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On Global Univalence Theorems

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PREFACE

This volume of lecture notes contains results on global univalent mappings. Some of the material of this volume had been given as seminar talks at the Department of Mathematics, University of Kansas, Lawrence during 1978-79 and at the Indian Statistical Institute, Delhi Centre during 1979-80.

Even though the classical local inverse function theorem is well-known, Gale-Nikaido's global univalent results obtained in (1965) are not known to many mathematicians that I have sampled. Recently some significant contributions have been made in this area notably by Garcia-Zangwill (1979), Mas-Colell (1979) and Scarf-Hirsch-Chilnisky (1980). Global univalent results are as important as local univalent results and as such I thoughtit is worthwhile to make these results well-known to the mathematical community at large. Also I believe that there are very many interesting open problems which are worth solving in this branch of Mathematics. I have also included a number of applications from different disciplines like Differential Equations, Mathematical Economincs, Mathematical Programming, Statistics etc. Some of the results have appeared only in Journals and we are bringing them to-gether in one place.

These notes contain some new results. For example Proposition 2, Theorem 4 in Chapter II, Theorem 4, Theorem 5 in Chapter III, Theorem 2" in Chapter V, Theorem 8 in Chapter VI, Theorem 2 in Chapter VII, Theorem 9 in Chapter VIII are new results.

It is next to impossible to cover all the known results on global univalent mappings for lack of space and time. For example a notable omission could be the role played by univalent mappings whose domain is complex numbers. We have also not done enough justice to the problem when a PL-function will be a homeomorphism in view of the growing importance of such functions. We have certainly given references where an interested reader can get more information.

I am grateful to Professors : Andreu Mas-Colell, Ruben Schramm, Albrecht Dold and an anonymous referee for their several constructive suggestions on various parts

of this material. I am also grateful to Professor David Gale for the example given at the end of Chapter II and Professor L. Salvadori for some useful discussion that I had with him regarding Chapter VII.

Moreover I wish to thank the Indian Statistical Institute, Delhi Centre for providing the facilities and the atmosphere necessary and conducive for such work. Finally I express my sincere thanks to Mr. V.P. Sharma for his excellent and painstaking work in typing several revisions of the manuscript, Mr. Mehar Lal who typed a preliminary version of this manuscript and Mr. A.N. Sharma who helped me in filling many symbols.

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INTRODUCTION

Let Ω be a subset of R^n and let F be a differentiable function from Ω to R^n. We are looking for nice conditions that will ensure the equation $F(x) = y$ to have at most one solution for all $y \in R^n$. In other words we want the equation $F(x) = y$ to have a unique solution for every y in the range of F.

Classical inverse function theorem says that if the Jacobian of the map does not vanish then if $F(x) = y$ has a solution x^*, then x^* is an isolated solution, that is, there is a neighbourhood of x^* which contains no other solution. In the global univalence problem, we demand x^* to be the only solution throughout Ω.

It is a fascinating fact, why the global univalence problem had not been posed or at any rate solved before Hadamard in 1906, which of course is a very late stage in the development of Analysis. It is funny and actually baffling, how much misunderstanding associated with the global univalence problem survived right into the middle of the twentieth century. A brief history of this may not be out of place here.

Paul Samuelson in (1949) gave as sufficient condition for uniqueness, that the Jacobian should not vanish and it was pointed out by A. Turing that this statement was faulty. However Paul Samuelson's economic intuition was correct and in his case all the elements of the Jacobian were essentially one-signed and this condition combined with the non-vanishing determinant, turns out to be sufficient to guarantee uniqueness in the large.

Paul Samuelson (1953) then stated that non-vanishing of the leading minors will suffice for global univalence in general. But Nikaido produced a counter example to this assertion and he went on to show that global univalence prevails in any convex region provided the Jacobian matrix is a quasi-positive definite matrix. Later, Gale proved that it is sufficient for uniqueness in any rectangular region provided the Jacobian matrix is a P-matrix, that is, every principal minor is positive. In fact this culminated in the well-known article of Gale-Nikaido (1965) which is the main source of inspiration for the present writer.

I should mention two other articles. The article of Banach-Mazur (1934) gives probably the first proof of a relevant result formulated with the demands of rigour still valid to-day. The more recent article by Palais (1959) covers a much wider area than the article of Banach and Mazur.

There are several approaches one can consider to the global univalent problem. For example the approach could be via linear inequalities, monotone functions or PL functions. Throughout we have followed more or less the approach through linear inequalities.

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In most of the theorems the conditions for global univalence are very stringent and therefore often not satisfied in applications. Another problem is to verify the conditions of the theorem in practice. In general it is hard to obtain necessary and sufficient conditions for global univalence results. There is lot of room for further research in this area. Gale-Nikaido's global univalent theorem is valid even if the partial derivatives are not continuous whereas Mas-Colell's results as well as Garcia-Zangwill's results demand the partial derivatives to be continuous. One of the major open problems in this area is the following: Can continuity of the derivatives in Mas-Colell's results be dispensed with (altogether or at least in part) or alternatively - are there counter examples? Another problem is the following: Is the fundamental global univalent result due to Gale-Nikaido valid in any compact convex region?

As already pointed out in some of the applications complete univalence is not warranted but in which some weaker univalence enunciations can nevertheless be made. In this connection I would like to cite at least two important papers one by Chua and Lam and the other by Schramm.

Because of the lack of a text on the global univalence and since the results are available only in articles scattered in various journals or in texts devoted to other subjects (for example Economics), I felt the need for writing this notes on global univalent mappings. In the next ten chapters with the exception of the first two chapters, various results on global univalent mappings as well as their applications are discussed. Also many examples are given and several open problems are mentioned which I believe will interest research workers.

Prerequisites needed for reading this monograph are real analysis and matrix theory. Here are a few suggestions.

[1]. W.Rudin (1976), Principles of Mathematical Analysis, Third Edition (International Student Edition) McGraw-Hill, Koyakusha Ltd.

[2]. F.R.Gantmacher (1959), The Theory of Matrices Vols. I and II, Chelsea Publishing Company, New York.

[3]. C.R. Rao (1974), Linear Statistical Inference and its Applications, Second Edition, Wiley Eastern Private Limited, New Delhi (Especially Chapter I dealing with 'Algebra of vectors and matrices').

[4]. G.S. Rogers (1980), Matrix derivatives, Marcel Dekker, New York and Basel (Actually only chapters 13 and 14 have the Jacobian and its properties as their central topic while 11 and 12 refer to the general theory).

[5]. W.Fleming (1977), Functions of several variables, Second Edition, Springer-Verlag, Heidelberg-New York.

Some knowledge of algebraic topology will be useful (especially degree theory) and we have mentioned a few references in Chapter IV.

CONTENTS

CHAPTER I

PRELIMINARIES AND STATEMENT OF THE PROBLEM

Abstract : In this chapter we will collect some well-known results like classical inverse function theorem, domain invariance theorem etc for ready reference (without proof). We will then give the statement of the problem considered in this monograph cite a few results and make some remarks.

Classical inverse function theorem : Let F be a transformation from an open set $\Omega \subset R^n$ to R^n. We will say that F is locally univalent, if for every $x \in \Omega$ there exists a neighbourhood U_x of x such that $F|U_x$ (=F restricted to U_x) is one-one. Inverse function theorem gives a set of sufficient conditions for F to be locally univalent. We come across such problems in various situations. For example, suppose for a given y, there exists an x_o such that $F(x_o) = y$. We may like to know whether there are points x other than x_o, contained in a small neighbourhood around x_o satisfying $F(x) = y$. Classical inverse function theorem asserts that the solution is unique locally. In order to state the inverse function theorem we need the following.

Definition : A transformation F is differentiable at t_o if there exists a linear transformation L (depending on t_o) such that

$$\lim_{h \to 0} \frac{1}{||h||} [F(t_o+h)-F(t_o)-L(h)] = 0 .$$

Here $||h||$ stands for the usual vector norm. The linear transformation L is called the differential of F at t_o and is often denoted by $DF(t_o)$. Write $F = (f_1,f_2,\ldots,f_n)$ where each f_i is a real-valued function from Ω. We denote their partial derivatives as $f_i^j = \frac{\partial f_i}{\partial x_j}$.

Remark 1 : A transformation F is differentiable at t_o if and only if each of its components f_i is differentiable at t_o for $i = 1,2,\ldots,n$.

Remark 2 : If F is differentiable at t_o, then the matrix of the linear transformation L is simply the Jacobian matrix J of partial derivatives $f_i^j(t_o)$.

Definition : Call F a transformation of class q, $q \geq 0$ if each f_i is of class $C^{(q)}$. That is, for every $f_i (i = 1,2,\ldots,n)$ all the partial derivatives upto order q exist and are continuous over its domain.

We are now ready to state the (local) inverse function theorem.

Local inverse function theorem : Let F be a map of class $C^{(q)}$, $q \geq 1$ from an open set $\Omega \subset R^n$ into R^n. If the Jacobian at $t_o \in \Omega$ does not vanish, then there exists an open set $\Delta_o \subset \Omega$ containing t_o such that :

(i) $F|\Delta_o$ is one-one, that is, F restricted to Δ_o is univalent.

(ii) $F(\Delta_o)$ is an open set.

(iii) The inverse G of $F|\Delta_o$ is of class $C^{(q)}$.

(iv) $J_G(x) = (J_F(t))^{-1}$ where $F(t) = x$, $t \in \Delta_o$. Here $J_G(x)$ denotes the Jacobian matrix evaluated at x. Proof of this may be found in Fleming [17].

Remark 1 : In one dimension the situation is simpler. If F is a real-valued function with domain an open interval Ω, then F^{-1} (=inverse map of F) exists if F is strictly monotone. Also F will be strictly monotone if $F'(t) \neq 0$ for all $t \in \Omega$, and in fact $G'(x) = \frac{1}{F'(t)}$ where $x = F(t)$. In higher dimensions the Jacobian $J_F(t)$ takes the place of $F'(t)$. The situation here is more complicated. For example, the non-vanishing of the Jacobian does not guarantee that F has a (global) inverse as in the univariate case. However, if $J_F(t_o)$ does not vanish at t_o, we can find a small neighbourhood Δ_o containing t_o such that F restricted to Δ_o will have an inverse. In other words we can only assert local inverse. This is precisely part of the statement of inverse function theorem.

If one is interested in just the local univalence we have the following theorem (proof may be found in [44]).

Local univalent theorem : Let $F:\Omega \subset R^n \to R^n$ be a mapping where Ω is an open connected subset of R^n. We have the following:

(i) If F is differentiable at a point $t_o \in \Omega$ and $J_F(t_o) \neq 0$, then there is a neighbourhood U of t_o such that $F(y) = F(t_o)$, $y \in U \Rightarrow y = t_o$.

(ii) If F is continuously differentiable in a neighbourhood of an interior point t_o of Ω and $J_F(t_o) \neq 0$, then there is a neighbourhood U of t_o where F is univalent, that is, $F(y) = F(z)$, $y,z \in U \Longrightarrow y = z$.

We are now ready to state the following:

Theorem on invariance of interior points : Let $F:\Omega \to R^n$ be a differentiable map with non-vanishing Jacobian, where Ω is an open region in R^n. Then the image set $F(\Omega)$ is also an open region.

For a proof see Nikaido [44]. This result is true not only for differentiable mappings with nonvanishing Jacobians but also for homeomorphic mappings from a region of R^n into R^n. That is the content of the following classical theorem due to Brouwer.

Invariance of domain theorem: If Ω is open in R^n and $F:\Omega \to R^n$ is one-one and continuous, then $F(\Omega)$ is open and F is a homeomorphism. For a proof see [30].

Definition : A mapping $F:\Omega \to R^n$ is called a local homeomorphism if for each $t \in \Omega$, a neighbourhood of t is mapped homeomorphically by F onto a neighbourhood of F(t).

It is clear that if $F:\Omega \to R^n$ is a continuously differentiable function with non-vanishing Jacobian it follows from local inverse-function theorem or local univalent theorem that F is a local homeomorphism. We will introduce one more definition.

Definition : Let $F:\Omega \to R^n$ be a continuous mapping where Ω is an open region in R^n with the property that each $y \in F(\Omega)$ has a neighbourhood V such that each component of $F^{-1}(V)$ is mapped homeomorphically onto V by F. Then F is called a covering map and (Ω,F) is called a covering space for $F(\Omega)$. In this case, the cardinal number n of the set $F^{-1}(y)$ is the same for all $y \in F(\Omega)$. If n is a finite integer, then F is called a finite covering, or more specifically, an n-covering.

Remarks : It is well-known that every covering map $F:\Omega \to R^n$ is a homeomorphism if Ω is connected and that every homeomorphic onto function $F:\Omega \to R^n$ is a covering map and every covering map is a local homeomorphism. However the converse is not true. A local homeomorphism need not be a covering map and a covering map need not be a homeomorphic onto function. A 1-covering map is necessarily a homeomorphic onto function. The following result is well-known [48].

A theorem on covering space : Let X and Y be connected, locally pathwise connected spaces (for example $X = Y = R^n$). Furthermore suppose Y is simply connected. Then F is a homeomorphism of X onto Y if and only if (X,F) is a covering space of Y. [Here $F:X \to Y$ is a map from X to Y].

We need this result especially in chapter IV where sufficient conditions are given in order that a map F from R^n to R^n will be a homeomorphism onto R^n. For results on degree theory, we freely use from chapter VI in [48]. Other good references for degree theory are [13,59,63].

Statement of the problem : Let $F:\Omega \subset R^n \to R^n$ be a differentiable map. We want F to be globally one-one throughout Ω. What conditions should we impose on the map F and the region Ω so that F is globally one-one ?

Remark 1 : Non-vanishing of the Jacobians alone will not suffice except in the univariate case. See the example of Gale and Nikaido given in chapter III.

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Remark 2 : Even in R^1 non-vanishing of the derivative is not a necessary condition for global univalence. For example $f(x) = x^3$ is globally univalent throughout R^1 whereas its derivative vanishes at $x = 0$. In general it appears difficult or hopeless to derive necessary conditions whenever global univalence prevails.

We will cite now a few typical results to give the reader some idea about this monograph.

Fundamental global univalence theorem : (Gale-Nikaido-Inada) : Let $F:\Omega \subset R^n \to R^n$ be a differentiable mapping where Ω is a rectangular region in R^n. Then F is globally univalent in Ω if either one of the following conditions holds good.

(a) J(x) (= Jacobian of F at x) is a P-matrix for every $x \in \Omega$.

(b) J(x) is an N-matrix and the partial derivatives are continuous for all $x \in \Omega$.

A global univalent theorem in R^3 [Parthasarathy] : Let F be a differentiable map from a rectangular region $\Omega \subset R^3$ to R^3 with its Jacobian J having the following two properties for every $x \in \Omega$:

(a) diagonal entries are negative and off-diagonal entries are positive.

(b) Every principal minor of order 2×2 is negative.

Then F is univalent in Ω.

Plastock's theorem : Let $F:R^n \to R^n$ be a continuously differentiable map. Suppose J does not vanish at any $x \in R^n$. If

$$\int_0^\infty \inf_{||x||=t} (1/||J(x)^{-1}||)dt = \infty$$

then F is a homeomorphism of R^n onto R^n. In fact F is a diffeomorphism. (Here $||x||$ stands for the usual Euclidean distant norm and $||A|| = \sup||Au||$ for A an $n \times n$ matrix and u an n vector with norm one).

In order to state McAuley's theorem we need the following definition.

Definition : Call a continuous mapping $F:\Omega \to R^n$ light if $F^{-1}(F(x))$ is totally disconnected for each $x \in \Omega$. [Here we will assume Ω to be a unit ball]. Call F open if for each U open in Ω, F(U) is open relative to $F(\Omega)$. Denote by S_F the set of points $x \in \Omega$ such that F is not locally one-one at x.

McAuley's Theorem : Suppose that F is a light open mapping of a unit ball Ω in R^n onto another unit ball B in R^n such that (1) $F^{-1} F(\partial\Omega) = \partial\Omega$ (2) $F(\partial\Omega) = \partial B$ (3) $F|S_F$ is one-one (4) for each component C of $B-S_F$ there is a nonempty V in C open relative to B such that $F|F^{-1}(V)$ is one-one. Then F is a homeomorphism.

Scarf's conjecture : Let $F:\Omega \subset R^n \to R^n$ be continuously differentiable on a compact rectangle Ω with $\det J(x) > 0$ for every $x \in \Omega$. Further suppose $J(x)$ is a P-matrix for every $x \in \partial\Omega$ (= boundary of Ω). Then F is one-one throughout Ω.

This conjecture was proved by three different set of researchers Garcia-Zangwill, Mas-Colell and Scarf et al. This result is an significant generalization of Gale-Nikaido's theorem.

Schramm's theorem : Let Ω be an x-simple domain in the (x,y)-plane, ℓ its boundary. Let $F = (f,g) : \bar{\Omega} \to R^2$ be a differentiable map, α the minimum and β the maximum of f on ℓ. Suppose the Jacobian of F is an NVL matrix for each $z \in \Omega$ and for each $u \in (\alpha,\beta)$, Suppose at most two points $z \in \ell$ satisfy $f(z) = u$. Then F restricted to $\bar{\Omega} \setminus (A(\alpha) \cup A(\beta))$ is univalent where $A(u) = \{z : z \in \bar{\Omega} \text{ and } f(z) = u\}$.

Remark 1 : Results obtained so far on global univalence are not complete and we have mentioned several interesting open problems throughout the monograph. For example it is not known whether Gale-Nikaido's result holds good in any compact convex regions. In chapter VIII and IX we have given various applications of univalent results in other areas like differential equations, Economics, Mathematical programming, Algebra etc.

Remark 2 : All the theorems cited above with the exception of McAuley's theorem depend on the choice of a fixed coordinate system. This is so because we place conditions on the Jacobian matrix. Though one may argue that this may not be the most natural approach to the problem under consideration, the present writer feels that this method yields useful results in many problems that arise in practice. See Chapter VII and Chapter IX in this connection. Also in some special cases the matrix conditions turn out to be necessary as well - see for example theorem 1 and theorem 6 in chapter VIII. Also, the present writer feels that it is not difficult to check these matrix conditions in a given problem.

CHAPTER II

P-MATRICES AND N-MATRICES

Abstract : In this chapter we will give a geometric characterization of P-matrices. We will give some properties of N-matrices. These facts we need later to prove global univalence results due to Gale, Nikaido and Inada. We will also see the interrelation between P-matrices and positive quasi-definite matrices. Finally we examine the question whether P-property holds good under multiplication (sum) of two P-matrices - this kind of result is useful in determining when the composition F o G (sum, F+G) of two univalent functions is univalent.

Let A be an $n \times n$ matrix with entries real numbers. We will not consider matrices with complex entries. If A is a symmetric matrix then A is positive definite if the associated quadratic form $x'Ax > 0$, for any x different from 0. Here prime denotes the transpose of the vector x. It is well known that a symmetric matrix A is positive definite if and only if every principal minor of A is positive. Suppose we drop the symmetric assumption from A. In such situations can we prove similar results? In other words, suppose A has the following property, namely $x'Ax > 0$ for every $x \neq 0$. They can we assert that every principal minor of A is positive? Another interesting question is to characterize matrices whose principal minors are positive. Next we will answer these questions.

Characterization of P-matrices : We will start with a few definitions. Let A be a not necessarily symmetric real $n \times n$ matrix.

Definition : Call A a P-matrix if every principal minor of A is positive.

Definition : Call A a positive quasi-definite matrix if $x'Ax > 0$ for every $x \neq 0$.

Definition : Call A an N-matrix if every principal minor of A is negative. Further N-matrices are divided into two categories:

(i) An N-matrix is said to be of the first category if A has at least one positive element.

(ii) An N-matrix is said to be of the second category if every element of A is non-positive.

Definition : Call A a Leontief-type matrix if the off-diagonal entries are non-positive.

We will make a few quick remarks.

Remark 1 : Every positive quasi-definite matrix is necessarily a P-matrix (we will give a proof of this fact after characterizing the class of P-matrices). But the converse is not necessarily true as the following example shows. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Then $(Au,u) = u_1^2+2u_1u_2+u_2^2$ (where $u = (u_1,u_2)$) and $(Au,u) = 0$ whenever $u_1 = -u_2$. Thus A is a P-matrix but not positive quasi-definite. Also observe that A is a positive quasi-definite matrix if and only if $\left(\frac{A+A'}{2}\right)$ is a positive definite matrix. In this example $\left(\frac{A+A'}{2}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is a singular matrix.

Remark 2 : The following example shows that every positive quasi-definite matrix need not be positive definite. Let $A = \begin{bmatrix} 2 & 2 \\ 3 & 8 \end{bmatrix}$. Clearly $(Au,u) = u_1^2+5u_1u_2+8u_2^2 > 0$ for any $u \neq 0$. Hence A is positive quasi-definite but not a positive definite matrix as A is not symmetric.

Remark 3 : First category N-matrices share some properties in common with P-matrices as we shall see below. However there is a nice characterization for symmetric second category N-matrices. In order to do that we need the following definition. Call a matrix A, merely positive definite if (i) there exists some vector x such that $x'Ax < 0$ and (ii) whenever $x'Ax < 0$, this will imply $Ax \leq 0$ or $Ax \geq 0$-in other words Ax is onesigned. The result then is the following. If A is a symmetric N-matrix of the second kind then A is merely positive definite. Furthermore A has exactly one (simple) negative eigenvalue. Proofs of these results may be found in Rao [62]. We are now ready to prove some results on P-matrices.

Theorem 1 : Let A be a P-matrix or an N-matrix of the first category. Then the system of linear inequalities

$$Ax \leq 0 \quad \text{and}$$

$$x \geq 0$$

has only the trivial solution $x = 0$.

Game theoretic interpretation of theorem 1 : Theorem 1 says that the minimax value of the matrix game A (as well as the minimax value of every principal submatrix C of A) is positive, provided A is a P-matrix or an N-matrix of the first kind. This can be seen as follows. Suppose von Neumann value of the matrix game is less than or equal to zero. (We will assume minimizer chooses rows and maximizer chooses columns). We have a probability vector y for the minimizer such that $y'A \leq 0$ or $A'y \leq 0$ (prime denotes transpose). If A is a P-matrix or an N-matrix so is A'. Thus we have got a nontrivial non-negative vector y satisfying $A'y \leq 0$ which contradicts theorem 1 and consequently value of A must be positive. It is also clear that value of A' is positive. See [49, 54] for details regarding game theory and [61] for results relating game theory and M-matrices. We follow the proof as given in

Nikaido [44].

Proof of Theorem 1 : First we will prove when A is a P-matrix. We will use induction principle. For n = 1, clearly Theorem 1, is true. So assume theorem 1 for n = k, prove that it holds good for n = k+1 (Here n refers to the order of the square matrix A). Let $x \geq 0$. That is,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1,k+1}\, x_{k+1} \leq 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{k+1,1}x_1 + a_{k+1,2}x_2 + \cdots + a_{k+1,k+1}x_{k+1} \leq 0$$

Since $a_{ii} > 0$, we can increase if necessary x_i such that one of the inequality becomes an equality. We will assume without loss of generality

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1,k+1}\, x_{k+1} = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2,k+1}\, x_{k+1} < 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{k+1,1}x_1 + a_{k+1,2}x_2 + \cdots + a_{k+1,k+1}x_{k+1} \leq 0 .$$

Using the first equality, one can eliminate x_1 from the other inequalities. The resulting inequalities can be written as

$$a^*_{22}\, x_2 + \cdots + a^*_{2,k+1}\, x_{k+1} \leq 0$$

$$a^*_{k+1,1}x_2 + \cdots + a^*_{k+1,k+1}\, x_{k+1} \leq 0$$

where $a^*_{ij} = \dfrac{a_{11}}{a_{i1}} \times a_{ij} - a_{1j}$ where i,j = 2,3,...,k+1. Plainly the matrix $C = (a^*_{ij})$ is a P-matrix of order k. Hence by induction hypothesis, $x_i = 0 \;\forall\; i \geq 2$. Substituting this in the first equality, $a_{11}x_1 = 0$. But $a_{11} > 0$ and hence $x_1 = 0$. This terminates the proof of theorem 1 when A is a P-matrix.

Now assume A is an N-matrix of first category. Clearly order of an N-matrix of first category should be at least 2 x 2 . Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a < 0$, $d < 0$, and $ad-bc < 0$. This means b and c should be of the same sign. Since A is of first category, $b > 0$ and $c > 0$. Consequently $A^{-1} \geq 0$. If $Ax \leq 0$ then $A^{-1}Ax = x \leq 0$. But $x \geq 0$ by hypothesis, therefore x = 0. This proves the theorem when n = 2. As before assume the result for n = k where $k \geq 2$ and prove it holds good for n = k+1. As A has at least one positive element, we can imitate the proof verbatim given for P-matrices till we get the matrix $C = (a^*_{ij})$i,j = 2,3,...,k+1. Observe that det A = a_{11} det C. Since $a_{11} < 0$, det A = a_{11} det C < 0, it follows that det C > 0. In fact one can check that C is a P-matrix. Hence it follows from the first part of the proof $x_i = 0 \;\forall\; i \geq 2$. Since $a_{11} \neq 0, x_1 = 0$ from the first equality. This terminates the proof of theorem 1 for N-matrices of first kind.

Remark 1 : Geometrically, theorem 1 says the following: Any non-trivial non-negative vector cannot be mapped to a vector in the negative orthant when A is a P-matrix or an N-matrix of the first kind.

Remark 2 : Theorem 1 is valid for any matrix A which has non-negative inverse - that is $A^{-1} \geq 0$. Characterization results are available in the literature for such class of matrices. In particular if A is a Leontief type matrix then $A^{-1} \geq 0$ if and only if there exists some $x \geq 0$ such that $Ax > 0$.

Remark 3 : A result on linear inequalities asserts the following [See 18, pp. 49]. For any given matrix D not necessarily a square matrix exactly one of the following alternatives holds. Either the inequalities $x'D \leq 0$ has a semipositive solution or the inequality $Dy > 0$ has a non-negative solution.

In view of this result on linear inequalities, conclusion of theorem 1 can be viewed as follows: For any matrix A, suppose the system $Ax \leq 0$, $x \geq 0$ has only a trivial solution. This statement is equivalent to the fact that A has a left poverse, that is there exist non-negative matrices N,M such that NA = I+M. This observation is due to Charnes et. al. [8]. Another feature of the result on linear inequalities is the following, which says that von Neumann value of a P-matrix game is positive.

Theorem 2 : Suppose A is a P-matrix or an N-matrix of first category. Then there exists a positive vector $y_o \geq 0$ such that $Ay_o > 0$.

Proof : From theorem 1, the matrix $D = (A,-I)$ has no semi-positive solution x with $x'D \leq 0$ and consequently from the above remark it follows that $Dy = (A,-I)y > 0$ for some $y \geq 0$. Here $y = (y_1,y_2,\dots,y_n, y_{n+1}, y_{n+2},\dots,y_{2n})$. Define $y_o = (y_1,y_2,\dots y_n)$. Then clearly $Ay_o > z \geq 0$ where $z = (y_{n+1}, y_{n+2},\dots,y_{2n})$. This terminates the proof of theorem 2.

Corollary 1 : Let $S_n = \{x : x \geq 0, \sum_{i=1}^{n} x_i^2 = 1\}$. Let A be a P-matrix or

or an N-matrix of first category. Then there exists an $\alpha > 0$ such that $\max_{1 \leq i \leq n} (Ax)_i \geq \alpha$ for every $x \in S_n$.

Proof : From theorem 1, for every $x \in S_n$ it follows that Ax has at least one coordinate strictly positive. Since $\max_{1 \leq i \leq n} (Ax)_i$ is continuous in x and S_n is compact,

$$\min_{x \in S_n} \max_{1 \leq i \leq n} (Ax)_i = \max_{1 \leq i \leq n} (Ax^o)_i \text{ for some } x^o \in S_n. \text{ Set } \alpha = \max_{1 \leq i \leq n} (Ax^o)_i. \text{ This } \alpha$$

will satisfy the requirements of the corollary and the proof is complete.

In order to give a characterization theorem for P-matrix, we introduce the following definition:

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