Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Asymptotic Analysis II –

Surveys and New Trends

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F. Verhulst
State University of Utrecht
Mathematical Institute
Budapestlaan 6
3508 TA Utrecht, The Netherlands

AMS Subject Classifications (1980): 34 E, 35 C 20, 70 H

ISBN 3-540-12286-9 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-12286-9 Springer-Verlag New York Heidelberg Berlin Tokyo

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

PREFACE

That part of applied mathematics which we call asymptotic analysis or perturbation methods is a fast-moving field where new results are continually being published on both methods and applications. The papers in this field are spread throughout the literature and can be found in the established mathematical journals and in current publications connected with parts of physics, engineering sciences, biology etc. This demonstrates the importance of asymptotic analysis, having so many relations with various parts of mathematics and applied fields, but it doesnot make it easy to keep track of the new literature.
The realisation of this fact triggered off the idea to publish a Lecture Notes mainly of a surveying character and when sounded out at several international conferences during 1981 met with an unanimously positive reaction. Most people who were invited to contribute, sent in a paper and the result is not a complete, but a fairly representative survey of the modern literature. We also note that some of the subjects which are lacking, like the asymptotics of free boundary problems or singular perturbations of spectra, are covered by recent conferences on asymptotic analysis. See for instance the Oberwolfach Conference Proceedings on Singular Perturbations 1981, Lecture Notes in Mathematics 942, W. Eckhaus and E.M. de Jager, eds.
The patience which I had to maintain as editor was matched by the patience of quite a number of authors to whom I suggested alterations and additions. In a few cases I asked for and obtained external referee reports.
Of the three contributions from the Soviet-Union two were written in English (Bakai and Bogaevsky/Povzner). The third paper was written in Russian and it is a pleasure to acknowledge the assistance of Peter Schuur with the translation and editing of this text.

Ferdinand Verhulst

CONTENTS

PART 1

SURVEY PAPERS

DYNAMICAL SYSTEMS DRIVEN BY SMALL WHITE NOISE: ASYMPTOTIC ANALYSIS AND APPLICATIONS

by

Z. Schuss
Department of Mathematics
Tel Aviv University
Ramat Aviv, Israel

and

B. J. Matkowsky
Department of Engineering Sciences and Applied Mathematics
Northwestern University
Evanston, Illinois

1. First Passage Problems in Stochastic Differential Equations of Mathematical Physics

The notion of white noise was introduced by Einstein [30] as a model for a particle immersed in a fluid, and subjected to random forces, due to its collisions with the particles of the surrounding medium. The motion of such a particle is referred to as Brownian motion. This notion led Langevin [31] to introduce a white noise perturbation term into Newton's equations of motion, as a model for the dynamics of a particle which moves in an external field of forces, but which, in addition, is subjected to the irregular forces of the surrounding medium. Of particular interest is the situation in which the particle is originally caught in a potential well, but may escape in the course of time, by passing over a potential barrier. This situation was considered by Kramers [14], who introduced the motion of a Brownian particle in a potential well, as a model for chemical reactions. Kramers' model is a very convenient tool for modeling a variety of physical phenomena involving random collisions or thermal fluctuations. Thus the Langevin equation has been used as a model for atomic migration in crystals [26,32], ionic conductivity [20], the effect of fast electrons in laser plasma [33], chemical reations [14,21], the effect of thermal fluctuations on Josephson junction devices [2], to name but a few.

In the case of Josephson junctions, the dynamical system describing the junction is more complicated than that described by Kramers. The system is not trapped in a potential well, but rather is attracted by a stable limit cycle. It eventually breaks away from the domain of attraction of the limit cycle, due to the thermal white noise present in the system, and reachers a stable equilibrium state. This latter state is again described by Kramers' model.

In the theory of communication, in filtering signals from noisy measurements in particular, the dynamics of the system describing the process, is also of the type described above. Thus for example the dynamical system describing the estimation errors of the phase locked loop (PLL) (a demodulator for FM transmission) contains white noise type perturbation terms. These random terms drive the system away from one stable equilibrium state into another thus causing the PLL to occasionally lose its lock. This effect, called cycle slipping, leads to degradation in the performance of the PLL [4,24,25]. We now describe the various models.

In section 2 we describe Kramers' diffusion model mathematically. Here we describe various physical models leading to Kramers' problem. In 1940, H. A. Kramers [14] introduced a diffusion model for chemical reactions. In this model the reacting particle is caught in a potential well U(x) (which corresponds to the chemical bonding forces) and is subjected to random collisions with the surrounding medium. The particle will eventually be pushed over the potential barrier by the random forces due to collisions. The mean escape time $\bar{\tau}$ determines the reaction rate κ by

$$\kappa = \frac{1}{2\bar{\tau}} \quad . \tag{1.1}$$

Here κ is the fraction of particles entering the reaction per unit time. The factor $\frac{1}{2}$ expresses the fact that a particle reaching the barrier either returns or crosses with equal probabilities. In the simplest case of dissociation for example, the factor κ enters the equation for the reactant concentration c(t) in the form

$$-\frac{dc}{dt} / c = \kappa \quad .$$

The first expression for κ was given by Arrhenius in the form

$$\kappa = \nu e^{-Q/kT}$$

where Q is the height of the potential barrier, k is Boltzmann's constant, T is temperature and ν is a preexponential factor, characteristic of a given reaction. Kramers' purpose was to give a microscopic model of the motion of the reacting particle and thus to find the dependence of κ on the properties of the medium, e.g. on the viscosity β and temperature T. Kramers used the Langevin equation of motion

$$\ddot{x} + \beta\dot{x} + U'(x) = \sqrt{2\beta kT}\ \dot{w} \tag{1.2}$$

to describe the dynamics of the reaction. To compute κ he considered the Fokker-Planck equation for the transition probability density $p(x,\dot{x},t)$

$$\frac{\partial p}{\partial t} = U'(x)\frac{\partial p}{\partial \dot{x}} - \dot{x}\frac{\partial p}{\partial x} + \beta\frac{\partial}{\partial \dot{x}}\left(\dot{x}p + \frac{kT}{m}\frac{\partial p}{\partial \dot{x}}\right) \quad . \tag{1.3}$$

This method of determining κ from (1.3) is not easily generalized to dimensions higher than one, though in the case of large dissipation such generalization was given by Landauer and Swanson [15]. We introduce a new method for computing κ. It is based on a boundary value approach to the problem rather than on equation (1.3), and is readily generalized to higher dimensions.

The quantity $\bar{\tau}$ appears in many other physical problems. Thus, for example, $\bar{\tau}$ determines the diffusion constant for atomic migration in a crystal as follows. The potential $U(x)$ in a crystal is a periodic function of period λ, say. The thermal vibrations of the crystallic lattice create a random force acting on a diffusing particle so that equation (1.2) can be used to describe its motion. Due to this random force the particle performs a random walk between the equilibrium states in the potential wells by making jumps of size $\pm\lambda$ at time intervals $\bar{\tau}$ apart, on the average. Thus the probability of getting from x to y in time $t = n\bar{\tau}$ is given by

$$p(x,y,n\bar{\tau}) = \frac{1}{2}\, p(x+\lambda,y,(n-1)\bar{\tau}) + \frac{1}{2}\, p(x-\lambda,y,(n-1)\bar{\tau}) \quad .$$

Expanding in $\bar{\tau}$ and λ we obtain

$$\frac{\partial p}{\partial t} = \frac{\lambda}{2\bar{\tau}} \frac{\partial^2 p}{dx^2} \equiv D \frac{\partial^2 p}{\partial x^2} \quad .$$

Thus the diffusion coefficient D is given by

$$D = \frac{\lambda}{2\bar{\tau}} \quad . \tag{1.4}$$

In higher dimensions the diffusion coefficient is replaced by a diffusion tensor which depends on the structure of the crystallic lattice (cf. [26,32]).

To determine the diffusion tensor for nonisotropic lattices, e.g. the zinc lattice, we note that the jump frequencies will be different in different directions and the probability of passage through the saddle points connecting the potential wells may be different in different directions as well. As an example, the diffusion coefficient in the (x,y)-plane may differ from that in the z-direction, so that the diffusion equation will take the form

$$D_1\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + D_2 \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} \quad .$$

The different values of D_1 and D_2 may be caused by jump frequencies in the (x,y)-plane which differ from those in the z-direction. In addition, differences in the sizes of the jumps as well as differences in the probabilities of the jumps may cause the diffusion coefficients D_1 and D_2 to differ. This last situation is best illustrated by an example. Consider a lattice in which the interstitial particle may move from one cell to another along one of 6 possible paths as shown in Fig. 1.1.

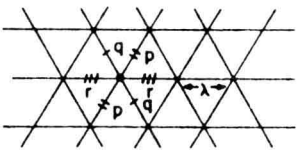


Fig. 1.1. Hexagonal lattice

Let the jump probabilities along lines with one, two or three hatches have probabilities q, p and r respectively. Thus $2(p+q+r) = 1$. The backward Kolmogorov equation for the transition probability of this random walk is given by

$$
\begin{aligned}
&P(x,y,a,b,(n+1)\bar{\tau}) \\
&\quad = (\text{Probability of reaching the point } (x,y) \text{ in the lattice in time } (n+1)\bar{\tau} \text{ given that initially the particle is at } (a,b)) \\
&\quad = r[P(x,y,a+\lambda,b,n\bar{\tau}) + P(x,y,a-\lambda,b,n\bar{\tau})] \\
&\qquad + p\left[P\left(x,y,a+\tfrac{1}{2}\lambda,b+\tfrac{\sqrt{3}}{2}\lambda,n\bar{\tau}\right) + P\left(x,y,a-\tfrac{1}{2}\lambda,b-\tfrac{\sqrt{3}}{2}\lambda,n\bar{\tau}\right)\right] \\
&\qquad + q\left[P\left(x,y,a-\tfrac{1}{2}\lambda,b+\tfrac{\sqrt{3}}{2}\lambda,n\bar{\tau}\right) + P\left(x,y,a+\tfrac{1}{2}\lambda,b-\tfrac{\sqrt{3}}{2}\lambda,n\bar{\tau}\right)\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
&P(x,y,a,b,(n+1)\bar{\tau}) - P(x,y,a,b,n\bar{\tau}) \\
&\quad = P_t(x,y,a,b,n\tau)\bar{\tau} + o(\bar{\tau}) \\
&\quad = \frac{\lambda^2}{2}\left[P_{aa}(x,y,a,b,n\bar{\tau})\left(2r + \tfrac{1}{2}p + \tfrac{1}{2}q\right) + P_{ab}(x,y,a,b,n\bar{\tau})\right. \\
&\qquad \left. \times \sqrt{3}(p-q) + P_{bb}(x,y,a,b,n\bar{\tau})\,\tfrac{3}{2}(p+q)\right] + o(\lambda^2) \quad .
\end{aligned}
$$

Thus

$$
\frac{\partial P}{\partial t} = \frac{\lambda^2}{2\bar{\tau}}\left[\frac{\frac{1}{2}+3r}{2}\frac{\partial^2 P}{\partial a^2} + \sqrt{3}(p-q)\frac{\partial^2 P}{\partial a \partial b} + \frac{3}{2}\left(\frac{1}{2}-r\right)\frac{\partial^2 P}{\partial b^2}\right]
$$

Therefore, the diffusion matrix has the form

$$
D = \frac{\lambda^2}{2\bar{\tau}}\begin{bmatrix} \frac{1}{2}\left(\frac{1}{2}+3r\right) & \frac{\sqrt{3}}{2}(p-q) \\ \frac{\sqrt{3}}{2}(p-q) & \frac{3}{2}\left(\frac{1}{2}-r\right) \end{bmatrix} .
$$

In principal axis coordinates, D is given by

$$
D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}
$$

where

$$
\lambda_{1,2} = \frac{\lambda^2}{4\bar{\tau}}\left[1 \pm \sqrt{1 - 12(pq+qr+pr)}\right] .
$$

This example illustrates how the diffusion coefficients may differ in different directions. In fact however, in-plane anisotropic effects in hexagonal lattices are not observed experimentally. They are observed in other lattices, e.g. in orthorhombic lattices.

More generally, let the possible sites into which a particle, initially located at the origin, can jump be given by the vectors $z_k = (z_k^1, z_k^2, z_k^3)$ $(k = 1, \cdots, 2n)$. Furthermore, assume that the jumps occur at time intervals $\bar{\tau}$ apart, with

probabilities $p_1,\cdots,p_{2n}$ respectively, where $\sum_i p_i = 1$. Then the transition probability of the random walk satisfies the backward equation

$$(1.5) \qquad P(x,y,(n+1)\bar{\tau}) = \sum_{j=1}^{2n} P(x+z_j,y,n\bar{\tau})p_j$$

where $x = (x_1,x_2,x_3)$, $y = (y_1,y_2,y_3)$. Hence, expanding the left and right hand sides (1.5) in Taylor series about $n\bar{\tau}$ and z_j respectively, and retaining only the leading terms we obtain

$$\frac{\partial P}{\partial t} = \sum_{i,j=1}^{3} D_{ij} \frac{\partial^2 P}{\partial x_i \partial x_j} \quad ,$$

where $t = n\bar{\tau}$ and

$$(1.6) \qquad D_{ij} = \sum_{k=1}^{n} p_k z_k^i z_k^j / \bar{\tau} \quad .$$

We have used the facts that $z_{n+j} = -z_j$ and $p_{n+j} = p_j (j=1,\cdots,n)$, which follows from the periodicity of the lattice, in order to cancel the first order spatial derivatives. Thus, the distribution of the exit points, as well as the expected exit time, determines the diffusion matrix in the lattice [24,25,32,18,26].

Next we consider a model for the conductivity of ionic crystals [20,25,40]. We thus consider the motion of an ion of charge q in a crystal which is subjected to a uniform electrostatic field. In a simplified one dimensional model, the potential U is given by

$$(1.7) \qquad U(x) = m^{-1}[-qEx + qK \sin \omega x] \quad , \quad A < x < T$$

and

$$U \to \infty \quad \text{as} \quad |x| \to \infty \quad ,$$

(cf. Fig. 1.2). Here the periodic term represents the internal potential of the crystal, and the linear term represents the potential of the external electrostatic field.

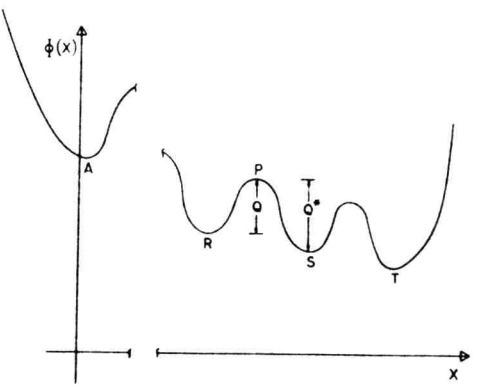


Fig. 1.2. Potential for ionic conductivity in crystals

In the absence of diffusion, a particle trapped at R cannot move to S. Therefore the mere presence of an external field E will not cause conductance, unless it is sufficiently strong. Nevertheless, ionic conductivity is observed even for weak electrostatic fields. Thus we attribute ionic conductivity to diffusion. We observe that the potential barrier to be overcome for a motion from R to S to occur is $Q = U(P) - U(R)$, which is lower than the potential barrier $Q^* = U(P) - U(S)$, which must be overcome for a motion from S to R. To compute the conductivity of the crystal, we compute the net current $I = I_{RS} - I_{SR}$ flowing from R to S, and employ it in

$$\sigma = \frac{\partial I}{\partial V} \quad , \tag{1.8}$$

where $V = U(S) - U(R)$, is the potential difference between S and R. We note that V is given by

$$V = Q - Q^* = -\frac{2\pi qE}{\omega m} \quad . \tag{1.9}$$

Now, the current flowing to the right is

$$I_{RS} = \frac{Cq}{\bar{\tau}_R} \quad , \tag{1.10}$$

while the current flowing to the left is

$$I_{SR} = \frac{Cq}{\bar{\tau}_L} \tag{1.11}$$

where $\bar{\tau}_R$ and $\bar{\tau}_L$ are the average times required to overcome the potential barriers Q and Q^* respectively, and C denotes the concentration of ions. Combining (1.8)-(1.11) with (3.5) we obtain

$$\sigma = \frac{\partial}{\partial E}\left(\frac{Cq\omega^2 K}{4\pi^2 m}\sqrt{1 - \left(\frac{E}{\omega k}\right)^2}\, e^{-mQ/kT}\left(1 - e^{-\frac{2\pi qE}{\omega kT}}\right)\right) \quad , \tag{1.12}$$

which is valid for $kT << mQ$. If $\frac{E}{\omega K} << 1$, then the leading term in the asymptotic expansion of (1.12) with respect to the parameter $\frac{E}{\omega K}$ is given by

$$\sigma_0 = \frac{Cq^2\omega K}{2\pi kTm} e^{-mH/kT} \quad , \tag{1.13}$$

where $H = \frac{2Kq}{m}$ is the value of Q when $E = 0$. We recall from (1.4) and (3.5) that the quantity $\frac{\omega K}{2\pi m} e^{-mH/kT}$ is the one-dimensional diffusion coefficient for atomic migration in crystals. Therefore (1.13) may be written as

$$\sigma_0 = \frac{Cq^2 D}{kT} \quad . \tag{1.14}$$

This formula for ionic conductance was derived by Nernst and Einstein (cf. [25]). Thus formula (1.12) is a generalization of the Nernst-Einstein formula, since (1.12) reduces to (1.14) for small $\frac{E}{\omega K}$. In nonisotropic crystals, $D = \{D_{ij}\}$ is a diffusion tensor so that $\sigma = \{\sigma_{ij}\}$ is a conductance tensor. A formula for $\{D_{ij}\}$ is given by (1.6).

The Nernst-Einstein formula (1.14) expresses the fact that the ratio of the current flow to the applied electric field E is a constant, depending on the properties of the crystal, but independent of E. Thus the current is a linear function of E so that Ohm's law is obeyed. In contrast, formula (1.12) exhibits a nonlinear dependence of σ on E, which for small applied electric fields reduces to (1.14). This nonlinear effect has been observed experimentally [39].

We now describe the Josephson junction tunnel, which leads to a model that is similar to, yet different from Kramers' model. A Josephson tunnel consists of two superconductors separated by a thin insulator. In its simplest form the Josephson effect claims that up to a certain current the voltage across a sufficiently thin tunnel junction is zero and the I-V characteristic is shown in Fig. 1.3.

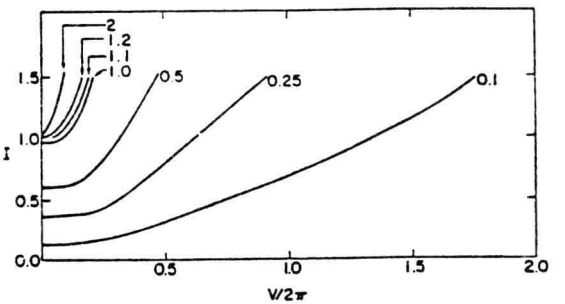


Fig. 1.3. I-V characteristics

At temperature sufficiently close to the transition temperature, thermal fluctuations can disrupt the coupling of the phases of the order parameters of the junction. The dc-Josephson current thereby acquires a noise voltage with a non-zero average value. Consider a Josephson junction in series with a large external resistance and battery, so that the junction is essentially being driven by a constant-current source. The equations of state are then

(1.15) $\quad d\theta/dt = 2eV/A$

(1.16) $\quad \dfrac{CdV}{dt} = I - I_J(T)\sin\theta - V/R + \tilde{L}(t)$

Equation (1.16) is the Josephson condition relating to θ, the difference in phases of the order parameter on opposite sides of the junction, and V, the potential difference. We assume that the area of the junction is sufficiently small so that in the absence of external magnetic fields the current is uniformly distributed over the area. Equation (1.16) expresses the condition of the conservation of charge: C is the capacitance of the junction; $I_J(T)$ is the maximum Josephson current at temperature T in the absence of noise; R is the resistance of the junction (assumed constant in the range of temperatures under consideration, i.e. a few degrees K^O); and $\tilde{L}(t)$ is a fluctuating noise current assumed to be $\tilde{L}(t) = \sqrt{\frac{2Tk}{R}}\,\dot{w}$, where $\dot{w}$ is

dependence of these lifetimes on the DC-SQUID parameters, on the external driving current I and on the external magnetic flux Φ_{ex}.

Here we find this dependence for the shunted DC-SQUID ($\beta_c < 1$). In this case we use the Smoluchowski approximation of the Fokker-Planck equation [25]. The resulting problem is equivalent to the exit problem of a particle out of a two dimensional potential well, a case for which Kramers' results [14] can not be used and have to be extended. Such an extension was done by Landauer and Swanson for a single saddle point [15]. Here, however, we use a new method based on the results of Matkowsky and Schuss [19-22]. Using this method we compute the mean lifetime in a two dimensional potential well with several saddle points on its boundary and the relative probability of exit through each saddle point. These results can be used to construct the I-V characteristics of the DC-SQUID. Numerical simulations for such a DC-SQUID were carried out by Tesche and Clarke [37]. We note that this method can be applied to elements containing more than two coupled Josephson junctions.

We consider a DC-SQUID which consists of two identical Josephson junctions and which is driven by an external current source I and an external magnetic flux Φ_{ex}, as shown in Fig. 1.4. Assuming the RSJ model [23,28] for the junctions we

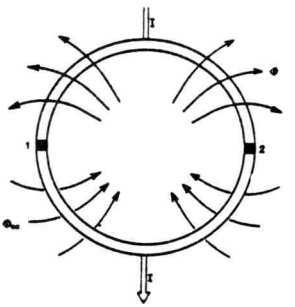


Fig. 1.4. The DC-SQUID model

obtain the following equations of motion for θ_i's, the phase differences across the junctions [37,38]

$$\ddot{\theta}_1 + G\dot{\theta}_1 + \sin\theta_1 = \frac{I}{2I_J} - K(\theta_1 - \theta_2 + \theta_{ex}) \tag{1.19}$$

$$\ddot{\theta}_2 + G\dot{\theta}_2 + \sin\theta_2 = \frac{I}{2I_J} + K(\theta_1 - \theta_2 + \theta_{ex})$$

where

$$G \equiv (\omega_J RC)^{-1} \equiv \beta_c^{-1/2} \quad ; \quad \omega_J^2 \equiv \frac{2eI_J}{hC}$$

$$K \equiv \frac{\Phi_0}{2\pi LI_J} \quad ; \quad \theta_{ex} \equiv \frac{2\pi\Phi_{ex}}{\Phi_0} \quad ; \quad \Phi_0 \equiv \frac{h}{2e} \tag{1.20}$$

standard white noise. Setting

$$\gamma = hI_1(T)/\sigma T \quad , \quad x = I/I_J(T) \quad ,$$

$$\Omega = RC(2eI_J/hC)^{1/2} \quad , \quad RC = \eta^{-1} \quad ,$$

$$p = (hC/2e)V \quad , \quad M = (h/2e)^2 C \quad , \quad L = \left(\frac{h}{2e}\right)\tilde{L}$$

$$U = -\frac{1}{2}\gamma T(x\theta + \cos\theta)$$

we obtain

$$\dot{\theta} = p/M \quad ; \quad \dot{p} = \frac{dU}{d\theta} - \eta p - L(t) \quad . \tag{1.17}$$

There are three cases to be considered. (i) η is large, i.e. $\eta > 1$, say (ii) $1 > \eta > \frac{kT}{Q}\omega$, where Q is the height of the potential barrier, and $\omega^2 = U''(\theta_0)$ where $\theta_0 = \pi - \arcsin x$, and (iii) η is small, that is $\eta < \frac{kT}{Q}\omega$.

(i) For large η the Smoluchowski approximation can be used [25]. It is given by

$$\theta'(s) = \frac{dU}{d\theta} + \sqrt{2kT}\, w'(s)$$

where $t = \eta M s$. In this case we have

$$Q = x(2\arcsin x - \pi) + 2\sqrt{1-x^2}$$

$$Q^* = x(\pi + 2\arcsin x) + 2\sqrt{1-x^2}$$

so that [20]

$$\left(\frac{1}{\tau_L} - \frac{1}{\tau_R}\right) = \frac{2}{\pi\gamma T}\sqrt{1-x^2}\,\sinh\frac{\pi x}{kT}\exp\left[-2\left(\sqrt{1-x^2} + x\arcsin x\right)\Big/kT\right] \equiv \frac{2}{\pi\gamma T} f(x). \tag{1.18}$$

Now, since the average voltage is proportional to $\langle\dot{\theta}\rangle$ we have

$$\bar{V} = \frac{\pi h}{e\eta M}\left(\frac{1}{\tau_L} - \frac{1}{\tau_R}\right)$$

so that

$$\bar{V} = Kf(x)$$

where K is a constant (cf. (1.12)). This agrees with [1] where a different method has been used. Their method is restricted to the one dimensional case, where explicit solutions can be obtained. Cases (ii) and (iii) are discussed in Section 7.

In the case of a system of several independent Josephson junctions and several independent thermal e.m.f.'s one obtains a system of stochastic differential equations so that a higher dimensional analysis is called for. This is the case, for example, of the DC-SQUID model [35].

The DC-SQUID with small coupling ($LI_J > h/2e$, where L is the self inductance and I_J is the junctions critical current) has several types of meta-stable states [36,38] and therefore it can be used as a logic element. At a finite temperature the thermal noise causes spontaneous transitions between the various states. Hence, the meta-stable states have finite mean lifetimes. It is of interest to know the

R and C are the resistance and capacitance of the junctions and time is measured in units of ω_J^{-1}. Assuming the ring is made of a super-conductor whose thickness is bigger than the Josephson penetration depth, we have

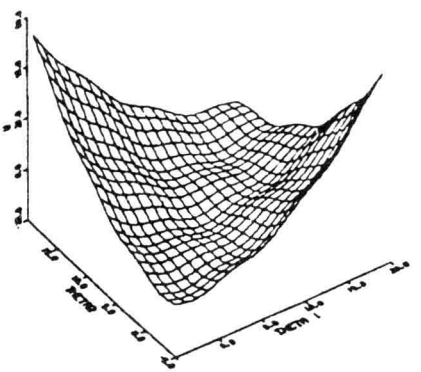
$$\theta_2 - \theta_1 = \frac{2\pi\Phi}{\Phi_0}$$

where Φ is the total magnetic flux through the ring.

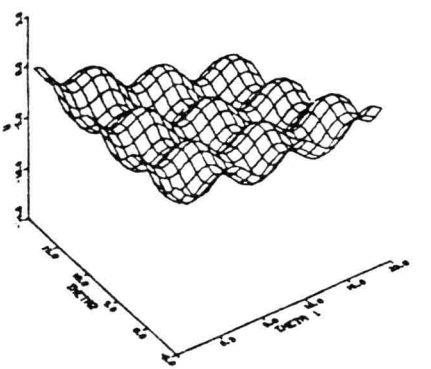
Equations (1.19) can be interpreted as the equations of motion of a particle with unit mass and dissipation G in a two dimensional field derived from the potential $U(\theta_1,\theta_2)$ given by

$$U(\theta_1,\theta_2) = \frac{1}{2}\kappa(\theta_1 - \theta_2 + \theta_{ex})^2 - \cos\theta_1 - \cos\theta_2 - \frac{I}{2I_J}(\theta_1+\theta_2) \quad . \tag{1.21}$$

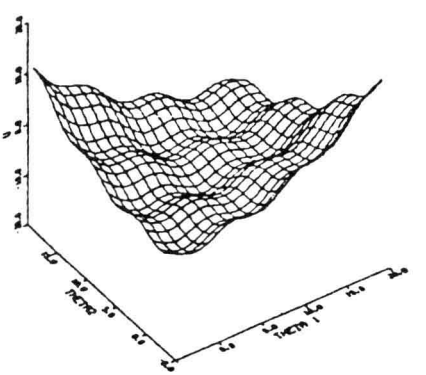
Note that the energy is measured in units of $\Phi_0 I_J/2\pi$. The potential surfaces $U(\theta_1,\theta_2)$ are shown in Fig. 1.5.



(a)



(b)



(c)

Fig. 1.5. Potential energy surfaces for $I/I_J = 0.5$. (a) $K = 1/\pi$, $U(\theta_1,\theta_2)$, $\theta_{ex} = 0$; (b) $K = 0.2$, $\theta_{ex} = 0$; (c) $K = 0.1$, $\theta_{ex} = \frac{\pi}{2}$.