$

whose right hand side makes sense since f is smooth and has a compact support.

Now the crucial question of an asymptotic convergence of the series on the right hand side of (9) to u_ε arises.

Let us make the following basic additional assumption:

$$\inf_{x \in \mathbb{R}^3} q(x) = q_0 > 0. \tag{10}$$

Under this assumption (10) (which is an ellipticity condition for the singular perturbation A^ε) one can easily show the asymptotic convergence in (9).

Indeed, integrating by part after multiplication of (7) by u_ε, using the Cauchy-Schwarz inequality and the basic assumption (10), one gets the following *a priori* estimate

$$\left(||u_\varepsilon||^2_{L^2(\mathbb{R}^3)} + \varepsilon^2 ||\nabla u_\varepsilon||^2_{L^2(\mathbb{R}^3)}\right)^{1/2} \leq \gamma(q)||f||_{L^2(\mathbb{R}^3)}, \tag{11}$$

where

$$\gamma(q) = \max\left\{1, \left(\sup_{x \in \mathbb{R}^3} q(x)\right)\left(\inf_{x \in \mathbb{R}^3} q(x)\right)^{-1}\right\} < \infty. \tag{12}$$

Introducing the norms of vectorial order $s = (s_1, s_2, s_3) \in \mathbb{R}^3$,

$$||u||_{(s),\varepsilon} := ||\varepsilon^{-s_1}(1 + |\xi|^2)^{s_2/2}(1 + \varepsilon^2|\xi|^2)^{s_3/2}\widehat{u}||_{L^2(\mathbb{R}^3)}, \tag{13}$$

where $\widehat{u}(\xi) = F_{x \to \xi} u$ is the Fourier transform of u, one can rewrite (11) in the form:

$$||u||_{(0,0,1),\varepsilon} \leq \gamma(q)||f||_{(0,0,0),\varepsilon}.$$

Actually,using (13) and estimating more accurately by the Cauchy-Schwarz inequality, one gets the following sharp a priori estimate:

$$(14) \qquad ||u_\varepsilon||_{(0,0,1),\varepsilon} \le \gamma(q)||f||_{(0,0,-1),\varepsilon}, \quad \forall\varepsilon > 0,$$

where $\gamma(q) < \infty$ is defined by (12).

Differentiating (7) with respect to x and using the same argument, one gets for any integer $s_2 \geq 0$, $s_3 > 0$ the following estimate:

$$(15) \qquad ||u_\varepsilon||_{(s_1,s_2,s_3),\varepsilon} \le C(s_1, s_2, s_3, \varepsilon_0, q)||f||_{(s_1,s_2,s_3-2),\varepsilon}, \quad \forall\varepsilon \in (0, \varepsilon_0),$$

where the constant $C(s_1, s_2, s_3, \varepsilon_0, q)$ may only depend on $s_1, s_2, s_3, \varepsilon_0, \gamma(q)$ and some derivatives of $q(x)$.

An estimate like (15) can be established for any $s = (s_1, s_2, s_3) \in \mathbf{R}^3$.

Using (15) for each given $s \in \mathbf{R}^3$, one finds:

(16)

$$||u_\varepsilon - \sum_{0\le k<N} \varepsilon^{2k} u_{2k}||_{(s),\varepsilon} \le C\varepsilon^{2N}||f||_{(s_1,s_2+2N,s_3-2),\varepsilon}, \quad \forall N > 0, \ \ \forall\varepsilon \in (0, \varepsilon_0),$$

where the constant $C > 0$ may only depend on $N, s, \varepsilon_0, \gamma(q)$ and some derivatives of $q(x)$.

Thus, (16) implies the asymptotic convergence in (9) and one may differentiate the asymptotic relation (9) with respect to x any number of times without loosing the asymptotic convergence, provided that $f \in C_0^\infty(\mathbf{R}^3)$, i.e. f is smooth and has compact support.

Note that (9) is not satisfactory in the sense, that it provides the asymptotic approximations to u_ε, which all have their support coinciding with the support of f. This is not the case for the solution u_ε of (6), (7) under the assumptoin (10), the support of u_ε being the entire $\mathbf{R}^3$.

A more appropriate asymptotic formula for the solution u_ε of (6), (7) under the assumption (10) is provided by the following argument. First, assume that

$q(x) \equiv q > 0$ is a constant. Then the solution of (6), (7) is given by the formula:

$$u_\varepsilon(x) = (4\pi\varepsilon^2 q)^{-1} \int_{\mathbb{R}^3} f(y)|x-y|^{-1} \exp(-|x-y|/(\varepsilon q^{1/2}))dy = \\ = (F^{-1}_{\xi\to x}(1+\varepsilon^2 q|\xi|^2)^{-1} F_{x\to\xi} f)(x), \tag{17}$$

where $F_{x\to\xi}$ and $F^{-1}_{\xi\to x}$ are the direct and inverse Fourier transform, respectively.

Now, if $q(x)$ is not a constant but still satisfies the conditions hereabove and, especially, the condition (10), one can still define the function:

$$u_\varepsilon^{(0)} := (4\pi\varepsilon^2 q(x))^{-1} \int_{\mathbb{R}^3} f(y)|x-y|^{-1} \exp(-|x-y|/\varepsilon(q(x))^{1/2}))dy = \\ = (F^{-1}_{\xi\to x}(1+\varepsilon^2 q(x)|\xi|^2)^{-1} F_{x\to\xi} f)(x) := (S^\varepsilon f)(x). \tag{18}$$

It turns out that

$$A^\varepsilon u_\varepsilon^{(0)} = f - \varepsilon g_\varepsilon, \tag{19}$$

where

$$||g_\varepsilon||_{(s_1,s_2,s_3-2),\varepsilon} \le C||f||_{(s_1,s_2,s_3-2),\varepsilon}, \quad \forall\varepsilon \in (0,\varepsilon_0), \tag{20}$$

with a constant $C > 0$, which does not depend on $\varepsilon, g_\varepsilon$ and f.

In other words, S^ε being the operator defined by (18) and introducing the spaces $H_{(s),\varepsilon}(\mathbb{R}^3)$ of the functions u whose Fourier transforms are locally integrable and have the norms (13) finite, $\forall\varepsilon \in (0,\varepsilon_0)$, one can rewrite (19), (20) as follows:

$$A^\varepsilon S^\varepsilon = I - \varepsilon Q^\varepsilon, \quad I = \text{identity}, \tag{21}$$

where the family of linear mappings Q^ε,

$$Q^\varepsilon : H_{(s-\nu),\varepsilon}(\mathbb{R}^3) \to H_{(s-\nu),\varepsilon}(\mathbb{R}^3), \quad \nu = (0,0,2), \quad \varepsilon \in (0,\varepsilon_0), \tag{22}$$

is equicontinuous, i.e. the norm of Q^ε is uniformly bounded with respect to $\varepsilon \in (0,\varepsilon_0)$, $\forall\varepsilon_0 < \infty$.

Thus, the norm of $\varepsilon Q^\varepsilon$ $(0 < \varepsilon < \varepsilon_0)$ is strictly less than 1, i.e. $\varepsilon Q^\varepsilon$ $(0 < \varepsilon < \varepsilon_0)$ is an equicontraction, provided that $\varepsilon_0 > 0$ is sufficiently small.

Hence, (21) yields for $\varepsilon_0 > 0$ sufficiently small:

$$(A^\varepsilon)^{-1} = S^\varepsilon(I - \varepsilon Q^\varepsilon)^{-1} = S^\varepsilon \sum_{k \geq 0} \varepsilon^k (Q^\varepsilon)^k, \tag{23}$$

the series on the right hand side of (23) being convergent with respect to the operator norm in $\mathcal{L}(H_{(s-\nu),\varepsilon}(\mathbf{R}^3))$ uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$.

One can show that in fact one has also the following asymptotic relation:

$$S^\varepsilon A^\varepsilon = I - \varepsilon Q_1^\varepsilon, \quad \varepsilon \in (0, \varepsilon_0),$$

where the family of linear mappings:

$$Q_1^\varepsilon : H_{(s),\varepsilon}(\mathbf{R}^3) \to H_{(s),\varepsilon}(\mathbf{R}^3), \quad \varepsilon \in (0, \varepsilon_0),$$

is again equicontinuous.

Of course, for the zero approximation $u_\varepsilon^{(0)}$ defined by (18) one has:

$$\|u_\varepsilon - u_\varepsilon^{(0)}\|_{(s),\varepsilon} \leq C\varepsilon \|f\|_{(s-\nu),\varepsilon} \tag{24}$$

and moreover, both u_ε and $u_\varepsilon^{(0)}$ are supported by the entire $\mathbf{R}^3$.

Introducing

$$u_\varepsilon^{(N-1)} = S^\varepsilon \sum_{0 \leq k < N} \varepsilon^k (Q^\varepsilon)^k f, \quad N > 0 \text{ integer}, \tag{25}$$

one finds:

$$\|u_\varepsilon - u_\varepsilon^{(N-1)}\|_{(s),\varepsilon} \leq C\varepsilon^N \|f\|_{(s-\nu),\varepsilon}, \quad \forall \varepsilon (0, \varepsilon_0), \tag{26}$$

where the constant $C > 0$ does not depend on ε, u_ε and f.

The basic difference between (16) and (26) is the fact, that the norm of f on the right-hand side of (26) is the same, i.e. $u_\varepsilon^{(N-1)}$ defined by (25) converges at order $O(\varepsilon^N)$ to u_ε in $H_{(s),\varepsilon}(\mathbf{R}^3)$, $\forall f \in H_{(s-\nu),\varepsilon}(\mathbf{R}^3)$, thus, also for *non-smooth* second members f, while in (16) the larger is $N > 0$ the smoother the second member must be in order to have the asymptotic convergence of order $O(\varepsilon^N)$.

Now, let us turn to the situation when $q(x)$ does not satisfy condition (10), for instance, let us consider the case of $q(x) \equiv -1$. Note that the formal asymptotic expansion (9) in this case takes the form:

$$u_\varepsilon \sim \sum_{k \geq 0} (-1)^k \varepsilon^{2k} \Delta^k f, \tag{27}$$

where Δ is the Laplace operator, $\Delta = \sum \partial_j^2$, $\partial_j = \partial/\partial x_j$.

However, equation (7), which in this case takes the form:

$$(1 + \varepsilon^2 \Delta) u_\varepsilon(x) = f(x), \quad x \in \mathbf{R}^3, \tag{28}$$

does not have a unique vanishing at infinity solution for $f \in C_0^\infty(\mathbf{R}^3)$.

One can consider two different classes of solutions satisfying at infinity the so-called Sommerfeld radiation conditions:

$$u_\varepsilon^\pm(x) = O(r^{-1}), \quad (i\varepsilon\partial_r \pm 1) u_\varepsilon^\pm(x) = O(r^{-2}), \quad \text{as } |x| = r \to \infty, \tag{29}$$

where $\partial_r = \partial/\partial r$ is the derivative with respect to $r = |x|$.

Each solution $u_\varepsilon^\pm$ is given by the formulae:

$$u_\varepsilon^\pm(x) = (4\pi\varepsilon^2)^{-1} \int_{\mathbf{R}^3} f(y)|x-y|^{-1} \exp(\pm i|x-y|/\varepsilon) dy. \tag{30}$$

Thus, the right hand side of (27) does not converge asymptotically, since it does not 'know' to which solution u_ε^+ or u_ε^- it should converge.

In fact, different methods are needed, in order to get convergent asymptotic approximations for the solutions $u_\varepsilon^\pm$ of (6), (7) when $q(x) \leq q_0 < 0$, $\forall x \in \mathbf{R}^3$, of course, $u_\varepsilon^\pm(x)$ being defined as solutions of (6), (7) satisfying the respective Sommerfeld radiation conditions (29).

Example 2. One of the efficient methods for solving approximately differential (and pseudodifferential) equations is the use of their finite difference approximations, which are, of course, perturbations of the approximated operators, the small parameter being the mesh-size of the uniform grid where the discretized difference equations are considered.

Let us have a look at the boundary value problem:

$$\text{(31)} \qquad \begin{cases} -\partial_x^2 u(x) = f(x), & x \in U = (0,1) \\ u(x') = \varphi(x'), & x' \in \partial U = \{0,1\}, \end{cases}$$

where $\partial_x = d/dx$, f is a given smooth function of $x \in \overline{U}$ and $\varphi(x')$, $x' \in \partial U$, are given real or complex numbers.

Since the solution $u(x)$ of (31) is a smooth function of $x \in \overline{U}$, too, one is templed to use a higher order approximation of (31) on the grid $\overline{U}_h = \{x{=}kh,\ k{=}0,1,\ldots,N\}$ with integer $N = h^{-1} > 0$.

For instance, the following finite difference approximation of $-\partial_x^2$ has the accuracy $O(h^4)$ on smooth functions:

$$\text{(32)} \qquad a(h, \partial_{x,h}, \partial_{x,h}^*) := -\partial_{x,h}\partial_{x,h}^* + (h^2/12)(\partial_{x,h}\partial_{x,h}^*)^2,$$

where $\partial_{x,h}$ and $\partial_{x,h}^*$ are the forward and backward finite difference derivatives, respectively, i.e.

$$\text{(33)} \quad (\partial_{x,h}u)(x) = h^{-1}\big(u(x{+}h) - u(x)\big), \quad (\partial_{x,h}^* u)(x) = h^{-1}\big(u(x) - u(x{-}h)\big).$$

Indeed, a straightforward computation shows that for any smooth function $u(x)$, $x \in \overline{U}$, one has:

$$\big(a(h, \partial_{x,h}, \partial_{x,h}^*) + \partial_x^2\big)u(x) = O(h^4), \quad \text{as } h \to 0.$$

One is tempted to use (32) for solving numerically (31).

However, one can not use (32) for all points on the grid $\overline{U}_h$ but only for the points $x_k = kh$ with $1 < k < N$. Of course at the points $x_0 = 0$ and $x_N = 1$

one can use the boundary condition in (31). Still one extra boundary condition, say at the points $x_1 = h$ and $x_{N-1} = 1-h$, is missing and one has to find this boundary condition in an appropriate way, since otherwise a discrete version of a boundary layer behavior in a neighborhood of the boundary $\partial U = \{0,1\}$ will appear, i.e. solutions of the homogeneous equation

$$a(h, \partial_{x,h}, \partial^*_{x,h})v_h(x) = 0$$

of the form $C_+ q^{-x/h} + C_- q^{-(1-x)/h}$, with $q = (7+\sqrt{48})$ will emerge with non-negligible coefficients $C_\pm$.

Thus, the approximation (32) along with appropriate boundary conditions is a singular perturbation of (31), h being the corresponding small parameter.

Some perturbations by finite differences might destroy the basic structure and make disappear the fundamental properties of the operators which are being approximated. For instance, the basic property of the differentiation operator $\partial_x = d/dx$ ($x \in \mathbf{R}$) is the fact that all the solutions of the homogeneous equation $\partial_x u(x) = 0$ are smooth (in fact, constants). This property of ∂_x is preserved on the grid $\mathbf{R}_h = h\mathbf{Z} = \{x{=}kh,\ k \in \mathbf{Z}\}$ by both the forward $\partial_{x,h}$ and the backward $\partial^*_{x,h}$ finite difference approximations of ∂_x. However, the centered finite difference derivative $\check{\partial}_{x,h} = (1/2)(\partial_{x,h} + \partial^*_{x,h})$ does not enjoy such a property anymore, since, besides the constants, also the non-smooth meshfunction $(-1)^{x/h}$ is a solution of the homogeneous equation $\check{\partial}_{x,h}u(x) = 0$ on the grid $\mathbf{R}_h$. The centered finite difference derivative is a *non-elliptic* approximation of ∂_x.

For the Hilbert transform

$$(Hu)(x) := (\pi i)^{-1} \text{ v.p. } \int_{\mathbf{R}} f(y)(x-y)^{-1}dy$$

the approximation

$$(H_h u)(x) := (\pi i)^{-1} h \sum_{y \in \mathbf{R}_h \setminus \{x\}} f(y)(x-y)^{-1}, \quad x \in \mathbf{R}_h$$

is *non-elliptic* (H_h is no longer invertible in $l_2(\mathbf{R}_h)$), while the approximation:

$$(\mathcal{H}_h u)(x) := (\pi i)^{-1} h \sum_{y \in \mathbf{R}_h \setminus \{x\}} f(y)(x-y)^{-1}(1 - \cos(\pi h^{-1}(x-y)), \quad x \in \mathbf{R}_h$$

preserves on the grid $\mathbf{R}_h$ all the fundamental properties of the Hilbert transform.

Example 3. Finite difference approximations of the heat equation provide a wealth of interesting situations from the point of view of singular perturbations.

Let us consider the Cauchy problem:

$$(34) \qquad \begin{cases} (\partial_t - \Delta)u(x,t) = f(x,t), & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = \varphi(x) \end{cases}$$

where $\varphi(x)$, $f(x,t)$ are given smooth functions with compact support.

Using the backward finite difference discretization as an approximation of $\partial_t = \partial/\partial t$ on the grid $\mathbf{R}_\tau^+ = \{t{=}k\tau,\ k{>}0\}$, one gets for the corresponding approximation $u_\tau(x,t)$, $t \in \mathbf{R}_\tau^+$ of $u(x,t)$ the following singularly perturbed recursion equations (implicit finite difference scheme):

$$(35) \qquad \begin{cases} (1 - \tau\Delta)u_\tau(x,t) = \tau f(x,t) + u_\tau(x,t{-}\tau), & t \in \mathbf{R}_\tau^+ \\ u_\tau(x,0) = \varphi(x). \end{cases}$$

For finding $u_\tau(x,t)$ on each step of the recursion one has to invert the singular perturbation $(1 - \varepsilon^2\Delta)$, $\tau = \varepsilon^2$, which is precisely of the elliptic type discussed hereabove.

One might be tempted to use the centered finite difference $\check{\partial}_{t,\tau}$ for approximating ∂_t since the accuracy in this case is $O(\tau^2)$. This is the so-called Richardson's scheme. However, $\check{\partial}_{t,\tau}$ being a *non-elliptic* approximation of ∂_t, one should not expect to have a finite difference scheme, reflecting the basic properties of the heat operator. First, there is a problem of imposing (or not) an extra initial condition, since the Richardson's approximation is a three-step finite difference scheme. An easy Fourier analysis shows that actually one has to consider the corresponding discretized problem in this case on $\mathbf{R}_\tau^+ \times \mathbf{R}_x^n$ not as an initial, but as a boundary value problem with just one boundary condition at $t = 0$ and another one either at $t = \infty$ (the solution vanishes as $t \to +\infty$) or at $t = T < \infty$. In both cases the smooth part of the solution will be 'polluted' by the non-smooth part generated by the solutions of the homogeneous equation having the form:

$$v_\tau(x,t) = (-1)^{(T-t)/\tau}\big((1+\tau^2\Delta^2)^{1/2} - \tau\Delta\big)^{-(T-t)/\tau}\psi(x) =$$
$$= (-1)^{(T-t)/\tau}\Big(F^{-1}_{\xi\to x}\big((1+\tau^2|\xi|^4)^{1/2} + \tau|\xi|^2\big)^{-(T-t)/\tau}F_{x\to\xi}\psi\Big)(x,t),$$

for any finite $T > 0$, the oscillatory non-smooth factor $(-1)^{(T-t)/\tau}$ being always present.

A good scheme having the accuracy $O(\tau^2)$ is the following one:

$$(36)\qquad \begin{cases} \big(\partial_{t,\tau} + \big((\tau/2)\Delta^2 - \Delta\big)\Theta_{t,\tau}\big)u_\tau(x,t) = \big(1 + (\tau/2)(\partial_t - \Delta)\big)f(x,t), \\ u(x,0) = \varphi(x), \qquad x \in \mathbf{R}^n,\ t \in \overline{\mathbf{R}}^+_\tau = \tau\mathbf{Z}_+, \end{cases}$$

where $\mathbf{Z}_+$ is the set of all non-negative integers and $\Theta_{t,\tau}$ is the shift operator on $\overline{\mathbf{R}}^+_\tau$, i.e. $(\Theta_{t,\tau}v)(t) = v(t+\tau)$.

Note that the corresponding singular perturbation to be inverted on each step of solving the implicit finite difference problem (36), is again an elliptic singular perturbation having the form:

$$(37)\qquad A^\varepsilon = 1 - \varepsilon^2\Delta + (1/2)\varepsilon^4\Delta^2, \quad \varepsilon^2 = \tau,\ x \in \mathbf{R}^n.$$

Of course, hereabove one may use the usual finite difference approximation Δ_h of the Laplacian on the grid $\mathbf{R}^n_h = h\mathbf{Z}^n$:

$$(38)\qquad \Delta_h := \sum_{1\le k\le n} \partial_{x_k,h}\partial^*_{x_k,h},$$

thus getting (by using the schemes hereabove) unconditionally stable time-space finite difference approximations of the heat equation with one condition at $t = 0$ (also for the Richardson's scheme), i.e. the approximations hereabove with Δ replaced by Δ_h given by (38), are stable (in the sense of non-accumulation of the errors), whatever the mesh-sizes τ and h are.